XXIII.—*On the Stresses due to Compound Strains.* By Professor C. NIVEN. Communicated by Professor TAIT.

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§ 1. The general problem in elasticity, as usually presented for solution, supposes the elastic substance to pass initially from a state without strain. But important cases exist where it must be conceived to start from a state already under considerable stress. When this is the case, the constitution of the solid undergoes great change, as is shown by the fact that strained glass loses its isotropic property, and becomes doubly refractive. This subject was long ago attacked by CAUCHY, who, by means of the theory of molecular actions, deduced the existence, in the expressions for the stresses due to the secondary strains, of terms proportional to the initial or primary stresses. The problem has been since discussed by MM. DE ST VENANT and BOUSSINESQ, who have applied to it GREEN's expression for the energy stored up during the strain. But the question, in their hands, still retains traces of CAUCHY's hypothetical element, inasmuch as their expression for the potential energy was deduced by means of the molecular theory. M. DE ST VENANT even considers it a strong argument for the truth of the latter, that it is indispensable in the discussion of this problem. These authors have also failed to see in what way the remaining part of the potential depends on the original strain.

In the present paper the treatment of the matter is grounded solely on the laws according to which one set of distortions may be superposed on another. The resultant strains, it is shown, differ from the primary by linear functions of the secondary ones, whether the latter be small or large. The general expression for the potential energy is thus found independently of any hypothesis, and so far coincides with the result of M. BOUSSINESQ. In the further development of the subject, I have confined myself to the case where the potential due to the primary strain is a quadratic function of the corresponding strains, and also where the substance was originally isotropic.* In this case, it appears that the increase of the potential energy, so far as it involves the primary strain, depends only on six quantities called *quasi-strains.* The primary stresses, each multiplied by the dilated unit-volume, also depend only on these six functions.

* See, however, the note added 16th February 1876.

From the formulæ which determine the small motions of such a substance, when strained homogeneously at first, it follows that the vibrations in a plane wave are not generally in its front, and that for each position of the latter there are three real and different velocities of transmission. If the primary stress be symmetrical with respect to an axis, the wave surface breaks up into an ellipsoid of revolution and a surface of the fourth class.*

In these investigations it was necessary to determine expressions for the stresses due to distortions of any magnitude. My results for these, though not their symbolical form, have, as I find, been already published in the "Comptes Rendus," t. lxxi. p. 400, by M. BOUSSINESQ. In the present paper two demonstrations are offered, one derived from the rules for compounding two strains.

The present paper contains also a general theory of the laws according to which strains and certain other physical magnitudes are transformed with respect to different sets of rectangular axes.

§ 2. Let the three intersecting edges of a rectangular element-parallelopiped PH, PK, PL, be called h, k, l, and let them be strained into P′H′, P′K′, P′L′, the displacements of P being u, v, w. The co-ordinates of H′, K′, L′, relative to P′ are h, k, l, multiplied respectively by the members of the successive columns

of the determinant $\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \nabla,$

where $$u_1 = \frac{d}{dx}(x+u) = 1 + \frac{du}{dx},\ u_2 = \frac{d}{dy}(x+u) = \frac{du}{dy},\ \ldots\ w_3 = 1 + \frac{dw}{dz}.$$

The minors of its constituents $u_1 \ldots w_3$ we denote by $U_1 \ldots W_3$; and the geometrical elements of the strain are expressed by these laws:—

(1) Any infinitesimal volume, dV, strains into dV', where $dV' = \nabla . dV$.

(2) If any infinitesimal surface-element $d\Sigma$, whose projections on the co-ordinate planes are A_x, A_y, A_z, strains into $d\Sigma'$ having corresponding projections A'_x, A'_y, A'_z, then

$$\left.\begin{aligned} A'_x &= A_x . U_1 + A_y . U_2 + A_z . U_3 \\ A'_y &= A_x . V_1 + A_y . V_2 + A_z . V_3 \\ A'_z &= A_x W_1 + A_y W_2 + A_z W_3 \end{aligned}\right\} \quad . \quad . \quad . \quad (2).$$

(3). The six *strains* are given by half the sums of the squares of the elements in the 3 columns of ∇ less 1, and by the sums of the products of the elements

* If the solid be "incompressible" for small strains, and if the primary strains be small, the equation giving the velocity of transmission of a plane wave is similar in form to that given by FRESNEL in his Theory of Double Refraction.

in each pair of columns. They are denoted by s_{xx}, s_{yy}, s_{zz}, s_{yz}, s_{zx}, s_{xy}. Somewhat further on, in dealing with long formulæ, I shall write them simply $\frac{1}{2}$A, $\frac{1}{2}$B, $\frac{1}{2}$C, D, E, F. They are readily expressible in terms of the edges of the tetrahedron P′H′K′L′.

The infinitesimal strained sphere $x_1'^2 + y_1'^2 + z_1'^2 = K^2$ was produced by straining the ellipsoid

$$x_1^2 + y_1^2 + z_1^2 + 2(s_{xx}.x_1^2 + s_{yy}.y_1^2 + \ . \ . \ . \ + s_{xy}\,.\,x_1 y_1) = K^2. \quad . \quad . \quad (3).$$

[$(x_1\ y_1\ z_1)$ are the relative co-ordinates of a point very near to P].

This result will be afterwards useful in determining the laws of resolution of a strain-system from one set of axes to another.

Equation of Energy.

§ 3. The energy required to strain any element $dx\,dy\,dz$ into its final form is W $dx\,dy\,dz$, where W is a function of the strains which, as will be found, cannot be of a lower degree than the second.

Let S_{xx}, S_{yy}, . . . S_{xy} be the six stresses at any point considered as tractions. The work on any triangular element, and so on any element $d\Sigma'$ of the strained surface, due to an infinitesimal displacement δu, δv, δw is the sum of the quantities of work done in its projections on the three co-ordinate planes. Its value is therefore

$$A_x'(S_{xx}\delta u + S_{xy}\delta v + S_{xz}\delta w) + A_y'(S_{yx}\delta u + S_{yy}\delta v + S_{yz}\delta w) + A_z'(S_{zx}\delta u + S_{zy}\delta v + S_{zz}\delta w).$$

The work done over the surface of a strained body is therefore

$$\begin{aligned} &\iint dy\,dz\,\{(S_{xx}\,.\,U_1 + S_{xy}V_1 + S_{xz}\,.\,W_1)\,\delta u + \text{two other similar terms}\} \\ &+ \iint dz\,dx\,\{\text{three similar terms}\} + \iint dx\,dy\,\{\text{three similar terms}\} \quad . \quad (4). \end{aligned}$$

Now the work done by internal stresses is of course $\iiint \delta W\,.\,dx\,dy\,dz$.

$$\left.\begin{aligned} \text{And} \qquad & \delta W = \frac{dW}{ds_{xx}}\delta s_{xx} + \frac{dW}{ds_{xy}}\delta s_{xy} + \ . \ . \ . \\ \text{and} \qquad & \delta s_{xx} = u_1\delta u_1 + v_1\delta v_1 + w_1\delta w_1 \\ & \delta s_{yz} = u_3\delta u_2 + v_3\delta v_2 + w_3\delta w_2 + u_2\delta u_3 + v_2\delta v_3 + w_2\delta w_3 \end{aligned}\right\} \quad . \quad (5).$$

The work done by the impressed forces $= \rho_0 \iiint (X\delta u + Y\delta v + Z\delta w)\,dx\,dy\,dz$.

That consumed in producing motion $= \rho_0 \iiint \left(\frac{d^2u}{dt^2}\delta u + \frac{d^2v}{dt^2}\delta v + \frac{d^2w}{dt^2}\delta w\right) dx\,dy\,dz$.

On substituting these values in the equation which expresses the conservation of energy, and equating according to LAGRANGE's principles the coefficients of δu . . . to zero, both throughout the interior and over the surface, we obtain

the equation of equilibrium, and expressions for the elastic forces. It is of course to be noticed that the expression for the work due to the potential energy of strain must be first integrated by parts throughout the volume of the solid under consideration. This volume may be any portion whatever of the whole solid, or the whole body itself.

Determination of Stresses and Equations of Equilibrium.

§ 4. On equating separately to zero the co-efficients of δu . . . taken over the three separate boundary integrals, we obtain nine equations to find the six stresses. But this difficulty is explained by the symmetry of the expressions for the tangential stresses which indicate that they may be derived from different sets of equations by different routes.

The general type of these fundamental boundary conditions is given by

$$\left.\begin{aligned} S_{xx} \,.\, U_1 + S_{xy}\, V_1 + S_{xz} \,.\, W_1 &= \frac{dW}{ds_{xx}} \,.\, u_1 + \frac{dW}{ds_{xy}} u_2 + \frac{dW}{ds_{xz}} \,.\, u_3 \\ S_{xx} \,.\, U_2 + S_{xy} \,.\, V_2 + S_{xz} \,.\, W_2 &= \frac{dW}{ds_{yx}} \,.\, u_1 + \frac{dW}{ds_{yy}} u_2 + \frac{dW}{ds_{yz}} \,.\, u_3 \end{aligned}\right\} \quad . \quad . \quad (6).$$

From these we may derive either the stresses in terms of the rate of change of W or conversely. The former are given by the general formulæ—

$$\left.\begin{aligned} S_{xx} \cdot \nabla &= \frac{dW}{ds_{xx}} \,.\, u_1^2 + \frac{dW}{ds_{yy}} u_2^2 + \ldots + 2\,\frac{dW}{ds_{xy}} \,.\, u_1 u_2 \\ &\vdots \\ S_{xy} \,.\, \nabla &= \frac{dW}{ds_{xx}} u_1 v_1 + \frac{dW}{ds_{yy}} u_2 v_2 + \ldots + \frac{dW}{ds_{xy}} (u_1 v_2 + u_2 v_1) \end{aligned}\right\} \quad . \quad . \quad (7).$$

These results are manifestly capable of expression in a symbolical form, which can be best explained by studying any one form and comparing it with its more complete development. The form is

$$S_{\lambda\mu} \,.\, \nabla = \frac{dW}{ds} (\delta_{1,x} + \delta_{2,y} + \delta_{3,z})^{\lambda} \,.\, (\delta_{1,x} + \delta_{2,y} + \delta_{3,z})^{\mu} \quad . \quad . \quad (\alpha),$$

where $u = \delta^x$, $v = \delta^y$, $w = \delta^z$.

A similar form exists for the expression of $\frac{dW}{ds_{\lambda\mu}}$ in terms of the stress. It is

$$\frac{dW}{ds_{\lambda\mu}} \,.\, \nabla = S \,.\, (D^x_{i,x} + D^y_{i,y} + D^z_{i,z}) \,.\, (D^x_{j,x} + D^y_{j,y} + D^z_{j,z}) \quad . \quad (\beta),$$

where $U = D^x$, $V = D^y$, $W = D^z$, and the suffixes i, j mean 1, 2, 3 according as λ, μ mean x, y, z.

The typical form of the equation of motion or equilibrium is

$$\rho_0 F_\lambda + \left(\frac{d}{dx} \cdot \frac{dW}{ds_x} + \frac{d}{dy} \cdot \frac{dW}{ds_y} + \frac{d}{dz} \cdot \frac{dW}{ds_z}\right) \,.\, (\delta_{1,x} + \delta_{2,y} + \delta_{3,z})^{\lambda} = \rho_0 \frac{d^2\delta^\lambda}{dt^2} \quad . \quad (\gamma).$$

§ 5. In cases where the displacements are exceedingly small, it will be sufficient to neglect terms in $S_{\lambda\mu}$ of the order $\frac{d\,(u\,,\,v\,,\,w)}{d\,(x\,,\,y\,,\,z)}$ compared to unity whenever they enter. Thus, in this case,

$$S_{\lambda\mu} = \frac{dW}{ds_{\lambda\mu}} \quad . \quad . \quad . \quad . \quad . \quad (8).$$

In investigating strains to this order of magnitude, it will be sufficient to use the unstrained area $d\Sigma$ instead of the strained $d\Sigma'$ in the foregoing process. It is also obvious that W must be at least a quadratic function of the strains, for otherwise S might be finite while $(s_{xx}\ldots)$ were zero. This can only happen in the case of

§ 6. *Compound Strains.*—Let the displacements, strains, and stresses due to the first strains be $(\alpha\ \beta\ \gamma)$, $(s^0{}_{xx}\ldots.)$, $(S^0{}_{xx}\ldots)$; and let the new position of $(x\,y\,z)$ be $(\xi\ \eta\ \zeta)$, where $\xi = x + \alpha$, $\eta = y + \beta$, $\zeta = z + \gamma$.

Let displacements and strains in second strain be $(u\ v\ w)$, $(\sigma_{xx}\ldots)$; then in the state of stress produced by superposing the latter strain on the former, the displacements are $\alpha + u$, $\beta + v$, $\gamma + w$, while the strain and stress systems will be denoted by s and S. The values of $s_{xx}\ldots$ may be found in terms of $\sigma_{xx}\ldots$ and $\alpha_1\ldots\gamma_3$ either by direct differentiation or by the general method employed by THOMSON and TAIT for compounding strain-displacements (§ 185).

Let the co-ordinates of any unstrained point very near P be, relatively to P, $p\ q\ r$, and let them become, after the first and second strains $(p'\,q'\,r')$ $(p_1\,q_1\,r_1)$, then

$$\left.\begin{aligned} p' &= \alpha_1 p + \alpha_2 q + \alpha_3 r \\ q' &= \beta_1 p + \ .\ \ .\ \ .\ \ . \\ r' &= \ .\ \ .\ \ .\ \ + \gamma_3 r \end{aligned}\right\} \qquad \left.\begin{aligned} p_1 &= u_1 p' + u_2 q' + u_3 r' \\ q_1 &= v_1 p' + v_2 q' + v_3 r' \\ r_1 &= w_1 p' + w_2 q' + w_3 r' \end{aligned}\right\};$$

whence finally,

$$\left.\begin{aligned} p_1 &= (\alpha_1 u_1 + \beta_1 u_2 + \gamma_1 u_3)\,p + (\alpha_2 u_1 + \beta_2 u_2 + \gamma_2 u_3) q + \ldots \\ q_1 &= (\alpha_1 v_1 + \beta_1 v_2 + \gamma_1 v_3)\,p + (\alpha_2 v_1 + \beta_2 v_2 + \gamma_2 v_3) q + \ldots \\ r_1 &= (\alpha_1 w_1 + \beta_1 w_2 + \gamma_1 w_3)\,p + (\alpha_2 w_1 + \beta_2 w_2 + \gamma_2 w_3) q + \ldots \end{aligned}\right\}.$$

We thus find

$$\left.\begin{aligned} s_{xx} &= s^0_{xx} + \alpha_1^2\sigma_{xx} + \beta_1^2\sigma_{yy} + \gamma_1^2\alpha_{zz} + \beta_1\gamma_1\sigma_{yz} + \gamma_1\alpha_1\sigma_{zx} + \alpha_1\beta_1\sigma_{xy} \\ s_{yy} &= s^0_{yy} + \ldots \\ s_{xy} &= s^0_{xy} + 2\alpha_1\alpha_2\sigma_{xx} + 2\beta_1\beta_2\sigma_{yy} + 2\gamma_1\gamma_2\sigma_{zz} + (\beta_1\gamma_2 + \gamma_1\beta_2)\sigma_{yz} + (\gamma_1\alpha_2 + \gamma_2\alpha_1)\sigma_{zx} + (\alpha_1\beta_2 + \alpha_2\beta_1)\sigma_{xy} \end{aligned}\right\}(9).$$

The values of $s_{xx} - s^0_{xx}$, $\ldots$ $s_{xy} - s^0_{xy}$ we shall denote by $D_{xx}\ldots D_{xy}$.

It may be noted that these expressions, which are rigorous, show that the order in which the strains take place is not indifferent, and that $D_{\lambda\mu}$ is a linear function of the superposed strains.

The energy due a compound strain may be readily found, for let W^0 correspond to the initial strain, and W to the final,

$$W^0 = \phi(s^0), \quad W = \phi(s^0 + D).$$

Whence by TAYLOR's theorem,

$$W - W^0 = D_{xx}\frac{dW^0}{ds^0_{xx}} + \ldots + \frac{1}{2}\left\{\frac{d^2W^0}{ds^{0\,2}_{xx}}D^2_{xx} + 2\frac{d^2W^0}{ds^0_{xx}ds^0_{xy}}D_{xx}D_{xy} + \ldots\right\}.$$

If ∇^0 correspond to the first strain, $dx\,dy\,dz\ \nabla^0 = d\xi\,d\eta\,d\zeta$; substituting which, and remembering the expressions given in equations (7) for $S_{\lambda\mu}$, it follows that

$$W dx\,dy\,dz = W^0 dx\,dy\,dz + (S^0_{xx}\sigma_{xx} + \ldots + S^0_{xy}\sigma_{xy})d\xi\,d\eta\,d\zeta + \frac{1}{2\nabla^0}\left(\frac{d^2W^0}{ds^{02}_{xx}}D^2_{xx} + \ldots\right)d\xi\,d\eta\,d\zeta \qquad (10).$$

The two last parts of this expression I shall usually write $W' d\xi\,d\eta\,d\zeta$ and $w' d\xi\,d\eta\,d\zeta$ respectively.

[§ 7. Without assuming a knowledge of (7), the second member of (10) may be written

$$W' = T^0_{xx}\sigma_{xx} + \ldots + T^0_{xy}\sigma_{xy}$$

where the forms of $T^0_{\lambda\mu}$ are known. If we now suppose the second strains to be very small, and after differentiation, according to the formula (8), which are well known, put $u = v = w = 0$, we obtain

$$T^0_{\lambda\mu} = S^0_{\lambda\mu},$$

which furnishes a new proof of the forms for S_{xy}.]

§ 8. The system of additional stresses $(S - S^0)$ introduced by the second strain, which we shall always suppose to be *small* and of the first order, can now be found. The part of it which arises from w' may be found from equation (8).

We have in general, on putting $\nabla = \nabla^0 . \nabla'$,

$$\left.\begin{aligned} \nabla'.\,S_{\lambda\mu} &= S^0(\delta_{1,x} + \delta_{2,y} + \delta_{3,z})^\lambda . (\delta_{1,x} + \delta_{2,y} + \delta_{3,z})^\mu + \frac{dw'}{d\sigma_{\lambda\mu}} \\ \nabla' &= 1 + \frac{du}{d\xi} + \frac{dv}{d\eta} + \frac{dw}{d\zeta}. \end{aligned}\right\} \qquad (11).$$

In particular,

$$\left.\begin{aligned} S_{xx} - S^0_{xx} &= -S^0_{xx}\left(\frac{du}{d\xi} + \frac{dv}{d\eta} + \frac{dw}{d\zeta}\right) + 2\left(\frac{du}{d\xi}S^0_{xx} + \frac{du}{d\eta}S^0_{xy} + \frac{du}{d\zeta}S^0_{xx}\right) + \frac{dw'}{d\sigma_{xx}} \\ S_{xy} - S^0_{xy} &= -S^0_{xy}.\frac{dw}{d\zeta} + \frac{dv}{d\xi}.S^0_{xx} + \frac{du}{d\eta}S^0_{yy} + \frac{dv}{d\zeta}S^0_{xz} + \frac{du}{d\zeta}S^0_{yz} + \frac{dw'}{d\sigma_{xy}} \end{aligned}\right\} \qquad (12).$$

The equations of motion of the solid, as has been remarked by ST VENANT, assume a very simple form, even in the case where (S^0) are not constant throughout the substance. If they were produced solely by external force acting on the surface of the body, the new equations have for their type

$$\rho X' + \left(\frac{d}{d\xi}\cdot\frac{dw'}{d\sigma_{xx}} + \frac{d}{d\eta}\cdot\frac{dw'}{d\sigma_{xy}} + \frac{d}{d\zeta}\cdot\frac{dw'}{d\sigma_{xz}}\right) + \left(S^0_{xx}\frac{d^2}{d\xi^2} + \ldots + 2\,S^0_{xy}\frac{d^2}{d\xi d\eta} - \rho\frac{d^2}{dt^2}\right)u = 0 \quad (13).$$

If the strains (S^0) were produced under the action of forces X_0, Y_0, Z_0, we must add to the left hand member of this equation the expression

$$-\rho\left(X_0 + X_0\frac{du}{d\xi} + Y_0\frac{du}{d\eta} + Z_0\frac{du}{d\zeta}\right).$$

§ 9. *Resolution of Strains.*—In the foregoing results nothing has been assumed respecting the constitution of the solid, and they all apply equally well whatever that may be. But in the reduction to the case of isotropism, it is important to know the laws according to which strains are resolved in different directions; or, to put more concisely a particular case of the general problem, to know what conditions must be satisfied that two systems of stresses, defined with reference to different sets of rectangular axes, may be equivalent. When the strains are of the first order of small quantities, the system $(2s_{xx}, 2s_{yy} \ldots s_{xy})$ follow the same laws of resolution as the stress system $(S_{xx} \ldots S_{xy})$. For in the case of isotropic media we have

$$S_{xx} = A\theta + 2B\,.\,s_{xx}\,,\; S_{xy} = Bs_{xy}\,,$$

when $\theta = s_{xx} + s_{yy} + s_{zz}$ and is an invariant, and A, B are constants.

But the same law holds good when the strains are not *small,* as I first found by actual transformation of co-ordinates. A simpler proof, however, is furnished by considering the strain ellipsoid, which as already shown may be written

$$x_1^2 + y_1^2 + z_1^2 = x^2 + y^2 + z^2 + 2\,(s_{xx}\,x^2 + \ldots + s_{xy}\,.\,xy) = K^2\,.$$

Expressed with regard to new axes this becomes

$$x_1'^2 + y_1'^2 + z_1'^2 = x'^2 + y'^2 + z'^2 + 2\,(s_{x'x'}\,x'^2 + \ldots + s_{x'y'}\,.\,x'y') = K^2\,,$$

and the new strains (s') are expressed *linearly* in terms of the old (s). But since the degree of any power of the distortions $\frac{d(u\,v\,w)}{d(x\,y\,z)}$ is unaltered by transformation, it follows that the parts of the first and second degrees separately, and consequently the whole (s), follow the same law of resolution, which is that of strains of the first order already given.

The sum of the squares of the members of the several rows of ∇^0, and the sum of the products of the corresponding members of each pair of rows, will be of great use to us in what follows. They are denoted by a, b, c, d, e, f. These quantities also follow the same laws of resolution as stresses, a property arising from the fact that $\frac{d}{dx}\frac{d}{dy}\frac{d}{dz}$ follow the ordinary parallelepiped law of resolution, just as $u\ v\ w$ do, when the axes are rectangular.

§ 10. *Reduction to Isotropic Media.*—The theory of the ellipsoid indicates directly that there are three invariant functions of the strains (s), of the first, second, and third degrees respectively. Calling these J_1, J_2, J_3 it follows that all possible invariants of these three degrees are J_1, $mJ_1^2 + nJ_2$, $m'J_1^3 + n'J_1J_2 + p'J_3$.

Also,

$$\left.\begin{aligned} J_1 &= s_{xx} + s_{yy} + s_{zz} \\ J_2 &= -4s_{yy}s_{zz} - 4s_{zz}s_{xx} - 4s_{xx}s_{yy} + s_{yz}^2 + s_{zx}^2 + s_{xy}^2 \\ J_3 &= 4s_{xx}s_{yy}s_{zz} + s_{yz}s_{zx}s_{xy} - s_{xx}s_{yz}^2 - s_{yy}s_{zx}^2 - s_{zz}s_{xy}^2 \end{aligned}\right\} \quad . \quad (14).$$

If we confine ourselves in W to terms of the second degree, we may take

$$2W = mJ_1^2 + nJ_2,$$

where $m = k + \frac{4n}{3}$ (See THOMSON and TAIT, art. 682).

The expressions for the stresses become very simple; for writing $2W = (m - 2n)J_1^2 + nJ_2^1$, where

$$J_2^1 = s_{yz}^2 + \ldots + 2s_{zz}^2.$$

$$\left.\begin{aligned} \nabla.S_{xx} &= (m-2n)J_1.a + n(2s_{xx}.u_1^2 + 2s_{yy}u_2^2 + \ldots + 2s_{xy}u_1u_2) \\ &\vdots \\ \Delta.S_{xy} &= (m-2n)J_1.f + n(2s_{xx}.u_1v_1 + 2s_{yy}u_2v_2 + \ldots + s_{xy}(u_1v_2 + u_2v_1)) \end{aligned}\right\} \quad (15).$$

in which $a, b \ldots f$, have the values in the last paragraph of § 9, but now refer to ∇.

These expressions admit also of the following transformations.

$$\left.\begin{aligned} \nabla\,.\,S_{xx} &= (\overline{m - 2n}\,.\,J_1 - n)a + n(a^2 + e^2 + f^2) \\ &\vdots \\ \nabla\,.\,S_{xy} &= (\overline{m - 2n}\,J_1 - n)\,.\,f + n(\overline{a + b}.f + de) \end{aligned}\right\} \quad . \quad (15a),$$

where also

$$2J_1 + 3 = a + b + c\,. \quad . \quad . \quad . \quad (15b);$$

or, with reference to the developments which follow in the next article, into these forms

$$\left.\begin{aligned} \nabla\,.\,S_{xx} &= (mJ_1 + 2n)a + n(\overline{e^2 - ac} + \overline{f^2 - ab} \\ &\vdots \\ \nabla\,.\,S_{xy} &= (mJ_1 + 2n).f + n(de - cf) \end{aligned}\right\} \quad (15c).$$

There is a remarkable corollary from these equations. There exists, as we shall see in § 13, or as we may deduce from the laws of transformation of $a \ldots f$, one set of rectangular axes at each point for which $d = e = f = 0$. If the solid be homogeneous after the strain, these have the same direction at every point. It follows from the above equations that for these axes there is no tangential stress, and the normal stresses are the sums of two parts of which one is directly as the corresponding a, b, c, and the other part as its square. And, generally, at every point of an isotropic solid the stresses (each multiplied by ∇) are functions of $a \ldots f$, which we may call the *quasi-strains.*

§ 11. The calculation of w' will usually be a very long and troublesome matter; for it contains 21 terms of the form $D_{\lambda\mu} \cdot D_{\lambda'\mu'}$, each of which terms $D_{\lambda\mu}$ contains 21 terms of the form $\sigma_{\lambda\mu}$. We should thus have, in general, to take account of 441 terms. But if we narrow the problem, as in last article, by taking W to consist of terms of the second degree only in (s), and by supposing it isotropic with regard to these terms, w' appears reduced to a form of remarkable and rather unexpected symmetry.

When $W = \phi(s)$ is homogeneous and of the second degree in (s), then also

$$\left.\begin{aligned} \nabla^0 . w' &= \phi(D_{xx} \ldots) \\ &= \tfrac{1}{2}(mI_1^2 + nI_2) \\ &\text{where } I_1 \text{ and } I_2 \text{ are the invariants of (D)} \end{aligned}\right\} \quad . \quad (16).$$

If for $2\sigma_{xx}$, $2\sigma_{yy} \ldots \sigma_{xy}$ we write A, B ... F, we find

$$I_1 = \frac{Aa + Bb + Cc}{2} + Dd + Ee + Ff \quad . \quad . \quad (17).$$

The calculation of I_2 is most easily performed by find finding $I_2 + 2I_1^2$; when this work, which naturally is not devoid of symmetry, is gone through, we arrive finally at

$$I_2 = -\left\{\begin{aligned} &(BC-D^2)(bc-d^2)+(CA-E^2)(ca-e^2)+(AB-F^2)(ab-f^2) \\ &+2(EF-AD)(ef-ad)+2(FD-BE)(fd-be)+2(DE-CF)(de-ef) \end{aligned}\right\} \quad (18).$$

In these results $a \ldots f$ are the initial *quasi-strains.*

[§ 12. The strain ellipsoid corresponding to the second strain may be written

$$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy = 1 \quad . \quad (\delta),$$

while the quasi-strain ellipsoid corresponding to the initial strain is

$$ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy = 1 \quad . \quad (\delta').$$

Let the discriminants of these surfaces be $S = 2J_3^0$ and s, and let them be

reciprocated with regard to the spheres whose radii are $\frac{1}{\sqrt[4]{S}}$ and $\frac{1}{\sqrt[4]{s}}$; thence we obtain

$$\left.\begin{aligned}(BC - D^2)x^2 + \ldots + 2(DE - CF)xy &= 1\\ (bc - d^2)x^2 + \ldots + 2(de - cf)xy &= 1\end{aligned}\right\} \quad . \quad . \qquad (\epsilon).$$

$2I_1$ and $-I_2$ are the invariants of the first order of the first and second pairs of surfaces].

§ 13. If the body after the first strain be homogeneous, and we choose the co-ordinate axes parallel to these of the quasi-strain ellipsoid, we may put $d = e = f = 0$, and a, b, c will be constant throughout.

Thus,

$$\left.\begin{aligned}I_1 &= a\sigma_{xx} + b\sigma_{yy} + c\sigma_{zz}\\ I_2 &= bc(\sigma_{yz}^2 - 4\sigma_{yy}\sigma_{zz}) + ca(\sigma_{zx}^2 - 4\sigma_{zz}\sigma_{xx}) + ab(\sigma_{xy}^2 - 4\sigma_{xx}\sigma_{yy})\end{aligned}\right\} \quad (19.)$$

We are now in a position to determine the plane waves which can be propagated unchanged through such a solid.

[*Inserted* 16*th February* 1876, *and following the notation of* §§ 16, 18.

The equations of motion are as follows:—

Putting
$$\nabla_1 = a\frac{d^2}{dx^2} + b\frac{d^2}{dy^2} + c\frac{d^2}{dz^2}$$
$$\nabla_2 = P^2\frac{d^2}{dx^2} + Q^2\frac{d^2}{dy^2} + R^2\frac{d^2}{dz^2}$$

we find, after some reduction, that the equations take the following forms:

$$\left.\begin{aligned}\rho\frac{d^2u}{dt^2} &= n(\nabla_2 + a\nabla_1)u + \tfrac{1}{2}\mathfrak{m}\frac{d\mathfrak{J}}{dx}\\ \rho\frac{d^2v}{dt^2} &= n(\nabla_2 + b\nabla_1)v + \tfrac{1}{2}\mathfrak{m}\frac{d\mathfrak{J}}{dy}\\ \rho\frac{d^2w}{dt^2} &= n(\nabla_2 + c\nabla_1)w + \tfrac{1}{2}\mathfrak{m}\frac{d\mathfrak{J}}{dz}\end{aligned}\right\} \quad . \quad . \qquad (20.)$$

where $\mathfrak{m} = 4m - n$.

To solve these, put

$$u, v, w = (u_0, v_0, w_0)\sin\frac{2\pi}{\lambda}(px + qy + vz - Vt),$$

where
$$p^2 + q^2 + r^2 = 1,$$
and let
$$\left.\begin{aligned}D_1 &= ap^2 + bq^2 + cr^2\\ D_2 &= P^2p^2 + Q^2q^2 + R^2r^2\\ \rho\frac{V^2}{n} - D_2 &= \psi\\ \theta &= apu_0 + bqv_0 + crw_0\end{aligned}\right\}$$

we arrive at the following:—

$$\left.\begin{aligned} n(\psi-a\mathrm{D}_1)u_0 &= \mathfrak{m}\,.\,ap\theta \\ n(\psi-b\mathrm{D}_1)v_0 &= \mathfrak{m}\,.\,bq\theta \\ n(\psi-c\mathrm{D}_1)w_0 &= \mathfrak{m}\,.\,cr\theta\,, \end{aligned}\right\} \quad . \quad . \quad . \quad (21.)$$

and on the elimination of $u_0\, v_0\, w_0$, the following equation for ψ

$$\frac{a^2p^2}{\psi-a\mathrm{D}_1}+\frac{b^2q^2}{\psi-b\mathrm{D}_1}+\frac{c^2r^2}{\psi-c\mathrm{D}_1}=\frac{n}{\mathfrak{m}}. \quad . \quad . \quad . \quad (22.)$$

It appears from this result that there are three values of V^2 for each plane wave, and the wave which spreads out from a centre will, therefore, possess three sheets.

But if the solid be incompressible, and the primary stress be finite, we shall have $\mathfrak{m}=\infty$, $m=\infty$, and also $\mathrm{J}_0=0$; and in this case the equation which gives V reduces to

$$\frac{a^2p^2}{\psi-a\mathrm{D}_1}+\frac{b^2q^2}{\psi-b\mathrm{D}_1}+\frac{c^2r^2}{\psi-c\mathrm{D}_1}=0\,.$$

and the equations which give $u_0\, v_0\, w_0$ require us to suppose $\theta=0$, $\mathfrak{m}\theta=$finite.

Now $\theta=0$ is equivalent to $a\frac{du}{dx}+b\frac{dv}{dy}+c\frac{dw}{dz}=0$; and at first sight it seems difficult to understand the significance of this condition. We must remember, however, that the word "incompressible," as here used, is a relative term, and denotes that when the strains are indefinitely small, the stresses in such a solid required to produce a very small cubic compression, are indefinitely great compared to those necessary to produce a shear of the same magnitude. This implies that m is indefinitely great compared to n; and the preceding investigation shows that if such a solid, having previously undergone considerable strain, be still further subjected to stresses of small magnitude, the distortions which they produce are such as to make $a\frac{du}{dx}+b\frac{dv}{dy}+c\frac{dw}{dz}=0$.

With these explanations we now proceed to consider the case where $a-1$, $b-1$, $c-1$ are such that their squares and products may be neglected. I shall also suppose that $m\mathrm{J}_0=0$. Let therefore $a=1+\alpha$, then $\mathrm{P}^2=a(a-1)=\alpha$, and similarly $\mathrm{Q}^2=b-1=\beta$, $\mathrm{R}^2=c-1=\gamma$; thus $\mathrm{D}_2=\alpha p^2+\beta q^2+\gamma r^2$, $\mathrm{D}_2=1+\mathrm{D}_1$ and $a\mathrm{D}_1=(1+\alpha)\,(1+\mathrm{D}_2)=1+\alpha+\mathrm{D}_2$, . . .

The equation for V becomes $\dfrac{(1+\alpha)^2p^2}{\frac{\rho\mathrm{V}^2}{n}-2\mathrm{D}_2-\alpha}+\dfrac{(1+\beta)^2q^2}{\frac{\rho\mathrm{V}^2}{n}-2\mathrm{D}_2-b}+\dfrac{(1=\gamma)^2r^2}{\frac{\rho\mathrm{V}^2}{n}-2\mathrm{D}_2-c}=0\,.$

It appears therefore that $\frac{\rho\mathrm{V}^2}{n}-1$ is of the order α, β, γ, and we may write

therefore 1 for $(1+\alpha)^2\ldots$, whence we obtain

$$\frac{p^2}{\frac{\rho V^2}{n}-2D_2-a}+\frac{q^2}{\frac{\rho V^2}{n}-2D_2-b}+\frac{r^2}{\frac{\rho V^2}{n}-2D_2-c}=0,$$

an equation not unlike that occurring in the theory of double refraction.]

It cannot be said that our investigations, so far at least, throw much light on the true cause of double refraction in strained glass beyond the fact that such glass is no longer isotropic. Nor, indeed, was it to be much expected, in our ignorance of the true nature of the luminiferous medium and the mode of its connection with terrestrial bodies.

§ 14. I add the calculation of the constants for the case of a substance *under longitudinal stress F.*

Here

$$\alpha=(\epsilon-1)x,\ \beta=(\lambda-1)y,\ \gamma=(\lambda-1)z,\ \nabla^0=\begin{vmatrix}\epsilon & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{vmatrix}.$$

We have

$$0=(m-2n)J_1-n+n\lambda^2,$$

$$\epsilon\lambda^2F=[(m-2n)J_1-n]\,\epsilon^2+n\epsilon^4,$$

whence

$$\left.\begin{aligned}&(3k-2n)\epsilon^2+2(3k+n)\lambda^2=9k\\ &F=\frac{n\epsilon(\epsilon^2-\lambda^2)}{\lambda^2}\end{aligned}\right\}.$$

In the case of glass, for which we have, very nearly, $3k=8n$, we find

$$\epsilon^2+3\lambda^2=4,\ \epsilon^3+\frac{F}{n}\epsilon^2-\epsilon-\frac{4F}{n}=0,$$

which latter equation has only one real root, as might have been expected.

On the Laws of Resolution of Forces and Stresses.

§ 15. These investigations may be fitly terminated by some general considerations on the law of resolution of stresses, which has been proved in § 9 to apply to strains, and to what I have called quasi-strains. It would seem that physical magnitudes possessing direction divide themselves naturally into two classes, in one of which the law of resolution along rectangular axes is that of ordinary forces, and in the other that of elastic stresses. To each class pertains a series of divariants and invariants, and each possess groups of derived directed magnitudes in some of which the law of resolution is that of their primitives, and in others the opposite law. We may term these shortly the law of forces and the law of stresses, and the corresponding groups force-groups and stress-groups respectively.

Def. 1. Let x, y, z be rectangular co-ordinates, and (u, v, w) a ternary group such that for rectangular axes,

$$ux + vy + wz = \text{an invariant, J not} = 0 \quad . \quad . \quad (23),$$

then $(u\ v\ w)$ constitute a force group.

Def. 2. Let (a, b, c, d, e, f) be a sextic group such that

$$ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy = \text{an invariant, J}' \quad (24),$$

where J′ differs from zero, then $(a, \ldots, f)$ from a stress group.

The known properties of planes and quadries permit us at once to enunciate.

Theorem (*a*). For an infinite number of sets of rectangular axes, one axis of which is fixed, the ternary group (u, v, w) becomes $(u', 0, 0)$, and the axis of x' is the axis of the group; so for *one* set of rectangular axes the sextic group becomes $(a', b', c', 0, 0, 0)$, and the axes of x', y', z' may be called the axes of the group.

Since in the above definitions the quantities $uv \ldots a \ldots$ are involved linearly, we derive the following

Theorem (*b*). From two force- or two stress-groups we may derive new force- or stress-groups by taking the sums or differences of the corresponding constituents of the groups for constituents of the new group.

Since $x.x + y.y + z.z = r^2$, an invariant, it follows that the characteristic law of resolution of (x, y, z) is itself that of forces; and since $x^2.x^2 + y^2.y^2 + z^2.z^2 + 2yz.yz + 2zx.zx + 2xy.xy = r^4$ an invariant, it follows the group $(x^2, y^2 \ldots xy)$ is itself a stress-group, and has the corresponding characteristic law of resolution. Hence follows,

Theorem (*c*). If (u, v, w), $(\mathrm{U}, \mathrm{V}, \mathrm{W})$ be two force-groups, then $u\mathrm{U} + v.\mathrm{V} + w.\mathrm{W}$ is an invariant J; and conversely if $u\mathrm{U} + v\mathrm{V} + w\mathrm{W}$ be an invariant, and one of the groups be a force-group, so also is the other: Also if $(a \ldots . f)$ $(\mathrm{A} \ldots . \mathrm{F})$ be stress-groups, then is $\mathrm{A}a + \mathrm{B}b + \mathrm{C}c + 2\mathrm{D}d + 2\mathrm{E}e + 2\mathrm{F}f$ an invariant; and conversely, if this expression be an invariant, and one of these groups be a stress-group, so must the other.

The theory of planes and quadries allows at once to write down the fundamental invariants of each group, as we may do in

Theorem (*d*). The invariant of $(u\ v\ w)$ is $\mathrm{I} = u^2 + v^2 + w^2$; and the invariants of the stress-group $(\mathrm{A} \ldots . \mathrm{F})$ are

$$\mathrm{J}_1 = \mathrm{A} + \mathrm{B} + \mathrm{C}, \mathrm{J}_2 = \mathrm{BC} + \mathrm{CA} + \mathrm{AB} - \mathrm{D}^2 - \mathrm{E}^2 - \mathrm{F}^2, \mathrm{J}_3 = \mathrm{ABC} + 2\,\mathrm{DEF} - \mathrm{AD}^2 - \mathrm{BE}^2 - \mathrm{CF}^2.$$

The following are a few of the applications of this theory in various depart-

ments of mathematics: in most cases the results run parallel in the two groups. We may denote force-groups in general by the symbols $(\alpha\ \beta\ \gamma)$, (u, v, w), &c., while stress-groups will be expressed by $(A \ldots F)$, $(a \ldots f)$, &c. Invariant functions will be represented by the characteristic symbols J, J^1.

Theorem (e).

Force-Groups.	*Stress-Groups.*
Lines (x, y, z)	$(x^2, y^2, z^2, yz, zx, xy)$ Since $J.x^2 + Jy^2 + J.z^2 = Jr^2$ we have the groups $(J, J, J, 0\ 0\ 0)$, $(r^2, r^2, r^2, 0\ 0\ 0)$.
Moments of a particle with regard to three planes.	$(y^2+z^2, z^2+x^2, x^2+y^2, -yz, -zx, -xy)$.
Moments of a solid with regard to three planes.	Moments and negative products of inertia of a solid.
Forces.	*Stresses.*
Displacements.	Strains and quasi-strains.

Theorem (f). Since $dJ = \frac{dJ}{dx}dx + \frac{dJ}{dy}dy + \frac{dJ}{dz}dz$, and $dJ^1 = \frac{dJ^1}{dA}dA + \ldots + \frac{dJ^1}{dF}dF$, it follows that $\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right) J$ form a force-group and $\left(\frac{d}{dA}, \ldots \frac{1}{2}\frac{d}{dF}\right) J^1$ a stress-group. Moreover, as these characteristic laws of resolution of $\frac{d}{dx} \ldots \frac{1}{2}\frac{d}{dF}$ depend on the relative directions only of the new and old fixed axes, and not on the subject of operation, we may remove the restriction that the latter is to be an invariant, and extend them to the case where it may be any directed quantity. In this way we establish the invariancy of such expressions as $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$, $\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$, and $\frac{d^2A}{dx^2} + \frac{d^2B}{dy^2} + \ldots + 2\frac{d^2F}{dx\,dy}$.

Theorem (g). If we choose $J' = \frac{J_1^2}{2} - J_2$ we reproduce the group $(A \ldots F)$, while by choosing $J' = J_3$ we see that $(BC - D^2, \ldots, DE - CF)$ form a stress-group. This result enables us also to deduce the new invariants

$$J^1_{-1} = a(BC - D^2) + \ldots\ldots + 2f(DE - CF)$$
$$J^1_{-2} = (bc - d^2)\ (BC - D^2) + \ldots + 2(de - cf)\ (DE - CF).$$

The application of (f) to J^1_{-1} produces the stress-group

$$(Bb + Cc - Dd, \ldots\ldots, De + dE - Cf - Fc).$$

Theorem (h). If we choose $J = \nabla = \begin{vmatrix} x & y & z \\ u & v & w \\ \alpha & \beta & \gamma \end{vmatrix}$ we obtain the force-group $(\gamma v - \beta w, \ldots \beta u - \alpha v)$; while if we choose $J^1 = \nabla^2$ and put $\alpha^2 = a$, $u^2 = A\ldots$, $\alpha\beta = f$, $uv = F$, we reproduce the stress-group last found.

By putting $u = \frac{d}{dx}$, $v = \frac{d}{dy}$, $w = \frac{d}{dz}$ we arrive at the force-group $\left(\frac{dv}{dz} - \frac{dw}{dy}, \ldots\ldots\right)$ and the stress-group $\left(\frac{d^2B}{dy^2} + \frac{d^2C}{dz^2} - \frac{d^2D}{dy\,dz}, \ldots\ldots\right)$.

Theorem (i). $J = (u\alpha + v\beta + w\gamma)\ (\alpha x + \beta y + \gamma z)$ and $J^1 = (ux + vy + wz)\ (\alpha x + \beta y + \gamma z)$ generate respectively the force- and stress-groups $(uA + vF + wE, \ldots)$, $(u\alpha, \ldots\ldots, \frac{1}{2}\overline{u\beta + v\alpha})$.

These include as particular cases the groups $\left(\frac{dA}{dx} + \frac{dF}{dy} + \frac{dE}{dz}, \ldots\right)$ and $\left(\frac{d\alpha}{dx}, \ldots, \frac{1}{2}\left(\frac{d\beta}{dx} + \frac{d\alpha}{dy}\right)\right)$.

Theorem (j). $J = \nabla\,.\,(u\alpha + v\beta + w\gamma)$ generates the force-group $(\overline{b - c}D - d\overline{B - C} + fE - eF, \ldots)$; while $J^1 = \nabla\,.\,(\alpha x + \beta y + \gamma z)$ generates the stress-group $(vE - wF, \ldots \frac{1}{2}\,.\,\overline{wA - B - uE - vF})$.

These include as particular cases those in which $\alpha = \frac{d}{dx}$, $a = \frac{d^2}{dx^2} \ldots f = \frac{d^2}{dx\,dy}$.

[*Added* 16*th February* 1876.]

§ 16. The reduction of the work necessary to produce a compound strain in an isotropic solid has been effected, in the foregoing paper, on the hypothesis that the work necessary to produce a state of strain from perfect freedom, is a quadratic function of the component strains. But it may be useful to show how the same problem may be solved when the work is a function of any degree in these strains.

Calling xyz the relative co-ordinates of two particles of the unstrained solid, $x_1y_1z_1$ those of the same particles in the state of strain, it has been shown that

$$x_1^2+y_1^2+z_1^2-(x^2+y^2+z^2)=\mathrm{A}x^2+\mathrm{B}y^2+\ldots+2\mathrm{F}xy, \quad . \qquad (25).$$

where $\mathrm{A}=2s_{xx},\ \mathrm{B}=2s_{yy},\ \ldots\ \mathrm{F}=s_{xy}.$

From this result the following invariants result—

$$\mathrm{J}=\mathrm{A}+\mathrm{B}+\mathrm{C}$$
$$\mathrm{H}=\mathrm{D}^2+\mathrm{E}^2+\mathrm{F}^2-\mathrm{BC}-\mathrm{CA}-\mathrm{AB}$$
$$\mathrm{K}=\mathrm{ABC}+2\mathrm{DEF}-\mathrm{AD}^2-\mathrm{BE}^2-\mathrm{CF}^2.$$

And there can be no more invariant functions of the strains; for these equations are sufficient to determine the three principal strains in terms of J, H, K; and if there were a fourth invariant, it must be a function of these principal strains, and consequently of J, H, K.

The work done in producing from freedom any state of strain must, therefore, be a function of J, H, K of the form—

$$\mathrm{W}=\tfrac{1}{2}(m\mathrm{J}^2+n\mathrm{H})+p\mathrm{J}^3+q\mathrm{JH}+r\mathrm{K}+\ldots, \quad . \quad . \qquad (26).$$

[The n here used is the same as that used by THOMSON and TAIT, p. 710, but the m is different. If the m there used be written $\mathfrak{m}$, we shall have $\mathfrak{m}=4m-n$].

We have now to find how J, H, K for a compound system depend on its component elements; and to do so, I use capital letters with suffix $_0$ to define the primary strains, letters without suffix to denote the secondary, and those with suffix $_1$ to indicate the final state; while a, b, c, d, e, f denote, as formerly, the primary quasi-strains. The corresponding invariants will also be similarly distinguished. We have, therefore—

$$\mathrm{J}=\mathrm{A}+\mathrm{B}+\mathrm{C},\ \mathrm{J}_0=\mathrm{A}_0+\mathrm{B}_0+\mathrm{C}_0,\ \mathrm{J}_1=\mathrm{A}_1+\mathrm{B}_1+\mathrm{C}_1,\ j=a+b+c,$$

and the remaining invariants $\mathrm{H}, \mathrm{H}_0, \ldots . k$ may be similarly written down.

The following "mixed concomitants," which make their appearance, are thus denoted:—

$$\left.\begin{aligned} \mathfrak{J} &= Aa+Bb+Cc+2Dd+2Ee+2Ef \\ \mathfrak{J}_1 &= A(bc-d^2)+B(ca-e^2)+\ldots\ldots+2F(de-cf) \\ -\mathfrak{J}_2 &= (BC-D^2)(bc-d^2)+\ldots\ldots+2(DE-CF)(de-cf).^{*} \end{aligned}\right\} \quad (27).$$

§ 17. We now proceed to find J_1, H_1, K_1. If we turn to equations (9), we find they may be written—

$$1+A_1=\alpha_1^2(1+A)+\beta_1^2(1+B)+\ldots+2\alpha_1\beta_1F$$
$$\vdots$$
$$F_1=\alpha_1\alpha_2(1+A)+\beta_1\beta_2(1+B)+\ldots+(\alpha_1\beta_2+\alpha_2\beta_1)F.$$

I. To find J_1 we have

$$3+J_1=a(1+A)+b(1+B)+\ldots+2fF,$$

whence

$$\left.\begin{aligned} J_1 &= J_0+\mathfrak{J} \\ J_0 &= j-3 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (28).$$

II. To find H_1, it is most convenient to calculate, in the first place, $(1+A_1)^2+(1+B_1)^2+(1+C_1)^2+2D_1^2+2E_1^2+2F_1^2$, which is found, after a little reduction, to be equal to

$$\Sigma a^2(1+A)^2+2\Sigma(bc+d^2)D^2+2\Sigma d^2(1+B)(1+C)+4\Sigma af(1+A)F$$
$$+4\Sigma ef(1+A)D+4\Sigma(ad+ef)EF,$$

where Σ indicates summation extended to all terms of the same type. On expanding and substituting for $A_1+B_1+C_1$ its value already found, we obtain

$$\left.\begin{aligned} H_1 &= 2j+h-3+hJ+2\mathfrak{J}+\mathfrak{J}_1+\mathfrak{J}_2, \\ \text{and on putting } A=B=\ldots=0 \text{ we find also} \\ H_0 &= 2j+h-3 \end{aligned}\right\} \quad . \quad . \quad (29).$$

III. To find K_1 we observe that K is the discriminant of the covariant (25), and that the problem of finding $1+A_1, \ldots.$ is the ordinary one of linear transformation, the modulus being ∇_0; hence

$$(1+A_1)(1+B_1)(1+C_1)+2D_1E_1F_1-\ldots=\nabla_0^2\Big((1+A)(1+B)(1+C)+2DEF-\ldots\Big)$$

Now, it is clear that, if ∇_0^2 be expressed as a determinant, we shall have $\nabla_0^2=k$,

* It is perhaps worth noting that there is still a fourth invariant, which depends on the systems $(AB\ldots F)$, $(a,b\ldots f)$, namely, $\mathfrak{J}^1=a(BC-D^2)+b(CA-E^2)+\ldots+2f(DE-CF)$, but it does not present itself in these investigations. It may be observed, however, that it is not really independent of the nine magnitudes $J_1\,H_1\,K_1\,j, h, k, \mathfrak{J}, \mathfrak{J}_1, \mathfrak{J}_2$. For every invariant relating to two systems of strains, or to a system of strains and one of quasi-strains, can depend on only nine elements,—the six principal strains, and the three magnitudes which determine one set of principal strain-axes with regard to the other. In fact, referring one set of strains to their principal axes, we see that the ten invariants involve only nine independent quantities.

This is a special case of the more general theorem that n sets of magnitudes cogredient with strains give rise to $n(2n+1)$ invariants apparently independent, but of which only $6n-3$ are actually independent, the remaining $2n^2-5n+3$ being functions of these.

Hence the above result furnishes the following

$$\left.\begin{array}{l} K_1 - H_1 + J_1 + 1 = k(K - H + J + 1) \\ \text{and by putting } A = B = \ldots = 0\text{, this also,} \\ K_0 = j + h + k - 1 \end{array}\right\} \quad . \quad . \quad (30.)$$

§ 18. We are now in a position to find the value of the work W_1, which corresponds to the compound strain, in terms of its components; to do this write $W_1 = W_0 + w$, and analyse w into the terms of the first, second ... degrees in the secondary strains; thus—

$$w = w_1 + w_2 + w_3 + \ldots$$

On substituting for J_1, H_1, K_1 their values, we can readily find w_1, w_2, ... If we confine ourselves to the case where

$$W_1 = \tfrac{1}{2}(mJ_1^2 + nH_1)$$

we shall have

$$\left.\begin{array}{l} w_1 = \tfrac{1}{2}\{2mJ_0\mathfrak{J} + n(hJ + 2\mathfrak{J} + \mathfrak{J}_1)\} \\ w_2 = \tfrac{1}{2}(m\mathfrak{J}^2 + n\mathfrak{J}_2) \end{array}\right\} \quad . \quad . \quad (31.)$$

Now w_1 may be written in the form

$$\left.\begin{array}{l} \frac{n}{2}\left\{(a)A + (b)B + (c)C + 2(d)D + 2(e)E + 2(f)F\right\} \\ \text{where} \quad n(a) = 2(mJ_0 + n)\,a + n(e^2 - ca + f^2 - ab) \\ \qquad\qquad \vdots \\ \qquad\quad n(f) = 2(mJ_0 + n)f + n(de - cf)\,, \end{array}\right\} \quad . \quad (32.)$$

These expressions may be simplified by supposing the axes of co-ordinates to be those of the primary quasi-strain ellipsoid; and if the primary stress be homogeneous throughout the solid, these axes will have the same direction throughout. In this case we shall write

$$\begin{array}{l} (a) = P^2, (b) = Q^2, (c) = R^2, (d) = (e) = (f) = 0 \\ P^2 = 2(\frac{m}{n}J_0 + 1)a - (ab + ac), \ldots \quad . \quad . \quad . \quad . \quad (33.) \\ w_1 = \frac{n}{2}(P^2A + Q^2B + R^2C)\,. \end{array}$$

The simplified value of w_2 has been already found.

The expressions found above (32) may be used to furnish the expressions for the stresses given in my paper at art. 10, by observing that

$$W_1 dV = W_0 dV + \frac{w}{\nabla_0}\delta V$$

where dV and δV are corresponding elements of the solid in its free state, and after the primary strain has been produced.

We may also apply the method given in art. 7 to find expressions for the stresses, whatever may be the nature of the function which expresses W_1 in

terms of J_1 H_1 and K_1. To do this we must first arrange K_1 according to the terms which are of the first, second, third degrees, in terms of the secondary strains, as follows:—

$$K_1 = K_0 + (h+k)J + \mathfrak{J} + \mathfrak{J}_1 + \mathfrak{J}_2 - kH + kK \,. \quad . \quad . \quad (34.)$$

The stress S^0_{xx} is given by $\nabla_0 S^0_{xx} = 2\frac{dW_1}{dA}$ where, after differentiation, the secondary strains are to be put equal to zero. When, therefore, we write

$$\frac{dW_1}{dA} = \left(\frac{dW}{dJ}\right)_0 \frac{dJ_1}{dA} + \left(\frac{dW}{dH}\right)_0 \frac{dH_1}{dA} + \left(\frac{dW}{dK}\right)_0 \frac{dK_1}{dA},$$

we need only attend to those terms in H_1 and K_1, which are linear in A, B, C....

Substituting for J_1, H_1 and K_1 we find

$$\nabla_0 S^0_{xx} = 2\left[\left(\frac{dW}{dH}\right)_0 h + \left(\frac{dW}{dK}\right)_0 \cdot \overline{h+k}\right] + \left[\left(\frac{dW}{dJ}\right)_0 + 2\left(\frac{dW}{dH}\right)_0 + \left(\frac{dW}{dK}\right)_0\right] a + 2\left[\left(\frac{dW}{dH}\right)_0 + \left(\frac{dW}{dK}\right)_0\right](bc - d^2) \,.$$

If we write this result,

$$S_{xx} = L + Ma + N(bc - d^2) \,,$$

the type of the tangential stresses is

$$S_{xy} = Mf + N(de - cf) \,.$$