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Commentatio de corporum coelestium perturbationibus,

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Quamquam methodum perturbationum determinandarum in commentatione prima, quam de hoc argumento scripsi, ita exposui ut quisque rerum mathematicarum haud ignarus ejus rationem plane intelligere possit, mihi tamen persuasum habeo, demonstrationem aliam formulis fundamentalibus Mechanicae innixam ad Methodum hanc certissime stabilendam maxime collaturam esse. Quam rem quum aliquamdiu animo volvissem, demonstrationem excogitavi, cui prima hujus commentationis paragraphus dicata est. Momenta praecipua hujus demonstrationis haec sunt. Theorematis nonnullis, quae extensio methodi variationis constantium arbitrariarum spectari possunt, demonstratis, adjumento hujus methodi formulas, quae perturbationes longitudinis verae, radii vectoris et quantitatum duarum, a quibus situs orbitae in spatio pendet, praebent ex aequationibus fundamentalibus Mechanicae deduco, quibus inventis, transformationem coordinatarum generaliore expono, quae immediate ad expressiones eas, quas alibi dedi, perducit, cujus vero transformationis casus specialis in eo consistit, quod perturbationes ad longitudinem mediam addantur, ita ut in prima commentatione exposui.

Paragraphus secunda continet applicationem illarum expressionum ad computandas perturbationes, quas corpus coeleste in fluido resistenti se movens patitur. Cui quidem rei accurate constituendae jam antea viri doctissimi operam suam navaverunt. Illi *Euler*, *Lagrange* et *Laplace* imprimis, problemate hoc summa cum sagacitate tractato, invenerunt axem majorem et excentricitatem corporum coelestium in fluido resistenti se moventium semper decrescere debere, itaque et eorum orbitas ellipticas magis magisque ad circulos appropinquaturas, et corpora ipsa demum in solem prolapsura esse.

Quum vero observationum veterum cum observationibus novissimis comparatio nos certiores fecisset, tales motuum anomalias nondum exstitisse, dubium erat, utrum spatium sit vacuum, an fluido tam subtili impletum, ut ejus effectus in planetas inde a tempore observationum antiquissimarum usque ad praesens tempus a sensibus nostris percipi non

potuerit. Jam lumen quidem, sive per emissam materiam, sive per undulationes fluidi alicujus oritur, opinionem hanc alteram ex aliqua parte confirmaverat, quum ill. *Encke* in cometae cognominis motu anomaliam detexit, quam mox effectum fluidi resistentis in toto universo diffusi optime explicavit. Hinc factum est, ut problema illud perturbationes a fluido resistenti ortas determinandi, quod geometrae illi, nulla alia cogente necessitate, generaliter tantum solverant, summa cum cura in extenso resolvi oporteret, ut ad casus speciales applicari possit. Duas praesertim solutiones tempus recens genuit. Ill. *Encke* ipse perturbationes has eodem modo in calculum vocavit, quo perturbationes planetariae in motu cometarum computari solent, cl. *Sohnke* Regiomontanus vero easdem perturbationes per transcendentibus ellipticas quae dicuntur exprimere suscepit. Illa methodus in commentationibus academiae Regiae Berolinensis anni 1829 publici juris facta est, haec in Diarii ab ill. *Crelle* editi Tomo X reperitur. Quae commentatio auctoris solertiam comprobans applicationem pulcherrimam praeclarae transcendentium ellipticarum ab ill. *Jacobi*, *Legendre* et *Abel* expolita theoriae sine dubio ponit, sed formulae ad computandas perturbationes hoc modo inventae non nihil desiderandum relinquunt. Densitatem fluidi in universo diffusi accepit auctor esse in quoque spatii puncto in ratione inversa quadrati distantiae a sole et hanc solam hypothesin in formulis suis evolvendis respexit; sed quum adhuc lateat, utrum haec fluidi densitatis lex vera sit necne, imo utrum praevideri possit, hanc legem densitatis veram esse non posse, si fluidum attractionis universalis legi subjectum sit, usu evenire potest, ut methodus haec nos plane deficiat. Porro quum auctor repererit, quinque tantum elementa per expressiones finitas functionum ellipticarum exprimi posse, sextum vero elementum per series infinitas illarum functionum exprimere coactus fuerit, itaque a posteriori monstraverit, perturbationes has in summa per functiones finitas transcendentium ellipticarum exhiberi non posse: maxime interest perscrutari, possintne perturbationes hae ope serierum simpliciorum, e. g. serierum secundum sinus aut cosinus

arcus alicujus progredientium commodius exprimi? Applicatis formulis generalibus nostris ad casum eum de quo agitur, reperi, perturbationes has, manente lege densitatis fluidi resistentis quacunque, per series secundum sinus et cosinus anomaliae excentricae progredientes maxima cum simplicitate exhiberi posse, praeterea has series pro omnibus excentricitatis inter cifram et unitatem existentibus valoribus admodum convergere, etiamsi lex densitatis fluidi in ejusmodi seriem evoluta minime convergat. Quam theoriam nec non eius applicationem ad cometam Enckianum in paragrapho secunda explicabo.

Paragraphus tertia exponet novam tabularum motum planetarum heliocentricum exhibentium formam, cujus hoc praecipue proprium est, ut eorum ascensionem rectam et declinationem heliocentricam ad aequatorem mobilem relatas immediate suppeditet. Quibus vero coordinatis datis, ascensionem rectam et declinationem geocentricam, quae verae in theoria planetarum quantitates quaesitae sunt, formulis trigonometricis omnium simplicissimis computare licet.

§. I. Theoria generalis perturbationum corporum coelestium.

1.

Denotantibus x, y, z , aut planetae aut satellitis aut cometae alicujus coordinatas orthogonias, quarum origo in centro Solis sita sit, r radium vectorem, m massam partibus massae Solis expressam, k mensuram communem vis attractionis, ita ut habeatur

$$a^3 n^2 = k\mu$$

ubi a semiaxis major orbitae, n motus medius sidereus unitate temporis t absolutus et $\mu = 1 + m$ est, per principia Mechanicae inventae sunt formulae fundamentales hae

$$(1) \dots \begin{cases} \frac{d^2 x}{dt^2} + k\mu \frac{x}{r^3} = k\mu \left(\frac{d\Omega}{dx} \right) \\ \frac{d^2 y}{dt^2} + k\mu \frac{y}{r^3} = k\mu \left(\frac{d\Omega}{dy} \right) \\ \frac{d^2 z}{dt^2} + k\mu \frac{z}{r^3} = k\mu \left(\frac{d\Omega}{dz} \right) \end{cases}$$

quarum integratione motus relativus corporis m respectu Solis investigandus est. In his vero formulis habetur

$$\Omega = \frac{m'}{\mu} \left\{ \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{xx' + yy' + zz'}{r'^3} \right\} \\ + \frac{m''}{\mu} \left\{ \frac{1}{\sqrt{(x-x'')^2 + (y-y'')^2 + (z-z'')^2}} - \frac{xx'' + yy'' + zz''}{r''^3} \right\} \\ + \text{etc.}$$

ubi quantitates quibus lineae superne ad dextram affixae sunt ad corpora coelestia alia, quae corpus m secundum legem Newtonianam attrahunt et vice versa ab hoc corpore attrahuntur, spectant. Ad aequationes (1) integrandas inservit methodus variationis constantium arbitrariarum, quare praecipua hujus methodi elementa afferam.

2.

Duobus praesertim modis geometrae formulas analyticas, ad quas methodus haec perducit, exposuerunt; tum expressiones explicuerunt, quae differentialia illarum constantium ope quotientium differentialium partialium ipsius Ω immediate exhibent, tum vice versa quotientes istos per differentialia constantium arbitrariarum expresserunt. Illae formulae originem suam ab ill. *Poisson* trahunt, hae ill. *Lagrange* debentur. Quamvis in plurimis casibus illae formulae his praeferendae sint, tamen hae ad finem quem hic persequor magis accommodatae videntur. Ergo his utar. Ceterum utraque formulae, debita eliminatione facta, easdem expressiones pro differentialibus constantium arbitrariarum suppeditant.

Suppono aequationes (1) posito $\Omega = 0$ integratas esse, et hanc integrationem constantes arbitrarias sive elementa $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ introduxisse. Sunt igitur coordinatae x, y, z nec non differentialia earum respectu temporis functiones illarum $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ et t spectanda, ita ut

$$x = X, \quad y = Y, \quad z = Z \dots \dots \dots (2)$$

designantibus X, Y, Z functiones illarum septem quantitatum. Si expressiones (2) differentiantur, dum tempus solum variabile spectetur, aequationibus (1) posito $\Omega = 0$ satisfaciunt; ut vero iisdem aequationibus, manente Ω quacunque, satisfaciant, accipitur $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ variabilia et quidem functiones temporis esse. Si, hac conditione valente, aequationes (2) differentiantur et in aequationibus (1) substituuntur, tres aequationes evadunt, quibus differentialia ipsarum α, β etc. determinanda sunt. Quae vero aequationes, quum earum numerus modo tres, contra ea numerus incognitarum sex sit, ad problema solvendum non sufficiunt, quare tres novas aequationes eligere nobis opus est, quas, quum problema omni alia conditione careat, et revera tres coordinatae ope sex elementorum infinite variis modis determinari possint, ad libitum accipere licet. Hunc in finem geometrae semper statuerunt, differentialia aequationum (2) ita sumta ut non nisi elementa $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ variabilia tractentur, cifrae aequalia fieri, quo facto problema determinatum est, et elementa proitura quasi osculatoria evadunt. Conditio vero, quam modo exposui, versatur in aequationibus his

$$(3), \dots \left\{ \begin{aligned} \left(\frac{dx}{d\alpha} \right) d\alpha + \left(\frac{dx}{d\beta} \right) d\beta + \left(\frac{dx}{d\gamma} \right) d\gamma + \left(\frac{dx}{d\delta} \right) d\delta + \left(\frac{dx}{d\epsilon} \right) d\epsilon + \left(\frac{dx}{d\zeta} \right) d\zeta &= 0 \\ \left(\frac{dy}{d\alpha} \right) d\alpha + \left(\frac{dy}{d\beta} \right) d\beta + \left(\frac{dy}{d\gamma} \right) d\gamma + \left(\frac{dy}{d\delta} \right) d\delta + \left(\frac{dy}{d\epsilon} \right) d\epsilon + \left(\frac{dy}{d\zeta} \right) d\zeta &= 0 \\ \left(\frac{dz}{d\alpha} \right) d\alpha + \left(\frac{dz}{d\beta} \right) d\beta + \left(\frac{dz}{d\gamma} \right) d\gamma + \left(\frac{dz}{d\delta} \right) d\delta + \left(\frac{dz}{d\epsilon} \right) d\epsilon + \left(\frac{dz}{d\zeta} \right) d\zeta &= 0 \end{aligned} \right.$$

Adjumento harum aequationum aequationes (2) secundum t differentiatæ præbent

$$\frac{dx}{dt} = \left(\frac{dX}{dt} \right), \quad \frac{dy}{dt} = \left(\frac{dY}{dt} \right), \quad \frac{dz}{dt} = \left(\frac{dZ}{dt} \right)$$

et iterum differentiendo nanciscimur

$$\begin{aligned} \frac{d^2x}{dt^2} &= \left(\frac{d^2X}{dt^2} \right) + \left(\frac{dx_i}{d\alpha} \right) \frac{d\alpha}{dt} + \left(\frac{dx_i}{d\beta} \right) \frac{d\beta}{dt} + \left(\frac{dx_i}{d\gamma} \right) \frac{d\gamma}{dt} + \left(\frac{dx_i}{d\delta} \right) \frac{d\delta}{dt} + \left(\frac{dx_i}{d\epsilon} \right) \frac{d\epsilon}{dt} + \left(\frac{dx_i}{d\zeta} \right) \frac{d\zeta}{dt} \\ \frac{d^2y}{dt^2} &= \left(\frac{d^2Y}{dt^2} \right) + \left(\frac{dy_i}{d\alpha} \right) \frac{d\alpha}{dt} + \left(\frac{dy_i}{d\beta} \right) \frac{d\beta}{dt} + \left(\frac{dy_i}{d\gamma} \right) \frac{d\gamma}{dt} + \left(\frac{dy_i}{d\delta} \right) \frac{d\delta}{dt} + \left(\frac{dy_i}{d\epsilon} \right) \frac{d\epsilon}{dt} + \left(\frac{dy_i}{d\zeta} \right) \frac{d\zeta}{dt} \\ \frac{d^2z}{dt^2} &= \left(\frac{d^2Z}{dt^2} \right) + \left(\frac{dz_i}{d\alpha} \right) \frac{d\alpha}{dt} + \left(\frac{dz_i}{d\beta} \right) \frac{d\beta}{dt} + \left(\frac{dz_i}{d\gamma} \right) \frac{d\gamma}{dt} + \left(\frac{dz_i}{d\delta} \right) \frac{d\delta}{dt} + \left(\frac{dz_i}{d\epsilon} \right) \frac{d\epsilon}{dt} + \left(\frac{dz_i}{d\zeta} \right) \frac{d\zeta}{dt} \end{aligned}$$

ubi brevitatæ causa

$$\left(\frac{dX}{dt} \right) = \frac{dx}{dt} = x_i, \quad \left(\frac{dY}{dt} \right) = \frac{dy}{dt} = y_i, \quad \left(\frac{dZ}{dt} \right) = \frac{dz}{dt} = z_i,$$

posui. Quum vero functiones X, Y, Z aequationibus (1) satisfaciant quoties $\Omega = 0$, necesse est habeatur identice

$$\left(\frac{d^2X}{dt^2} \right) + k\mu \frac{x_i}{r^3} = 0; \quad \left(\frac{d^2Y}{dt^2} \right) + k\mu \frac{y_i}{r^3} = 0; \quad \left(\frac{d^2Z}{dt^2} \right) + k\mu \frac{z_i}{r^3} = 0.$$

Si igitur valores ipsarum $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ ex præcedentibus aequationibus sumti in (1) substituuntur, emergunt

$$(4), \dots \left\{ \begin{aligned} \left(\frac{dx_i}{d\alpha} \right) \frac{d\alpha}{dt} + \left(\frac{dx_i}{d\beta} \right) \frac{d\beta}{dt} + \left(\frac{dx_i}{d\gamma} \right) \frac{d\gamma}{dt} + \left(\frac{dx_i}{d\delta} \right) \frac{d\delta}{dt} + \left(\frac{dx_i}{d\epsilon} \right) \frac{d\epsilon}{dt} + \left(\frac{dx_i}{d\zeta} \right) \frac{d\zeta}{dt} &= k\mu \left(\frac{d\Omega}{dx} \right) \\ \left(\frac{dy_i}{d\alpha} \right) \frac{d\alpha}{dt} + \left(\frac{dy_i}{d\beta} \right) \frac{d\beta}{dt} + \left(\frac{dy_i}{d\gamma} \right) \frac{d\gamma}{dt} + \left(\frac{dy_i}{d\delta} \right) \frac{d\delta}{dt} + \left(\frac{dy_i}{d\epsilon} \right) \frac{d\epsilon}{dt} + \left(\frac{dy_i}{d\zeta} \right) \frac{d\zeta}{dt} &= k\mu \left(\frac{d\Omega}{dy} \right) \\ \left(\frac{dz_i}{d\alpha} \right) \frac{d\alpha}{dt} + \left(\frac{dz_i}{d\beta} \right) \frac{d\beta}{dt} + \left(\frac{dz_i}{d\gamma} \right) \frac{d\gamma}{dt} + \left(\frac{dz_i}{d\delta} \right) \frac{d\delta}{dt} + \left(\frac{dz_i}{d\epsilon} \right) \frac{d\epsilon}{dt} + \left(\frac{dz_i}{d\zeta} \right) \frac{d\zeta}{dt} &= k\mu \left(\frac{d\Omega}{dz} \right) \end{aligned} \right.$$

quæ cum aequationibus (3) junctæ problema solvunt. Ut elementum quodque per aequationes istas determinetur, inter sex aequationes (3) et (4) deinceps quinque elementa eliminari deberent, sed tali modo aequationes valde implicitæ evaderent. Formulas vero simpliciores geometræ invenerunt introducendo differentialia partialia ipsius Ω respectu elementorum loco differentialium partialium istius quantitatis respectu coordinatarum. Quod fit per ratiocinationes sequentes.

3.

Quum Ω quantitates x, y, z , non contineat, habetur

$$\begin{aligned} k\mu \left(\frac{d\Omega}{d\alpha} \right) d\alpha &= \left\{ \left(\frac{dx_i}{d\alpha} \right) \left(\frac{dx}{d\alpha} \right) + \left(\frac{dy_i}{d\alpha} \right) \left(\frac{dy}{d\alpha} \right) + \left(\frac{dz_i}{d\alpha} \right) \left(\frac{dz}{d\alpha} \right) \right\} d\alpha \\ &+ \left\{ \left(\frac{dx_i}{d\beta} \right) \left(\frac{dx}{d\alpha} \right) + \left(\frac{dy_i}{d\beta} \right) \left(\frac{dy}{d\alpha} \right) + \left(\frac{dz_i}{d\beta} \right) \left(\frac{dz}{d\alpha} \right) \right\} d\beta \\ &+ \text{etc.} \\ &+ \left\{ \left(\frac{dx_i}{d\zeta} \right) \left(\frac{dx}{d\alpha} \right) + \left(\frac{dy_i}{d\zeta} \right) \left(\frac{dy}{d\alpha} \right) + \left(\frac{dz_i}{d\zeta} \right) \left(\frac{dz}{d\alpha} \right) \right\} d\zeta \end{aligned}$$

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$$\begin{aligned} \left(\frac{d\Omega}{d\alpha} \right) &= \left(\frac{d\Omega}{dx} \right) \left(\frac{dx}{d\alpha} \right) + \left(\frac{d\Omega}{dy} \right) \left(\frac{dy}{d\alpha} \right) + \left(\frac{d\Omega}{dz} \right) \left(\frac{dz}{d\alpha} \right) \\ \left(\frac{d\Omega}{d\beta} \right) &= \left(\frac{d\Omega}{dx} \right) \left(\frac{dx}{d\beta} \right) + \left(\frac{d\Omega}{dy} \right) \left(\frac{dy}{d\beta} \right) + \left(\frac{d\Omega}{dz} \right) \left(\frac{dz}{d\beta} \right) \\ &\vdots \\ \left(\frac{d\Omega}{d\zeta} \right) &= \left(\frac{d\Omega}{dx} \right) \left(\frac{dx}{d\zeta} \right) + \left(\frac{d\Omega}{dy} \right) \left(\frac{dy}{d\zeta} \right) + \left(\frac{d\Omega}{dz} \right) \left(\frac{dz}{d\zeta} \right) \end{aligned}$$

Itaque si aequationes (4) adduntur postquam resp. per $\left(\frac{dx}{d\alpha} \right), \left(\frac{dy}{d\alpha} \right), \left(\frac{dz}{d\alpha} \right)$, multiplicatae sunt, obtinetur ope aequationum præcedentium

Sed multiplicando primam aequationem (3) per $\left(\frac{dx}{d\alpha}\right)$, secundam per $\left(\frac{dy}{d\alpha}\right)$, tertiam per $\left(\frac{dz}{d\alpha}\right)$ nanciscimur addendo

$$0 = \left\{ \left(\frac{dx}{d\alpha}\right) \left(\frac{dx}{d\alpha}\right) + \left(\frac{dy}{d\alpha}\right) \left(\frac{dy}{d\alpha}\right) + \left(\frac{dz}{d\alpha}\right) \left(\frac{dz}{d\alpha}\right) \right\} d\alpha + \left\{ \left(\frac{dx}{d\beta}\right) \left(\frac{dx}{d\alpha}\right) + \left(\frac{dy}{d\beta}\right) \left(\frac{dy}{d\alpha}\right) + \left(\frac{dz}{d\beta}\right) \left(\frac{dz}{d\alpha}\right) \right\} d\beta \\ + \text{etc.} + \left\{ \left(\frac{dx}{d\zeta}\right) \left(\frac{dx}{d\alpha}\right) + \left(\frac{dy}{d\zeta}\right) \left(\frac{dy}{d\alpha}\right) + \left(\frac{dz}{d\zeta}\right) \left(\frac{dz}{d\alpha}\right) \right\} d\zeta$$

Subtracta hac aequatione a praecedente, elicitur

$$k\mu \left(\frac{d\Omega}{d\alpha}\right) dt = (\alpha, \beta) d\beta + (\alpha, \gamma) d\gamma + (\alpha, \delta) d\delta + (\alpha, \varepsilon) d\varepsilon + (\alpha, \zeta) d\zeta$$

ubi brevitatis caussa

$$(5) \left\{ \begin{aligned} (\alpha, \beta) &= \left(\frac{dx}{d\alpha}\right) \left(\frac{dx}{d\beta}\right) + \left(\frac{dy}{d\alpha}\right) \left(\frac{dy}{d\beta}\right) + \left(\frac{dz}{d\alpha}\right) \left(\frac{dz}{d\beta}\right) \\ &\quad - \left(\frac{dx}{d\alpha}\right) \left(\frac{dx}{d\beta}\right) - \left(\frac{dy}{d\alpha}\right) \left(\frac{dy}{d\beta}\right) - \left(\frac{dz}{d\alpha}\right) \left(\frac{dz}{d\beta}\right) \\ (\alpha, \gamma) &= \left(\frac{dx}{d\alpha}\right) \left(\frac{dx}{d\gamma}\right) + \left(\frac{dy}{d\alpha}\right) \left(\frac{dy}{d\gamma}\right) + \left(\frac{dz}{d\alpha}\right) \left(\frac{dz}{d\gamma}\right) \\ &\quad - \left(\frac{dx}{d\alpha}\right) \left(\frac{dx}{d\gamma}\right) - \left(\frac{dy}{d\alpha}\right) \left(\frac{dy}{d\gamma}\right) - \left(\frac{dz}{d\alpha}\right) \left(\frac{dz}{d\gamma}\right) \\ &\quad \text{etc.} = \text{etc.} \end{aligned} \right.$$

resp. per $\left(\frac{dx}{d\beta}\right) \left(\frac{dy}{d\beta}\right) \left(\frac{dz}{d\beta}\right)$ et aequationes (3) per $\left(\frac{dx}{d\beta}\right) \left(\frac{dy}{d\beta}\right) \left(\frac{dz}{d\beta}\right)$ emergit.

$$k\mu \left(\frac{d\Omega}{d\beta}\right) dt = (\beta, \alpha) d\alpha + (\beta, \gamma) d\gamma + \text{etc.} + (\beta, \zeta) d\zeta$$

ubi

$$(\beta, \alpha) = -(\alpha, \beta)$$

$$(\beta, \gamma) = \left(\frac{dx}{d\beta}\right) \left(\frac{dx}{d\gamma}\right) + \left(\frac{dy}{d\beta}\right) \left(\frac{dy}{d\gamma}\right) + \left(\frac{dz}{d\beta}\right) \left(\frac{dz}{d\gamma}\right) \\ - \left(\frac{dx}{d\beta}\right) \left(\frac{dx}{d\gamma}\right) - \left(\frac{dy}{d\beta}\right) \left(\frac{dy}{d\gamma}\right) - \left(\frac{dz}{d\beta}\right) \left(\frac{dz}{d\gamma}\right) \\ \text{etc.} = \text{etc.}$$

scripsi. Eodem modo, scilicet multiplicando aequationes (4)

Quodsi calculi hi perducuntur, pervenimus ad aequationes has

$$(6) \dots \dots \dots \left\{ \begin{aligned} k\mu \left(\frac{d\Omega}{d\alpha}\right) dt &= (\alpha, \beta) d\beta + (\alpha, \gamma) d\gamma + (\alpha, \delta) d\delta + (\alpha, \varepsilon) d\varepsilon + (\alpha, \zeta) d\zeta \\ k\mu \left(\frac{d\Omega}{d\beta}\right) dt &= -(\alpha, \beta) d\alpha + (\beta, \gamma) d\gamma + (\beta, \delta) d\delta + (\beta, \varepsilon) d\varepsilon + (\beta, \zeta) d\zeta \\ k\mu \left(\frac{d\Omega}{d\gamma}\right) dt &= -(\alpha, \gamma) d\alpha - (\beta, \gamma) d\beta + (\gamma, \delta) d\delta + (\gamma, \varepsilon) d\varepsilon + (\gamma, \zeta) d\zeta \\ k\mu \left(\frac{d\Omega}{d\delta}\right) dt &= -(\alpha, \delta) d\alpha - (\beta, \delta) d\beta - (\gamma, \delta) d\gamma + (\delta, \varepsilon) d\varepsilon + (\delta, \zeta) d\zeta \\ k\mu \left(\frac{d\Omega}{d\varepsilon}\right) dt &= -(\alpha, \varepsilon) d\alpha - (\beta, \varepsilon) d\beta - (\gamma, \varepsilon) d\gamma - (\delta, \varepsilon) d\delta + (\varepsilon, \zeta) d\zeta \\ k\mu \left(\frac{d\Omega}{d\zeta}\right) dt &= -(\alpha, \zeta) d\alpha - (\beta, \zeta) d\beta - (\gamma, \zeta) d\gamma - (\delta, \zeta) d\delta - (\varepsilon, \zeta) d\varepsilon \end{aligned} \right.$$

ubi coefficientes ipsarum $d\alpha$, $d\beta$, $d\gamma$, etc. omnes ex aequationibus (5) mutatis mutandis petendi sunt.

4.

Qui coefficientes ipsarum $d\alpha$, $d\beta$, etc. proprietate insigni hac gaudent, quod tempus explicite non continent, quod theorema sequenti modo demonstrari potest.

$$\frac{d(\alpha, \beta)}{dt} = \left(\frac{d^2x}{d\alpha dt}\right) \left(\frac{dx}{d\beta}\right) + \left(\frac{dx}{d\alpha}\right) \left(\frac{d^2x}{d\beta dt}\right) + \left(\frac{d^2y}{d\alpha dt}\right) \left(\frac{dy}{d\beta}\right) + \left(\frac{dy}{d\alpha}\right) \left(\frac{d^2y}{d\beta dt}\right) + \left(\frac{d^2z}{d\alpha dt}\right) \left(\frac{dz}{d\beta}\right) + \left(\frac{dz}{d\alpha}\right) \left(\frac{d^2z}{d\beta dt}\right) \\ - \left(\frac{d^2x}{d\alpha dt}\right) \left(\frac{dx}{d\beta}\right) - \left(\frac{dx}{d\alpha}\right) \left(\frac{d^2x}{d\beta dt}\right) - \left(\frac{d^2y}{d\alpha dt}\right) \left(\frac{dy}{d\beta}\right) - \left(\frac{dy}{d\alpha}\right) \left(\frac{d^2y}{d\beta dt}\right) - \left(\frac{d^2z}{d\alpha dt}\right) \left(\frac{dz}{d\beta}\right) - \left(\frac{dz}{d\alpha}\right) \left(\frac{d^2z}{d\beta dt}\right)$$

Sed quum sit $\frac{dx}{dt} = x$, $\frac{dy}{dt} = y$, $\frac{dz}{dt} = z$, habetur

Quotientes differentiales $\left(\frac{dx}{d\alpha}\right)$, $\left(\frac{dx}{d\beta}\right)$, $\left(\frac{dx}{d\gamma}\right)$ etc. etc.

e quibus aequationes (5) compositae sunt, pro functionibus elementorum et temporis habendi sunt. Differentiando igitur primam aequationem (5) dum tempus, quatenus explicite in ea continetur, variabile spectatur, emergit

$$\begin{aligned} \left(\frac{dx}{d\beta}\right) &= \left(\frac{d^2x}{d\beta dt}\right), & \left(\frac{dx_i}{d\alpha}\right) &= \left(\frac{d^2x_i}{d\alpha dt}\right), & \left(\frac{dy_i}{d\beta}\right) &= \left(\frac{d^2y_i}{d\beta dt}\right) \\ \left(\frac{dy_i}{d\alpha}\right) &= \left(\frac{d^2y_i}{d\alpha dt}\right), & \left(\frac{dz_i}{d\beta}\right) &= \left(\frac{d^2z_i}{d\beta dt}\right), & \left(\frac{dz_i}{d\alpha}\right) &= \left(\frac{d^2z_i}{d\alpha dt}\right) \end{aligned}$$

Quodsi valores hi in aequationibus praecedentibus substituuntur, terminus primus et octavus, tertius et decimus, quintus et duodecimus mutuo se tollunt, quare erit

$$\begin{aligned} \frac{d(x, \beta)}{dt} &= \left(\frac{dx}{d\alpha}\right) \left(\frac{d^2x_i}{d\beta dt}\right) + \left(\frac{dy}{d\alpha}\right) \left(\frac{d^2y_i}{d\beta dt}\right) + \left(\frac{dz}{d\alpha}\right) \left(\frac{d^2z_i}{d\beta dt}\right) \\ &\quad - \left(\frac{d^2x_i}{d\alpha dt}\right) \left(\frac{dx}{d\beta}\right) - \left(\frac{d^2y_i}{d\alpha dt}\right) \left(\frac{dy}{d\beta}\right) - \left(\frac{d^2z_i}{d\alpha dt}\right) \left(\frac{dz}{d\beta}\right) \end{aligned}$$

Porro, posito

$$V = \frac{1}{r} \quad \text{et} \quad W = k\mu(V + \Omega)$$

aequationes (1) abeunt in

$$\frac{dx_i}{dt} = \left(\frac{dW}{dx}\right), \quad \frac{dy_i}{dt} = \left(\frac{dW}{dy}\right), \quad \frac{dz_i}{dt} = \left(\frac{dW}{dz}\right)$$

ubi animadvertendum est, functionem W quantitates x_i, y_i, z_i , non continere. Differentiatis igitur aequationibus praecedentibus tum secundum α tum secundum β , evadunt

$$\begin{aligned} \left(\frac{d^2x_i}{d\alpha dt}\right) &= \left(\frac{d^2W}{dx^2}\right) \left(\frac{dx}{d\alpha}\right) + \left(\frac{d^2W}{dx dy}\right) \left(\frac{dy}{d\alpha}\right) + \left(\frac{d^2W}{dx dz}\right) \left(\frac{dz}{d\alpha}\right) \\ \left(\frac{d^2y_i}{d\alpha dt}\right) &= \left(\frac{d^2W}{dx dy}\right) \left(\frac{dx}{d\alpha}\right) + \left(\frac{d^2W}{dy^2}\right) \left(\frac{dy}{d\alpha}\right) + \left(\frac{d^2W}{dy dz}\right) \left(\frac{dz}{d\alpha}\right) \\ \left(\frac{d^2z_i}{d\alpha dt}\right) &= \left(\frac{d^2W}{dx dz}\right) \left(\frac{dx}{d\alpha}\right) + \left(\frac{d^2W}{dy dz}\right) \left(\frac{dy}{d\alpha}\right) + \left(\frac{d^2W}{dz^2}\right) \left(\frac{dz}{d\alpha}\right) \\ \left(\frac{d^2x_i}{d\beta dt}\right) &= \left(\frac{d^2W}{dx^2}\right) \left(\frac{dx}{d\beta}\right) + \left(\frac{d^2W}{dx dy}\right) \left(\frac{dy}{d\beta}\right) + \left(\frac{d^2W}{dx dz}\right) \left(\frac{dz}{d\beta}\right) \\ \left(\frac{d^2y_i}{d\beta dt}\right) &= \left(\frac{d^2W}{dx dy}\right) \left(\frac{dx}{d\beta}\right) + \left(\frac{d^2W}{dy^2}\right) \left(\frac{dy}{d\beta}\right) + \left(\frac{d^2W}{dy dz}\right) \left(\frac{dz}{d\beta}\right) \\ \left(\frac{d^2z_i}{d\beta dt}\right) &= \left(\frac{d^2W}{dx dz}\right) \left(\frac{dx}{d\beta}\right) + \left(\frac{d^2W}{dy dz}\right) \left(\frac{dy}{d\beta}\right) + \left(\frac{d^2W}{dz^2}\right) \left(\frac{dz}{d\beta}\right) \end{aligned}$$

Si aequationes hae in valore praecedenti ipsius $\frac{d(x, \beta)}{dt}$ substituuntur, termini omnes evanescunt, itaque fit

$$\frac{d(x, \beta)}{dt} = 0$$

et simili modo demonstratur esse

$$\frac{d(\alpha, \gamma)}{dt} = 0 \quad \frac{d(\beta, \gamma)}{dt} = 0 \quad \text{etc. etc.}$$

Q. E. D. Ceterum theorema hoc casus tantum specialis theorematis cujusdam generalioris est, quod ex conservatione virium vivarum provenit, et in libri ill. *Lagrange: Mécanique analytique* inscripti Tomo primo inspicere potest.

5.

Jam satis notum est, loco elementorum ellipticorum in astronomia generaliter adhibitorum innumera aliorum elementorum systemata introduci posse et nominatim geometras in calculis analyticis eos coordinatarum et differentialium earum respectu temporis valores, qui aut tempori $t = 0$ aut tempori cuidam determinato respondent, tanquam elementa adhibuisse. Sint elementa haec per x, y, z, x_i, y_i, z_i , designata, ita ut x denotet valorem coordinatae x , qui tempori cuidam determinato respondet, x , quotientem differentialem ipsius x respectu t , pro eodem tempore et sic porro. Quibus positis, aequationes (6) abeunt in has

$$(7) \dots \dots \dots \begin{cases} k\mu \left(\frac{d\Omega}{dx}\right) dt = (x, y) dx + (x, z) dz + (x, x_i) dx_i + (x, y_i) dy_i + (x, z_i) dz_i, \\ \vdots \\ k\mu \left(\frac{d\Omega}{dz_i}\right) dt = -(x, z_i) dx - (y, z_i) dy - (z, z_i) dz - (x_i, z_i) dx_i - (y_i, z_i) dy_i, \end{cases}$$

et aequationes (5) in has

$$\begin{aligned}
 (8) \dots\dots\dots \left\{ \begin{aligned}
 (x, y) &= \left(\frac{dx}{dx}\right) \left(\frac{dx_i}{dy}\right) + \left(\frac{dy}{dx}\right) \left(\frac{dy_i}{dy}\right) + \left(\frac{dz}{dx}\right) \left(\frac{dz_i}{dy}\right) \\
 &\quad - \left(\frac{dx_i}{dx}\right) \left(\frac{dx}{dy}\right) - \left(\frac{dy_i}{dx}\right) \left(\frac{dy}{dy}\right) - \left(\frac{dz_i}{dx}\right) \left(\frac{dz}{dy}\right) \\
 &\quad \vdots \\
 (x, x_i) &= \left(\frac{dx}{dx}\right) \left(\frac{dx_i}{dx_i}\right) + \left(\frac{dy}{dx}\right) \left(\frac{dy_i}{dx_i}\right) + \left(\frac{dz}{dx}\right) \left(\frac{dz_i}{dx_i}\right) \\
 &\quad - \left(\frac{dx_i}{dx}\right) \left(\frac{dx}{dx_i}\right) - \left(\frac{dy_i}{dx}\right) \left(\frac{dy}{dx_i}\right) - \left(\frac{dz_i}{dx}\right) \left(\frac{dz}{dx_i}\right) \\
 &\quad \vdots \\
 (x, y_i) &= \left(\frac{dx}{dx_i}\right) \left(\frac{dx_i}{dy_i}\right) + \left(\frac{dy}{dx_i}\right) \left(\frac{dy_i}{dy_i}\right) + \left(\frac{dz}{dx_i}\right) \left(\frac{dz_i}{dy_i}\right) \\
 &\quad - \left(\frac{dx_i}{dx_i}\right) \left(\frac{dx}{dy_i}\right) - \left(\frac{dy_i}{dx_i}\right) \left(\frac{dy}{dy_i}\right) - \left(\frac{dz_i}{dx_i}\right) \left(\frac{dz}{dy_i}\right) \\
 &\quad \vdots
 \end{aligned} \right.
 \end{aligned}$$

Sed quum quantitates (x, y) etc. tempus explicite non contineant, in x, y, z, x_i, y_i, z_i , antequam differentiationes respectu elementorum x, y , etc. perficiuntur, valor quilibet pro t substitui potest. Licet nobis igitur eundem valorem ipsius t substituere per quem x, y , etc. in x, y , etc. transcunt, unde obtinetur

$$\begin{aligned}
 (x, y) &= \left(\frac{dx}{dx}\right) \left(\frac{dx_i}{dy}\right) + \text{etc.} \\
 &\quad - \left(\frac{dx_i}{dx}\right) \left(\frac{dx}{dy}\right) - \text{etc.} \\
 &\text{etc. etc.}
 \end{aligned}$$

Sed manifestae sunt aequationes

$$\begin{aligned}
 \left(\frac{dx}{dx}\right) &= 1, \left(\frac{dx}{dy}\right) = 0 \text{ etc. etc. } \left(\frac{dy}{dy}\right) = 1, \left(\frac{dy}{dz}\right) = 0 \text{ etc. etc.} \\
 \left(\frac{dz}{dz}\right) &= 1, \left(\frac{dz}{dx_i}\right) = 0 \text{ etc. etc. } \left(\frac{dx_i}{dx_i}\right) = 1, \left(\frac{dx_i}{dy_i}\right) = 0 \text{ etc. etc.} \\
 \left(\frac{dy_i}{dy_i}\right) &= 1, \left(\frac{dy_i}{dz_i}\right) = 0 \text{ etc. etc. } \left(\frac{dz_i}{dz_i}\right) = 1
 \end{aligned}$$

quare quantitates (x, y) etc. abeunt in

$$\begin{aligned}
 (x, y) &= 0, \text{ etc. } (x, x_i) = 1, (x, y_i) = 0 \text{ etc. } (y, x_i) = 1, \\
 (y, z_i) &= 0 \text{ etc. } (z, z_i) = 1, (x_i, y_i) = 0 \text{ etc.}
 \end{aligned}$$

et aequationes (7) fiunt

$$\begin{aligned}
 dx &= -k\mu \left(\frac{d\Omega}{dx_i}\right) dt, & dy &= -k\mu \left(\frac{d\Omega}{dy_i}\right) dt, & dz &= -k\mu \left(\frac{d\Omega}{dz_i}\right) dt \\
 dx_i &= k\mu \left(\frac{d\Omega}{dx}\right) dt, & dy_i &= k\mu \left(\frac{d\Omega}{dy}\right) dt, & dz_i &= k\mu \left(\frac{d\Omega}{dz}\right) dt
 \end{aligned}$$

6.

Aequationes praecedentes jam ab ill. *Lagrange* datae sunt, sed licet mihi hoc generalius proferre

T h e o r e m a.

Denotantibus $\xi, \eta, \zeta, \xi_i, \eta_i, \zeta_i$, coordinatas eas et

$$(9) \dots\dots\dots \left\{ \begin{aligned}
 d\xi &= -k\mu \left(\frac{d\Omega}{d\xi_i}\right) dt, & d\eta &= -k\mu \left(\frac{d\Omega}{d\eta_i}\right) dt, & d\zeta &= -k\mu \left(\frac{d\Omega}{d\zeta_i}\right) dt \\
 d\xi_i &= k\mu \left(\frac{d\Omega}{d\xi}\right) dt, & d\eta_i &= k\mu \left(\frac{d\Omega}{d\eta}\right) dt, & d\zeta_i &= k\mu \left(\frac{d\Omega}{d\zeta}\right) dt
 \end{aligned} \right.$$

Cujus theorematibus perfacilis est

D e m o n s t r a t i o.

Quum $\xi, \eta, \zeta, \xi_i, \eta_i, \zeta_i$, designent coordinatas eas et earum differentialia respectu temporis quae pro tempore indefinito τ locum habent, quantitates hae ex elementis ellipticis more communi receptis et ex quantitate indeterminata

differentialia earum respectu temporis, quae tempore cuidam indefinito, quod τ nominabo, respondent, habitisque $\xi, \eta, \zeta, \xi_i, \eta_i, \zeta_i$, pro elementis, id quod semper fieri potest, nanciscimur illis similes aequationes has

nata τ non aliter compositae sunt, quam coordinatae x, y, z , et earum differentialia respectu temporis x_i, y_i, z_i , iisdem elementis et tempore t constant; quum porro quantitates (x, y) etc. a tempore independentes sint, in x, y, z, x_i, y_i, z_i , ante differentiationes respectu elementorum $\xi, \eta, \zeta, \xi_i, \eta_i, \zeta_i$, peractas τ loco t ponere nobis licebit, quo facto x, y, z, x_i, y_i, z_i resp. in $\xi, \eta, \zeta, \xi_i, \eta_i, \zeta_i$ abeunt. Quae quum

ita sint, omnes etiam aequationes in fine articuli praecedentis allatae hoc in casu valent. Q. E. D.

7.

Habitis $\xi, \eta, \zeta, \xi, \eta, \zeta$, pro elementis ellipticis, ad computandas perturbationes coordinatarum x, y, z , hae coordinatae tamquam functiones ipsorum $\xi, \eta, \zeta, \xi, \eta, \zeta$, exhiberi et ope theorematis Tayloriani in series secundum incrementa $\delta\xi, \delta\eta$ etc. ipsorum ξ, η , etc. progredientes evolvi debent. Quibus factis, integralia aequationum (9) absque constantibus arbitrariis iis addendis loco $\delta\xi, \delta\eta$ etc. in his seriebus substitui debent. Sed computatio haec insignem

$$(10) \dots \left\{ \begin{array}{l} \delta x = \left(\frac{dx}{d\xi}\right) \delta\xi + \left(\frac{dx}{d\eta}\right) \delta\eta + \left(\frac{dx}{d\zeta}\right) \delta\zeta + \left(\frac{dx}{d\xi}\right) \delta\xi + \left(\frac{dx}{d\eta}\right) \delta\eta + \left(\frac{dx}{d\zeta}\right) \delta\zeta, \\ + \frac{1}{2} \left(\frac{d^2x}{d\xi^2}\right) \delta\xi^2 + \left(\frac{d^2x}{d\xi d\eta}\right) \delta\xi \cdot \delta\eta + \left(\frac{d^2x}{d\xi d\zeta}\right) \delta\xi \cdot \delta\zeta \text{ etc. } + \text{etc.} \\ + \frac{1}{6} \left(\frac{d^3x}{d\xi^3}\right) \delta\xi^3 + \frac{1}{2} \left(\frac{d^3x}{d\xi^2 d\eta}\right) \delta\xi^2 \cdot \delta\eta + \text{etc. } + \text{etc.} \\ + \text{etc.} \end{array} \right.$$

et similes aequationes pro δy et δz . Quum τ sit quantitas penitus arbitraria, loco τ in aequationibus (9) quaelibet quantitas substitui potest, si t excipis, post integrationes vero peractas quemlibet valorem loco τ , incluso t , substituere nobis licebit. Sed manifestum est, eundem valorem ipsius τ quem in integratis aequationibus (9) substitueris, te etiam in aequatione (10) et in similibus pro $\delta\eta$ et $\delta\xi$ substituere debere, unde theorema facile emergit. Fit enim, substituto t loco τ in aequatione (10) et in similibus pro δy et δz ,

$$\xi = x, \quad \eta = y, \quad \zeta = z, \quad \xi = x, \quad \eta = y, \quad \zeta = z,$$

sed quum t non minus quam τ sint quantitates quae in differentiationibus, quae ad aequationem (10) et similes pro δy et δz formandas requiruntur, constantes spectari debeant, nihil interest utrum ante differentiationes has peractas an post calculum hunc τ in t mutemus, mutato vero τ in t , antequam x, y, z , secundum ξ, η , etc. differentiantur loco $d\xi, d\eta$, etc. resp. dx, dy , etc. ponere licitum est. Hinc sua sponte evadit

$$\left(\frac{dx}{d\xi}\right) = \left(\frac{dx}{dx}\right) = 1, \quad \left(\frac{dx}{d\eta}\right) = \left(\frac{dx}{dy}\right) = 0 \text{ etc. etc. } \left(\frac{dy}{d\xi}\right) = 0 \\ \left(\frac{dy}{d\eta}\right) = 1, \quad \left(\frac{dy}{d\zeta}\right) = 0 \text{ etc. etc. } \left(\frac{dz}{d\xi}\right) = 0, \quad \left(\frac{dz}{d\eta}\right) = 0 \text{ etc.}$$

Quoties vero quotientes differentiales primi ordinis quantitatis cuiuspiam aut constantes aut cifrae aequales sunt, certa quotientes differentiales omnes ordinum altiorum respectu earundem quantitatum cifrae aequales sunt. Itaque sequitur esse

$$\delta x = \delta\xi, \quad \delta y = \delta\eta, \quad \delta z = \delta\zeta,$$

in modum abbreviatur per hoc

Theorema.

Designantibus $f\delta\xi, f\delta\eta, f\delta\zeta$ integralia trium primarum aequationum (9), habentur statim valores perturbati coordinatarum x, y, z , per has aequationes

$$x = f\delta\xi, \quad y = f\delta\eta, \quad z = f\delta\zeta$$

si in dextera earum parte post integrationem peractam τ in t mutatur, ita ut ultimae tres aequationes (9) ad efficiendas integrationes trium primarum aequationum tantum inserviant.

Demonstratio.

Habetur per theorema Taylorianum

si in dexteris harum aequationum partibus τ in t mutatur. Quum vero valor verus perturbatus ipsius x obtineatur, si ad δx valor pure ellipticus ipsius x additur, et praeterea ut integrale integrum primae aequationis (9) obtineatur, ad $\delta\xi$ valor ipsius ξ pure ellipticus, qui mutato τ in t in valorem ejusmodi ipsius x abibit, addendus sit: habetur valor verus perturbatus ipsius x per integrale $-k\mu \int \left(\frac{d\Omega}{d\xi}\right) dt$ sive quod idem est per integrale $f\delta\xi$ si in eo τ in t mutatur, nec non eodem modo valores ejusmodi ipsarum y et z . Q. E. D.

8.

Quasi corollaria ad theorema art. praec. spectari possunt quae hoc art. exponam. Quum eodem modo ut in art. praec. demonstrari possit esse

$$x = f\delta\xi, \quad y = f\delta\eta, \quad z = f\delta\zeta,$$

si $d\xi, d\eta$, et $d\zeta$, pro ultimis tribus aequationibus (9) habentur et si in dextris membris post integrationes τ in t mutatur: valores veros perturbatos ipsarum x, y, z praebent hae quoque aequationes

$$x = fdt\delta\xi, \quad y = fdt\delta\eta, \quad z = fdt\delta\zeta,$$

sive id quod idem est

$$x = k\mu fdt \int \left(\frac{d\Omega}{d\xi}\right) dt, \quad y = k\mu fdt \int \left(\frac{d\Omega}{d\eta}\right) dt, \\ z = k\mu fdt \int \left(\frac{d\Omega}{d\zeta}\right) dt$$

si post primas integrationes peractas τ in t mutatur. Porro

quum ξ, η, ζ , sint valores ipsarum x, y, z , si in iis t in τ mutatur, erit

$$\xi = \frac{d\xi}{d\tau}, \quad \eta = \frac{d\eta}{d\tau}, \quad \zeta = \frac{d\zeta}{d\tau}$$

atque tum, differentiando priores tres aequationes (9) secundum τ ,

$$\frac{d \left(\frac{d\Omega}{d\xi} \right)}{d\tau} = - \left(\frac{d\Omega}{d\xi} \right), \quad \frac{d \left(\frac{d\Omega}{d\eta} \right)}{d\tau} = - \left(\frac{d\Omega}{d\eta} \right),$$

$$\frac{d \left(\frac{d\Omega}{d\zeta} \right)}{d\tau} = - \left(\frac{d\Omega}{d\zeta} \right)$$

quae sunt aequationes memoratu dignae.

9.

Quoties loco elementorum $\xi, \eta, \zeta, \xi, \eta, \zeta$, elementa alia introducuntur, aequationes (9) locum non habent, sed semper elementum quodque per aequationem huius formae datur

$$(11) \dots dx = P \left(\frac{d\Omega}{d\beta} \right) dt + Q \left(\frac{d\Omega}{d\gamma} \right) dt + \text{etc.}$$

designantibus α, β, γ , etc. elementa electa atque P, Q , etc. functiones elementorum, quae tempus explicite non cohibent. Quoties vero inter sex elementa electa aut tria aut duo aut unum ex elementis more vulgari receptis et ex quantitate indeterminata τ tali modo composita sunt, quali aut tres aut duae aut una coordinatae ex iisdem elementis et ex tempore t componuntur, conclusiones art. 7 semper valent, neque res aliter se habet, quoties functiones quaecunque ex elementis illis et ex quantitate indeterminata τ compositae pro elementis habentur unde afferre licet hoc, quod in nostra theoria perturbationum fundamentale dici potest,

Theorema.

Quoties loco aliquot elementorum ab astronomis more vulgari receptorum eliguntur elementa Λ, Γ , etc. quae ex illis elementis et ex quantitate indeterminata τ tali modo composita sunt, quali quantitates quaelibet L, G , etc. ex iisdem elementis et ex tempore t constant: valores veri perturbati quantitatum L, G , etc. obtinentur, si in integris aequationibus differentialibus pro illis electis elementis Λ, Γ , etc. τ in t mutatur.

Demonstratio.

Sint a, b, c , etc. elementa more vulgari recepta, L functio quaedam temporis et istorum elementorum, et Λ functio illa ipsius τ in qua L transibit, si in ea t in τ mutatur. Sit

$$L = \psi(t, a, b, c, \text{etc.})$$

unde erit

$$\Lambda = \psi(\tau, a, b, c, \text{etc.})$$

denotante ψ signum functionis. Quum x, y, z nec non x, y, z , sint functiones ipsarum a, b, c , etc. et t , inter has functiones et Λ unum elementorum e.g. elementum a eliminare licet, ut emergat

$$x = \chi(t, \tau, \Lambda, b, c, \text{etc.})$$

ubi χ signum functionis est, et similes functiones obtinentur pro y, z, x, y, z . Quam ob rem pervenitur ope aequationum (5) et (6) ad aequationes sicut (11) pro dL, db, dc , etc. quarum integratio elementa ipsa Λ, b, c , etc. praebet. Quibus in aequationibus (11) quotientes differentiales ipsius Ω generaliter loquendo functiones ipsarum τ et t erunt, coefficientes vero P, Q , etc. quidem τ , sed nullo modo t continere possunt. Imo eliminatione inter functiones illas ipsas, quam modo commemoravi, non opus erit, nam finem eundem assequeris, si eliminationem inter differentiaalia harum functionum institueris. Inventis tali modo valoribus perturbatis ipsarum Λ, b, c , etc. valorem verum perturbatum ipsius L nanciscemur, exhibendo L tanquam functionem ipsarum Λ, b, c , etc. id quod fit per eliminationem ipsius a inter functiones duas supra ψ denotatas, et substituendo pro Λ, b, c , etc. valores eorum veros perturbatos eos quos modo explicavi. Idem vero valor ipsius L obtinetur ope theorematum Tayloriani. Denotando enim per $\delta\Lambda, \delta b, \delta c$ etc. integralia aequationum (11) absque constantibus arbitrariis, habetur

$$\delta L = \left(\frac{dL}{d\Lambda} \right) \delta\Lambda + \left(\frac{dL}{db} \right) \delta b + \left(\frac{dL}{dc} \right) \delta c + \text{etc.} \left\{ \dots (12) \right.$$

$$+ \frac{1}{2} \left(\frac{d^2 L}{d\Lambda^2} \right) \delta\Lambda^2 + \left(\frac{d^2 L}{d\Lambda db} \right) \delta\Lambda \cdot \delta b + \text{etc.} \left. \right\}$$

$$+ \text{etc.}$$

et hinc valor verus perturbatus ipsius L , si $\delta\Lambda$ ad valorem pure ellipticum ejusdem quantitatis additur. Inquiramus in inveniendos quotientes differentiales $\left(\frac{dL}{d\Lambda} \right), \left(\frac{dL}{db} \right)$ etc. $\left(\frac{d^2 L}{d\Lambda^2} \right)$ etc. etc. Differentiata hunc in finem functione supra ψ denotata, invenitur

$$dL = p da + r db + s dc + \text{etc.}$$

$$d\Lambda = \pi da + \rho db + \sigma dc + \text{etc.}$$

ubi coefficientes π, ρ, σ etc. ex τ et ex elementis a, b, c , etc. eodem modo constant, quo resp. p, r, s , etc. ex t et ex iisdem elementis compositi sunt. Eliminato da inter praecedentes aequationes fit

$$dL = \frac{p}{\pi} d\Lambda + \frac{\pi r - p\rho}{\pi} db + \frac{\pi s - p\sigma}{\pi} dc + \text{etc.}$$

itaque

$$\left(\frac{dL}{d\Lambda}\right) = \frac{p}{\pi}, \quad \left(\frac{dL}{db}\right) = \frac{\pi r - p\rho}{\pi}, \quad \left(\frac{dL}{dc}\right) = \frac{\pi s - p\sigma}{\pi}, \text{ etc.}$$

Differentialis his aequationibus, emergunt

$$\begin{aligned} d\left(\frac{dL}{d\Lambda}\right) &= \frac{\pi dp - p d\pi}{\pi^2} \\ d\left(\frac{dL}{db}\right) &= \frac{\pi^2 dr - \pi p d\rho - \pi \rho dp + p \rho d\pi}{\pi^2} \\ \text{etc.} &= \text{etc.} \end{aligned}$$

Sed quum p , r , etc. π , ρ , etc. sint functiones ipsarum a , b , c , etc. poni potest

$$\begin{aligned} d\left(\frac{dL}{d\Lambda}\right) &= \frac{p'\pi - \pi'p}{\pi^2} da + \frac{r'\pi - \rho'p}{\pi^2} db + \text{etc.} \\ d\left(\frac{dL}{db}\right) &= \frac{\pi(\pi r' - p\rho') + \rho(p\pi' - \pi\rho')}{\pi^2} da + \frac{\pi(\pi r'' - p\rho'') + \rho(p\rho' - \pi r')}{\pi^2} db + \frac{\pi(\pi s'' - p\sigma'') + \rho(p\sigma' - \pi s')}{\pi^2} dc + \text{etc.} \end{aligned}$$

quae, eliminato da ope aequationis hujus

$$d\Lambda = \pi da + \rho db + \sigma dc + \text{etc.}$$

suppeditant

$$\begin{aligned} d\left(\frac{dL}{d\Lambda}\right) &= A d\Lambda + B db + \text{etc.} \\ d\left(\frac{dL}{db}\right) &= B d\Lambda + A' db + B' dc + \text{etc.} \end{aligned}$$

ubi brevitatis gratia posui

$$\begin{aligned} A &= \frac{p'\pi - \pi'p}{\pi^3} \\ B &= \frac{\pi(r'\pi - \rho'p) + \rho(\pi'p - p'\pi)}{\pi^3} \\ A' &= \frac{\pi^2(\pi r'' - p\rho'') + 2\pi\rho(p\rho' - \pi r') + \rho^2(\pi p' - p\pi')}{\pi^3} \\ B' &= \frac{\pi^2(\pi s'' - p\sigma'') + \pi\sigma(p\rho' - \pi r') + \pi\rho(p\sigma' - \pi s') + \rho\sigma(\pi p' - p\pi')}{\pi^3} \end{aligned}$$

Itaque

$$\begin{aligned} \left(\frac{d^2L}{d\Lambda^2}\right) &= A, \quad \left(\frac{d^2L}{d\Lambda db}\right) = B, \quad \text{etc.} \\ \left(\frac{d^2L}{db^2}\right) &= A', \quad \left(\frac{d^2L}{db dc}\right) = B', \quad \text{etc.} \end{aligned}$$

Quum vero in integratis aequationibus quae ipsi (11) similes perturbationes ipsorum Λ , b , c , etc. sive $\delta\Lambda$, δb , δc , etc. praebent, τ sit quantitas penitus arbitraria, t loco τ ponere licitum est, quo facto in quotientibus differentialibus quas aequatio (12) continet etiam t loco τ poni debet, sed posita $\tau = t$ in expressionibus pro quotientibus his modo inventis, emergunt statim

$$\begin{aligned} \left(\frac{dL}{d\Lambda}\right) &= 1, \quad \left(\frac{dL}{db}\right) = 0, \quad \left(\frac{dL}{dc}\right) = 0, \quad \text{etc.} \\ \left(\frac{d^2L}{d\Lambda^2}\right) &= 0, \quad \left(\frac{d^2L}{d\Lambda db}\right) = 0, \quad \text{etc.} \\ \left(\frac{d^2L}{db^2}\right) &= 0, \quad \left(\frac{d^2L}{db dc}\right) = 0, \quad \text{etc.} \end{aligned}$$

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$$\begin{aligned} dp &= p' da + r' db + s' dc + \text{etc.} \\ d\pi &= \pi' da + \rho' db + \sigma' dc + \text{etc.} \\ dr &= p'' da + r'' db + s'' dc + \text{etc.} \\ d\rho &= \pi'' da + \rho'' db + \sigma'' dc + \text{etc.} \\ \text{etc.} &= \text{etc.} \end{aligned}$$

ubi p' , r' , etc. π' , ρ' , etc. tales sunt, ut transeat π' in p' , ρ' in r' , etc. π'' in p'' , ρ'' in r'' , etc. quoties in iis τ in t mutatur. Praeterea animadvertendum est, esse $p'' = r'$, $\pi'' = \rho'$.

Substitutis his valoribus ipsorum dp , etc. in aequationibus praecedentibus, nanciscimur has

et computatio ipsa, quae hos valores praebuit, jam indicat, fore ut quotientes differentiales ordinum altiorum omnes cifrae aequales invenirentur. Sed hanc rem per argumentationem subsequentem extra omnem dubitationem ponere licet.

Quum quotientes differentiales primi ordinis ipsius L modo inventi, mutato τ in t , partim constantes partim cifrae aequales sint, atque quum et τ et t sint quantitates, quae in omnibus his differentiationibus constantes spectari debeant: certo concludi potest, quotientes differentiales altiorum ordinum omnes, mutato τ in t , cifrae aequales fieri debere, quapropter theorema allatum demonstratum est, quoties loco unius elementi functio quaelibet veluti Λ quasi elementum introducitur.

Si insuper functio alia elementorum a , b , c , etc. et temporis t datur, quam G nominabo, ita ut sit

$$G = f(t, a, b, c, \text{ etc.})$$

ubi f signum functionis est, cujus valor in motu perturbato quaeritur, accipiat

$$\Gamma = f(\tau, a, b, c, \text{ etc.})$$

inter numerum elementorum et eliminantur elementa duo e. g. elementa a et b in expressionibus coordinatarum x , y , z et differentialium x , y , z , sive id quod ad eundem finem adducit, eliminantur differentialia da et db inter differentialia harum aequationum, unde per formulas supra traditas, pro quoque elementorum differentiali $d\Gamma$, $d\Lambda$, dc , etc. aequationem veluti (11) nancisceris, quarum aequationum integratione valores perturbati horum elementorum emergunt. Itaque theorema Taylorianum praebet

$$\delta L = \left(\frac{dL}{d\Lambda}\right) \delta\Lambda + \left(\frac{dL}{d\Gamma}\right) \delta\Gamma + \left(\frac{dL}{dc}\right) \delta c + \text{etc.}$$

$$+ \frac{1}{2} \left(\frac{d^2 L}{d\Lambda^2}\right) \delta\Lambda^2 + \left(\frac{d^2 L}{d\Lambda d\Gamma}\right) \delta\Lambda \cdot \delta\Gamma + \text{etc.} + \text{etc.}$$

$$\delta G = \left(\frac{dG}{d\Lambda}\right) \delta\Lambda + \left(\frac{dG}{d\Gamma}\right) \delta\Gamma + \left(\frac{dG}{dc}\right) \delta c + \text{etc.}$$

$$+ \frac{1}{2} \left(\frac{d^2 G}{d\Lambda^2}\right) \delta\Lambda^2 + \left(\frac{d^2 G}{d\Lambda d\Gamma}\right) \delta\Lambda \cdot \delta\Gamma + \text{etc.} + \text{etc.}$$

Jam habebamus

$$dL = p da + r db + s dc + \text{etc.}$$

$$d\Lambda = \pi da + \rho db + \sigma dc + \text{etc.}$$

$$dL = \frac{r\lambda - p\mu}{\lambda\rho - \pi\mu} d\Lambda + \frac{p\rho - r\pi}{\lambda\rho - \pi\mu} d\Gamma + \frac{\mu(p\sigma - \pi s) + \nu(r\pi - \rho p) + \lambda(s\rho - \sigma r)}{\lambda\rho - \pi\mu} dc + \text{etc.}$$

$$dG = \frac{m\lambda - \mu l}{\lambda\rho - \pi\mu} d\Lambda + \frac{l\rho - \pi m}{\lambda\rho - \pi\mu} d\Gamma + \frac{\sigma(l\mu - \lambda m) + \rho(n\lambda - \nu l) + \pi(m\nu - \mu n)}{\lambda\rho - \pi\mu} dc + \text{etc.}$$

Si igitur in $\delta\Lambda$, $\delta\Gamma$ nec non in praecedentibus aequationibus τ in t mutatur elicitur

$$\left(\frac{dL}{d\Lambda}\right) = 1, \quad \left(\frac{dL}{d\Gamma}\right) = 0, \quad \left(\frac{dL}{dc}\right) = 0, \quad \text{etc.}$$

$$\left(\frac{dG}{d\Lambda}\right) = 0, \quad \left(\frac{dG}{d\Gamma}\right) = 1, \quad \left(\frac{dG}{dc}\right) = 0, \quad \text{etc.}$$

et hinc eodem modo ut ante concluditur, obtineri valores perturbatos ipsarum L et G , quoties in integralibus $\int d\Lambda$ et $\int d\Gamma$ τ in t mutatur, unde theorema demonstratum est quoties duae functiones adsunt, et simili modo demonstratur idem quoties functiones plures dantur.

10.

Theorema art. praec., quamvis generaliter valeat, quomodocunque quantitates L , G , etc. compositae sunt, sensu tamen stricto ad earum compositionem modo referendum est, quantum vero ad earum significationem in ellipsi pura et in ellipsi perturbata discrimen essentielle interesse potest. Quae hujus theorematism conditio sequentibus illustratur.

Integralia aequationum (1) per methodum variationis constantium arbitrariorum in praecedentibus ita determinavimus, ut eandem formam habeant, sive Ω sit cifrae aequalis, sive non. Itaque coordinatae x , y , z et proinde functio quaelibet ipsarum x , y , z proprietate hac gaudent, quod eandem formam habent sive perturbationes adsunt, sive non. Item, aequationes (3) probant, quotientes differentiales primi ordinis respectu temporis ipsarum x , y , z in utroque casu ejusdem formae quoque esse. Aequationes vero (4) demonstrant, quotientes secundi ordinis $\frac{d^2 x}{dt^2}$,

et simili modo praebet functio G

$$dG = l da + m db + n dc + \text{etc.}$$

$$d\Gamma = \lambda da + \mu db + \nu dc + \text{etc.}$$

ubi coefficientium indoles talis est, ut λ in l , μ in m , ν in n , etc. transeat quoties τ in t mutatur. Eliminatis db et da inter aequationes praecedentes pro $d\Lambda$ et $d\Gamma$, inveniuntur hae

$$da = -\frac{\mu}{\lambda\rho - \pi\mu} d\Lambda + \frac{\rho}{\lambda\rho - \pi\mu} d\Gamma + \frac{\mu\sigma - \rho\nu}{\lambda\rho - \pi\mu} dc + \text{etc.}$$

$$db = \frac{\lambda}{\lambda\rho - \pi\mu} d\Lambda - \frac{\pi}{\lambda\rho - \pi\mu} d\Gamma + \frac{\pi\nu - \lambda\sigma}{\lambda\rho - \pi\mu} dc + \text{etc.}$$

His vero valoribus ipsarum da et db in aequationibus pro dL et dG substitutis, evadunt

$\frac{d^2 x}{dt^2}$, $\frac{d^2 z}{dt^2}$ et proinde quotientes differentiales ordinum altiorum omnes in utroque casu ejusdem formae non esse. Ergo, quoties L , G , etc. sunt aut eo ipso functiones ipsarum x , y , z , $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, aut ad functiones harum quantitatum reduci possunt, non modo earum compositio, sed etiam earum significatio in ellipsi pura et in ellipsi perturbata eadem est, et igitur integralia $\int d\Lambda$, $\int d\Gamma$, etc. post τ in t mutatum, non modo respectu eorum compositionis sed etiam respectu eorum significationis valores veros perturbatos ipsarum L , G , etc. praebunt. Quoties vero functiones L , G etc. aut praeter illas quantitates insuper $\frac{d^2 x}{dt^2}$, etc. $\frac{d^3 x}{dt^3}$ etc. etc. continent, aut solae functiones horum quotientium sunt, integralia illa modo quantum ad earum compositionem valores perturbatos ipsarum L , G , etc. subministrabunt, quod attinet ad earum significationem, haec in ellipsi perturbata ab earum significatione in ellipsi pura necessario differt.

Designante igitur L aut longitudinem veram, aut radium vectorem, aut functionem quandam longitudinis verae, radii vectoris et quotientium differentialium primi ordinis harum quantitatum respectu temporis, integrali $\int d\Lambda$, post τ in t mutatam, eadem significatio in motu perturbato attribui debet, quam L in motu imperturbato habet, quia longitudo vera, radius vector et quotientes eorum differentiales primi ordinis respectu temporis solae functiones ipsarum x , y , z , $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ sunt. Denotante vero e. g. L quantitatem $\frac{a^3 n^2}{r^3} e \sin f$, ubi a semiaxis major, n motus medius, e excentricitas, f anomalia vera et r radius vector

est, integrale $\int d\Lambda$ valorem quidem perturbatum quantitatis $\frac{a^3 n^2}{r^3} e \sin f$ praebebit, sed integrali huic significatio alia attribuenda est. Designat enim, si vires perturbantes desunt, $\frac{a^3 n^2}{r^3} e \sin f$ quantitatem $-\frac{1}{2} \frac{d^2 f}{dt^2}$, quae secundum praecedentia, quoties vires perturbantes adsunt, alius formae est. Quo in casu, ut significatio integralis $\int d\Lambda$ investigetur, quantitas $\frac{a^3 n^2}{r^3} e \sin f$ aut immediate per $x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, aut per functiones harum quantitatum exprimi debet, et quum facile reperiatur, esse $\frac{a^3 n^2}{r^3} e \sin f = \frac{d \log r}{dt} \times \frac{dv}{dt}$, ubi v longitudo vera, integrali $\int d\Lambda$ significatio $\frac{d \log r}{dt} \times \frac{dv}{dt}$ in motu perturbato attribuenda est.

Attamen accepta quantitate hac

$$-\frac{1}{2} \frac{d^2 \phi}{d\tau^2} = \frac{a^3 n^2}{r^3} e \sin \phi$$

inter elementorum numerum, formataque ad quantitatem $-2 \frac{\log r}{dt} \times \frac{dv}{dt}$, qualis in motu perturbato locum habet determinandam, aequatione hac veluti (11)

$$\frac{d^3 \phi}{d\tau^2} = P\left(\frac{d\Omega}{d\beta}\right) dt + Q\left(\frac{d\Omega}{d\gamma}\right) dt + \text{etc.}$$

adjumento ejusdem aequationis, ad ipsum $\frac{d^2 f}{dt^2}$, qualis in motu perturbato existit, determinandum, pervenire poteris. Nam, integrata aequatione praecedente respectu ipsius τ , si constans arbitraria huic integrali addenda, quae generaliter functio ipsius t erit, rite determinata adjectaque fuerit, aequationem nancisceris, quae quantitatem $\frac{d^2 \phi}{d\tau^2}$ per vires perturbantes et per differentiale temporis exhibet. Qua aequatione respectu ipsius t integrata, si τ in t mutatum fuerit, valorem perturbatum ipsius $\frac{df}{dt}$ habebis, qui secundum t differentiatius, valorem quaesitum ipsius $\frac{d^2 f}{dt^2}$ praebet.

11.

Si L denotat functionem ipsarum x, y, z , absque quantitibus $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, ea quae in art. praec. exposui, ad methodos duas differentialis $\frac{dL}{dt}$ determinandi viam muniant. Aut quotiens differentialis $\frac{d\Lambda}{d\tau}$ pro uno elementorum habitus per praecepta in praecedentibus exposita directe determinari potest, aut eadem quantitas ex valore ipsius $\int d\Lambda$ antequam τ in t mutatum erit per differentiationem respectu ipsius τ eruitur. Etenim aequatione hac veluti (11) formata

$$d\Lambda = P\left(\frac{d\Omega}{d\beta}\right) dt + Q\left(\frac{d\Omega}{d\gamma}\right) dt + \text{etc.}$$

quam brevitatis caussa ita exhibebo

$$d\Lambda = \frac{d\Lambda}{dt} dt$$

necessario habebitur

$$\frac{d^2 \Lambda}{d\tau^2} = \frac{d^2 \Lambda}{d\tau dt} dt$$

quum vero τ et t penitus a se invicem independentes sint, differentiatio respectu ipsius τ post integrationem respectu ipsius t peractam instituere licet, ite ut fiat

$$\frac{d\Lambda}{d\tau} = \frac{d \int \frac{d\Lambda}{dt} dt}{d\tau}$$

ubi in membro ad sinistram quantitas $\frac{d\Lambda}{d\tau}$ ita considerari potest, ac si directo modo inventa esset, postquam valor ejus pure ellipticus inter elementorum numerum acceptus fuerit. Quum vero significatio ipsius $\frac{dL}{dt}$ in utroque motus genere eadem sit, aequatio praecedens statim subministrat inter valores perturbatos aequationem hanc

$$\frac{dL}{dt} = \left(\frac{d \int \frac{d\Lambda}{dt} dt}{d\tau} \right)$$

sive

$$\frac{dL}{dt} = \left(\frac{d\Lambda}{d\tau} \right)$$

designante linea superposita τ in t mutari debere.

12.

Jam venio ad eorum quae in praecedentibus exposui applicationem. Sint l longitudo vera ad planum ipsarum xy reducta, b latitudo supra idem planum et r radius vector planetae, tum erit

$$x = r \cos b \cos l$$

$$y = r \cos b \sin l$$

$$z = r \sin b$$

$$x_1 = r_1 \cos b \cos l - b_1 r \sin b \cos l - l_1 r \cos b \sin l$$

$$y_1 = r_1 \cos b \sin l - b_1 r \sin b \sin l + l_1 r \cos b \cos l$$

$$z_1 = r_1 \sin b + b_1 r \cos b$$

designantibus r, b, l differentialia ipsarum r, b, l respectu temporis per dt divisa. Elementa accipio esse λ, ρ, β , λ, ρ, β , quae ex elementis more vulgari receptis et quantitate indeterminata τ tali modo composita sint, quali resp. l, r, b, l, r, b iisdem elementis et tempore t constant. Jam duae methodi aequationum pro $d\lambda, d\rho$, etc. computandarum se nobis offerunt; coordinatis enim ξ, η , etc. in sensu eodem, qui iis in art. 6 attributus est, acceptis, habebimus $\xi = \rho \cos \beta \cos \lambda, \eta = \rho \cos \beta \sin \lambda$, etc. unde aequationes (9) per algorithmum notum differentialium partialium in aequationes quaesitas pro $d\lambda, d\rho$, etc. transformare liceret, eadem vero aequationes calculo directo ope aequa-

tionum (5) et (6) obtineri possunt. Methodo posteriori quae illa simplicior videbatur utar.

13.

Aequationes art. praecedentis differentiatiae praebent

$$\begin{aligned}
 dx &= dr \cos b \cos l - db r \sin b \cos l - dl r \cos b \sin l \\
 dy &= dr \cos b \sin l - db r \sin b \sin l + dl r \cos b \cos l \\
 dz &= dr \sin b + db r \cos b \\
 dx_1 &= dr_1 \cos b \cos l - db_1 r_1 \sin b \cos l - dl_1 r_1 \cos b \sin l \\
 &\quad - dl(r_1 \cos b \sin l - b_1 r_1 \sin b \sin l + l_1 r_1 \cos b \cos l) \\
 &\quad - dr(b_1 \sin b \cos l + l_1 \cos b \sin l) \\
 &\quad - db(r_1 \sin b \cos l + b_1 r_1 \cos b \cos l - l_1 r_1 \sin b \sin l) \\
 dy_1 &= dr_1 \cos b \sin l - db_1 r_1 \sin b \sin l + dl_1 r_1 \cos b \cos l \\
 &\quad + dl(r_1 \cos b \cos l - b_1 r_1 \sin b \cos l - l_1 r_1 \cos b \sin l) \\
 &\quad + dr(l_1 \cos b \cos l - b_1 \sin b \sin l) \\
 &\quad - db(l_1 r_1 \sin b \cos l + r_1 \sin b \sin l + b_1 r_1 \cos b \sin l) \\
 dz_1 &= dr_1 \sin b + db_1 r_1 \cos b \\
 &\quad + dr b_1 \cos b \\
 &\quad + db(r_1 \cos b - b_1 r_1 \sin b)
 \end{aligned}
 \tag{13}$$

Ut quotientes differentiales nansciscamur, quibus ad computandas quantitates (λ, ρ) , (λ, β) etc. secundum aequationes (5) opus est, oporteret dl , dr , db , dl_1 , etc. nec non $d\lambda$, $d\rho$, $d\beta$, $d\lambda_1$, etc. per differentialia elementorum vulgariarum exhibita essent et proinde eliminando quodque differentialium dl , dr , etc. per differentialia $d\lambda$, $d\rho$, etc. exprimeretur. Quum vero in quantitatibus (λ, ρ) , (λ, β) etc. tempus evanescere debeat, licitum erit τ loco t in aequationibus praecedentibus nec non in aequationibus illis differentialia $[dl, dr, \text{etc.}]$ per differentialia elementorum vulgariarum exhibentibus ponere, unde haec cum differentialibus $d\lambda$, $d\rho$, identica evadunt. Fiunt igitur

$$\left(\frac{dl}{d\lambda}\right) = 1, \quad \left(\frac{dr}{d\rho}\right) = 1 \text{ etc.} \quad \left(\frac{dl}{d\rho}\right) = 0, \quad \left(\frac{dr}{d\lambda}\right) = 0, \text{ etc.}$$

et quotientes differentiales ad computandas quantitates (λ, ρ) , (λ, β) , etc. requisiti obtinentur ponendo $t = \tau$ in dextris membris aequationum (13). Ergo

$$\begin{aligned}
 \left(\frac{dx}{d\lambda}\right) &= -\rho \cos \beta \sin \lambda; & \left(\frac{dy}{d\lambda}\right) &= \rho \cos \beta \cos \lambda; & \left(\frac{dz}{d\lambda}\right) &= 0 \\
 \left(\frac{dx}{d\rho}\right) &= \cos \beta \cos \lambda; & \left(\frac{dy}{d\rho}\right) &= \cos \beta \sin \lambda; & \left(\frac{dz}{d\rho}\right) &= \sin \beta \\
 \left(\frac{dx}{d\beta}\right) &= -\rho \sin \beta \cos \lambda; & \left(\frac{dy}{d\beta}\right) &= -\rho \sin \beta \sin \lambda; & \left(\frac{dz}{d\beta}\right) &= \rho \cos \beta \\
 \left(\frac{dx}{d\lambda_1}\right) &= 0 & \left(\frac{dy}{d\lambda_1}\right) &= 0 & \left(\frac{dz}{d\lambda_1}\right) &= 0 \\
 \left(\frac{dx}{d\rho_1}\right) &= 0 & \left(\frac{dy}{d\rho_1}\right) &= 0 & \left(\frac{dz}{d\rho_1}\right) &= 0 \\
 \left(\frac{dx}{d\beta_1}\right) &= 0 & \left(\frac{dy}{d\beta_1}\right) &= 0 & \left(\frac{dz}{d\beta_1}\right) &= 0 \\
 \left(\frac{dx_1}{d\lambda}\right) &= -\rho_1 \cos \beta \sin \lambda + \beta_1 \rho \sin \beta \sin \lambda - \lambda_1 \rho \cos \beta \cos \lambda \\
 \left(\frac{dx_1}{d\rho}\right) &= -\beta_1 \sin \beta \cos \lambda - \lambda_1 \cos \beta \sin \lambda \\
 \left(\frac{dx_1}{d\beta}\right) &= \lambda_1 \rho \sin \beta \sin \lambda - \rho_1 \sin \beta \cos \lambda - \beta_1 \rho \cos \beta \cos \lambda \\
 \left(\frac{dx_1}{d\lambda_1}\right) &= -\rho \cos \beta \sin \lambda; & \left(\frac{dx_1}{d\rho_1}\right) &= \cos \beta \cos \lambda; & \left(\frac{dx_1}{d\beta_1}\right) &= -\rho \sin \beta \cos \lambda \\
 \left(\frac{dy_1}{d\lambda}\right) &= -\lambda_1 \rho \cos \beta \sin \lambda + \rho_1 \cos \beta \cos \lambda - \beta_1 \rho \sin \beta \cos \lambda \\
 \left(\frac{dy_1}{d\rho}\right) &= \lambda_1 \cos \beta \cos \lambda - \beta_1 \sin \beta \sin \lambda \\
 \left(\frac{dy_1}{d\beta}\right) &= -\lambda_1 \rho \sin \beta \cos \lambda - \rho_1 \sin \beta \sin \lambda - \beta_1 \rho \cos \beta \sin \lambda \\
 \left(\frac{dy_1}{d\lambda_1}\right) &= \rho \cos \beta \cos \lambda; & \left(\frac{dy_1}{d\rho_1}\right) &= \cos \beta \sin \lambda; & \left(\frac{dy_1}{d\beta_1}\right) &= -\rho \sin \beta \sin \lambda \\
 \left(\frac{dz_1}{d\lambda}\right) &= 0; & \left(\frac{dz_1}{d\rho}\right) &= \beta_1 \cos \beta; & \left(\frac{dz_1}{d\beta}\right) &= \rho_1 \cos \beta - \beta_1 \rho \sin \beta \\
 \left(\frac{dz_1}{d\lambda_1}\right) &= 0; & \left(\frac{dz_1}{d\rho_1}\right) &= \sin \beta; & \left(\frac{dz_1}{d\beta_1}\right) &= \rho \cos \beta
 \end{aligned}$$

14.

Jam omnia praesto adsunt quae ad computandas

(λ, ρ), (λ, β) etc. requiruntur, et aequationes (5) facile opera praebent

$$\begin{aligned}
 (\lambda, \rho) &= 2\lambda, \rho \cos^2 \beta; & (\lambda, \beta) &= -2\lambda, \rho^2 \sin \beta \cos \beta; & (\lambda, \lambda_r) &= \rho^2 \cos^2 \beta; & (\lambda, \rho_r) &= 0; & (\lambda, \beta_r) &= 0 \\
 (\rho, \beta) &= -2\beta, \rho; & (\rho, \lambda_r) &= 0; & (\rho, \rho_r) &= 1; & (\rho, \beta_r) &= 0 \\
 & & (\beta, \lambda_r) &= 0; & (\beta, \rho_r) &= 0; & (\beta, \beta_r) &= \rho^2 \\
 & & & & (\lambda, \rho_r) &= 0; & (\lambda, \beta_r) &= 0 \\
 & & & & & & (\rho, \beta_r) &= 0
 \end{aligned}$$

quibus valoribus aequationes (6) abeunt in

$$\begin{aligned}
 k\mu \left(\frac{d\Omega}{d\lambda} \right) dt &= 2\lambda, \rho \cos^2 \beta d\rho - 2\lambda, \rho^2 \sin \beta \cos \beta d\beta + \rho^2 \cos^2 \beta d\lambda, \\
 k\mu \left(\frac{d\Omega}{d\rho} \right) dt &= -2\lambda, \rho \cos^2 \beta d\lambda - 2\beta, \rho d\beta + d\rho, \\
 k\mu \left(\frac{d\Omega}{d\beta} \right) dt &= 2\lambda, \rho^2 \sin \beta \cos \beta d\lambda + 2\beta, \rho d\rho + \rho^2 d\beta, \\
 k\mu \left(\frac{d\Omega}{d\lambda_r} \right) dt &= -\rho^2 \cos^2 \beta d\lambda \\
 k\mu \left(\frac{d\Omega}{d\rho_r} \right) dt &= -d\rho \\
 k\mu \left(\frac{d\Omega}{d\beta_r} \right) dt &= -\rho^2 d\beta
 \end{aligned}$$

quae facili eliminatione facta, subministrant

$$\begin{aligned}
 d\lambda &= -\frac{k\mu}{\rho^2 \cos^2 \beta} \left(\frac{d\Omega}{d\lambda_r} \right) dt \\
 d\rho &= -k\mu \left(\frac{d\Omega}{d\rho_r} \right) dt \\
 d\beta &= -\frac{k\mu}{\rho^2} \left(\frac{d\Omega}{d\beta_r} \right) dt \\
 d\lambda_r &= \frac{k\mu}{\rho^2 \cos^2 \beta} \left(\frac{d\Omega}{d\lambda} \right) dt + 2\lambda, \frac{k\mu}{\rho} \left(\frac{d\Omega}{d\rho_r} \right) dt - 2\lambda, \rho \beta \frac{k\mu}{\rho^2} \left(\frac{d\Omega}{d\beta_r} \right) dt \\
 d\rho_r &= k\mu \left(\frac{d\Omega}{d\rho} \right) dt - 2\lambda, \frac{k\mu}{\rho} \left(\frac{d\Omega}{d\lambda_r} \right) dt - 2\beta, \frac{k\mu}{\rho} \left(\frac{d\Omega}{d\beta_r} \right) dt \\
 d\beta_r &= \frac{k\mu}{\rho^2} \left(\frac{d\Omega}{d\beta} \right) dt + 2\lambda, \rho \beta \frac{k\mu}{\rho^2} \left(\frac{d\Omega}{d\lambda_r} \right) dt + 2\beta, \frac{k\mu}{\rho} \left(\frac{d\Omega}{d\rho_r} \right) dt
 \end{aligned}$$

15.

Ut aequationes hae in usum vocentur, necesse est functiones ipsius r quas involvunt explicite adstent, idquod fit introducendo quotientes differentiales ipsius Ω respectu ipsarum l, r et b loco quotientium differentialium respectu ipsarum $\lambda, \rho, \beta, \lambda_r, \rho_r, \beta_r$. Quem in finem habetur

$$(14) \dots \dots \dots \left\{ \begin{aligned} d\Omega &= \left(\frac{d\Omega}{dl} \right) dl + \left(\frac{d\Omega}{dr} \right) dr + \left(\frac{d\Omega}{db} \right) db \\ d\Omega &= \left(\frac{d\Omega}{d\lambda} \right) d\lambda + \left(\frac{d\Omega}{d\rho} \right) d\rho + \left(\frac{d\Omega}{d\beta} \right) d\beta + \left(\frac{d\Omega}{d\lambda_r} \right) d\lambda_r + \left(\frac{d\Omega}{d\rho_r} \right) d\rho_r + \left(\frac{d\Omega}{d\beta_r} \right) d\beta_r \end{aligned} \right.$$

si Ω tum pro functione ipsarum l, r et b , tum pro functione ipsarum λ, ρ, β , etc. habetur. Denotantibus vero a, c, e, f, g, h elementa more vulgari recepta, emergunt

$$\begin{aligned}
d\lambda &= \left(\frac{d\lambda}{da}\right) da + \left(\frac{d\lambda}{dc}\right) dc + \left(\frac{d\lambda}{de}\right) de + \left(\frac{d\lambda}{df}\right) df + \left(\frac{d\lambda}{dg}\right) dg + \left(\frac{d\lambda}{dh}\right) dh \\
d\rho &= \left(\frac{d\rho}{da}\right) da + \left(\frac{d\rho}{dc}\right) dc + \left(\frac{d\rho}{de}\right) de \\
d\beta &= \left(\frac{d\beta}{da}\right) da + \left(\frac{d\beta}{dc}\right) dc + \left(\frac{d\beta}{de}\right) de + \left(\frac{d\beta}{df}\right) df + \left(\frac{d\beta}{dg}\right) dg + \left(\frac{d\beta}{dh}\right) dh \\
d\lambda_1 &= \left(\frac{d\lambda_1}{da}\right) da + \left(\frac{d\lambda_1}{dc}\right) dc + \left(\frac{d\lambda_1}{de}\right) de + \left(\frac{d\lambda_1}{df}\right) df + \left(\frac{d\lambda_1}{dg}\right) dg + \left(\frac{d\lambda_1}{dh}\right) dh \\
d\rho_1 &= \left(\frac{d\rho_1}{da}\right) da + \left(\frac{d\rho_1}{dc}\right) dc + \left(\frac{d\rho_1}{de}\right) de \\
d\beta_1 &= \left(\frac{d\beta_1}{da}\right) da + \left(\frac{d\beta_1}{dc}\right) dc + \left(\frac{d\beta_1}{de}\right) de + \left(\frac{d\beta_1}{df}\right) df + \left(\frac{d\beta_1}{dg}\right) dg + \left(\frac{d\beta_1}{dh}\right) dh \\
dl &= \left(\frac{dl}{da}\right) da + \left(\frac{dl}{dc}\right) dc + \left(\frac{dl}{de}\right) de + \left(\frac{dl}{df}\right) df + \left(\frac{dl}{dg}\right) dg + \left(\frac{dl}{dh}\right) dh \\
dr &= \left(\frac{dr}{da}\right) da + \left(\frac{dr}{dc}\right) dc + \left(\frac{dr}{de}\right) de \\
db &= \left(\frac{db}{da}\right) da + \left(\frac{db}{dc}\right) dc + \left(\frac{db}{de}\right) de + \left(\frac{db}{df}\right) df + \left(\frac{db}{dg}\right) dg + \left(\frac{db}{dh}\right) dh
\end{aligned}$$

Quibus aequationibus in (14) substitutis, terminisque per eadem differentialia multiplicatis comparatis, nanciscimur

$$\begin{aligned}
&\left(\frac{d\Omega}{dl}\right)\left(\frac{dl}{da}\right) + \left(\frac{d\Omega}{dr}\right)\left(\frac{dr}{da}\right) + \left(\frac{d\Omega}{db}\right)\left(\frac{db}{da}\right) \\
&= \left(\frac{d\Omega}{d\lambda}\right)\left(\frac{d\lambda}{da}\right) + \left(\frac{d\Omega}{d\rho}\right)\left(\frac{d\rho}{da}\right) + \left(\frac{d\Omega}{d\beta}\right)\left(\frac{d\beta}{da}\right) + \left(\frac{d\Omega}{d\lambda_1}\right)\left(\frac{d\lambda_1}{da}\right) + \left(\frac{d\Omega}{d\rho_1}\right)\left(\frac{d\rho_1}{da}\right) + \left(\frac{d\Omega}{d\beta_1}\right)\left(\frac{d\beta_1}{da}\right) \\
&\left(\frac{d\Omega}{dl}\right)\left(\frac{dl}{dc}\right) + \left(\frac{d\Omega}{dr}\right)\left(\frac{dr}{dc}\right) + \left(\frac{d\Omega}{db}\right)\left(\frac{db}{dc}\right) \\
&= \left(\frac{d\Omega}{d\lambda}\right)\left(\frac{d\lambda}{dc}\right) + \left(\frac{d\Omega}{d\rho}\right)\left(\frac{d\rho}{dc}\right) + \left(\frac{d\Omega}{d\beta}\right)\left(\frac{d\beta}{dc}\right) + \left(\frac{d\Omega}{d\lambda_1}\right)\left(\frac{d\lambda_1}{dc}\right) + \left(\frac{d\Omega}{d\rho_1}\right)\left(\frac{d\rho_1}{dc}\right) + \left(\frac{d\Omega}{d\beta_1}\right)\left(\frac{d\beta_1}{dc}\right) \\
&\left(\frac{d\Omega}{dl}\right)\left(\frac{dl}{de}\right) + \left(\frac{d\Omega}{dr}\right)\left(\frac{dr}{de}\right) + \left(\frac{d\Omega}{db}\right)\left(\frac{db}{de}\right) \\
&= \left(\frac{d\Omega}{d\lambda}\right)\left(\frac{d\lambda}{de}\right) + \left(\frac{d\Omega}{d\rho}\right)\left(\frac{d\rho}{de}\right) + \left(\frac{d\Omega}{d\beta}\right)\left(\frac{d\beta}{de}\right) + \left(\frac{d\Omega}{d\lambda_1}\right)\left(\frac{d\lambda_1}{de}\right) + \left(\frac{d\Omega}{d\rho_1}\right)\left(\frac{d\rho_1}{de}\right) + \left(\frac{d\Omega}{d\beta_1}\right)\left(\frac{d\beta_1}{de}\right) \\
&\left(\frac{d\Omega}{dl}\right)\left(\frac{dl}{df}\right) + \left(\frac{d\Omega}{db}\right)\left(\frac{db}{df}\right) = \left(\frac{d\Omega}{d\lambda}\right)\left(\frac{d\lambda}{df}\right) + \left(\frac{d\Omega}{d\beta}\right)\left(\frac{d\beta}{df}\right) + \left(\frac{d\Omega}{d\lambda_1}\right)\left(\frac{d\lambda_1}{df}\right) + \left(\frac{d\Omega}{d\beta_1}\right)\left(\frac{d\beta_1}{df}\right) \\
&\left(\frac{d\Omega}{dl}\right)\left(\frac{dl}{dg}\right) + \left(\frac{d\Omega}{db}\right)\left(\frac{db}{dg}\right) = \left(\frac{d\Omega}{d\lambda}\right)\left(\frac{d\lambda}{dg}\right) + \left(\frac{d\Omega}{d\beta}\right)\left(\frac{d\beta}{dg}\right) + \left(\frac{d\Omega}{d\lambda_1}\right)\left(\frac{d\lambda_1}{dg}\right) + \left(\frac{d\Omega}{d\beta_1}\right)\left(\frac{d\beta_1}{dg}\right) \\
&\left(\frac{d\Omega}{dl}\right)\left(\frac{dl}{dh}\right) + \left(\frac{d\Omega}{db}\right)\left(\frac{db}{dh}\right) = \left(\frac{d\Omega}{d\lambda}\right)\left(\frac{d\lambda}{dh}\right) + \left(\frac{d\Omega}{d\beta}\right)\left(\frac{d\beta}{dh}\right) + \left(\frac{d\Omega}{d\lambda_1}\right)\left(\frac{d\lambda_1}{dh}\right) + \left(\frac{d\Omega}{d\beta_1}\right)\left(\frac{d\beta_1}{dh}\right)
\end{aligned}$$

Quibus aequationibus, eliminatione instituta, $\left(\frac{d\Omega}{d\lambda}\right)$, $\left(\frac{d\Omega}{d\rho}\right)$, etc. per $\left(\frac{d\Omega}{dl}\right)$, etc. exprimentur. Quem in finem quotientes $\left(\frac{dl}{da}\right)$, etc. nec non $\left(\frac{d\lambda}{da}\right)$, etc. ante omnia per elementa a, c, e etc. et per l, λ etc. exhibendi sunt, id quod semper fieri potest. Sed quum inde expressiones pro ipsis $\left(\frac{d\Omega}{d\lambda}\right)$, etc. valde implicitae evadere videantur, si nihil negligere propositum est, me calculum hunc silentio praetermittere posse puto.

Nihilominus tamen calculus talis, qualem in praecedentibus exposui, ad aequationes simplicissimas perducit, [si loco elementorum $\lambda, \rho, \beta, \lambda_1, \rho_1, \beta_1$ elementa quaedam alia eliguntur, quibus subsequencia sint dicata.

16.

Sint planetae cujuslibet elementa

- asemiaxis major;
- aeexcentricitas;
- canomaliam media certae determinataeque temporis epochae respondens;

ωarcus inter locum perihelii et intersectionem plani
orbitae cum plano ipsarum xy interceptus;
 θlongitudo hujus intersectionis, sive longitudo nodi
ascendentis orbitae in plano fixo ipsarum xy ;
 iinclinatio ad idem planum;

sint porro uti in artt. praec.

$$\begin{aligned}x &= r \cos b \cos l \\y &= r \cos b \sin l \\z &= r \sin b\end{aligned}$$

qui valores coordinatarum x, y, z , pro integralibus aequationum (1) haberi possunt, si r, l et b motui in sectionibus conicis existenti subiunguntur. Introducendo longitudinem veram in orbita quam v denotabo, trigonometria sphaerica praebet

$$\begin{aligned}\cos b \cos(l-\theta) &= \cos(v-\theta) \\ \cos b \sin(l-\theta) &= \sin(v-\theta) \cos i \\ \sin b &= \sin(v-\theta) \sin i\end{aligned}$$

sive

$$\begin{aligned}\cos b \cos l &= \cos(v-\theta) \cos \theta - \sin(v-\theta) \sin \theta \cos i \\ \cos b \sin l &= \cos(v-\theta) \sin \theta + \sin(v-\theta) \cos \theta \cos i\end{aligned}$$

unde

$$\begin{aligned}x &= r \cos(v-\theta) \cos \theta - r \sin(v-\theta) \sin \theta \cos i \\ y &= r \cos(v-\theta) \sin \theta + r \sin(v-\theta) \cos \theta \cos i \\ z &= r \sin(v-\theta) \sin i\end{aligned}$$

introducta vero anomalia vera f , unde

$$v - \theta = f + \omega$$

praecedentes expressiones ipsarum x, y, z tali modo exhiberi possunt,

$$\begin{aligned}x &= \alpha \xi + \beta \eta \\ y &= \alpha_1 \xi + \beta_1 \eta \\ z &= \alpha_{11} \xi + \beta_{11} \eta\end{aligned}$$

ubi

$$(15) \dots \dots \begin{cases} \alpha = \cos \omega \cos \theta - \sin \omega \sin \theta \cos i \\ \beta = -\sin \omega \cos \theta - \cos \omega \sin \theta \cos i \\ \alpha_1 = \cos \omega \sin \theta + \sin \omega \cos \theta \cos i \\ \beta_1 = -\sin \omega \sin \theta + \cos \omega \cos \theta \cos i \\ \alpha_{11} = \sin \omega \sin i \\ \beta_{11} = \cos \omega \sin i \end{cases}$$

$$\xi = r \cos f, \quad \eta = r \sin f$$

Radius vector vero et anomalia vera elementis a, e, c , taliter constant

$$\begin{aligned}r \cos f &= a \cos u - ae \\ r \sin f &= a \sqrt{1-e^2} \sin u\end{aligned}$$

ubi

$$\begin{aligned}u - e \sin u &= nt + c \\ a^3 n^2 &= k \mu.\end{aligned}$$

17.

Loco elementorum a, e, c , introducā elementa ϕ, l, p , quae ex elementis a, e, c et quantitate indeterminata τ tali modo composita sint, quali f, l, r et $\frac{df}{dt}$ ex iisdem elementis et tempore t constant. Per litteram l ipsi r praefixam denotabo logarithmum hyperbolicum hujus quantitatis. Erunt igitur

$$\begin{aligned}p \cos \phi &= a \cos v - ae \\ p \sin \phi &= a \sqrt{1-e^2} \sin v \\ \phi &= \frac{a^2 n}{p^2} \sqrt{1-e^2}\end{aligned}$$

ubi

$$v - e \sin v = n\tau + c$$

Loco elementorum ω, i, θ introducā motus rotatorios circum axes tres sub angulis rectis se secantes. Animo enim concipi potest, planum orbitae dum a situ in spatio qui tempori t respondet ad situm qui tempori $t + dt$ respondet transgrediatur, motum rotationis circum axem quendam absolvisse, qui motus in motus tres circum axes coordinatarum dissolvi potest. Jam elegantissimo calculo analytico geometrae gallici motus hos rotatorios tractaverunt et deduxerunt, quare hoc loco eosdem graphice demonstrare mihi proposui, id quod nemo, quantum equidem teneo, hucusque fecit. Quae etiam demonstratio in lucem proferat, quid motus hi rotatorii circum axes coordinatarum designent.

18.

Sit in figura planum, cui circulus maximus $ExAy$ respondet planum orbitae tempore t , circulus maximus $FHx'Ay'$ planum orbitae tempore $t+T$. Ad utrumque orbitae planum erigantur plana duo perpendicularia, quae in figura per circulos maximos zx , zy , $z'x'$ et $z'y'$ repraesententur, sintque intersectiones horum planorum axes coordinatarum, ita ut intersectio inter plana zx et yz sit axis ipsarum z , intersectio inter plana $x'y'$ et $z'y'$ sit axis ipsarum y' et sic porro. Denique sit x locus perihelii tempore t et x' idem tempore $t+T$. Jam si orbitae planum per motum rotationis circum axem quendam a situ qui per circulum xy repraesentatur ad situm qui per $x'y'$ designatur transgressum est, dico motum rotatorium circum axem ipsarum z esse differentiam inter arcus Ax et Ax' ab intersectione utriusque orbitae sive ipsarum xy plani ad utrumque perihelii sive axium ipsarum x locum extensos, nec non motum rotatorium circum axem ipsarum y esse differentiam inter arcus Bz et Bz' , denique motum rotatorium circum axem ipsarum x esse differentiam inter arcus Cy et Cy' . Ponam igitur

$$\begin{aligned} Ax - Ax' &= \chi \\ Bz - Bz' &= \psi \\ Cy - Cy' &= \sigma \end{aligned}$$

Sit $DEFG$ planum fixum ad quod orbita refertur, et D punctum initiale longitudinum, eruntque, tempore t , $DE=\theta$, $AEF=i$, $xE=\omega$, et tempore $t+T$, $DF=\theta'$, $AFG=i'$, $x'F=\omega'$. Sed quum generaliter planum orbitae circum axem fixum non gyretur, res ita considerari debet, ac si orbita in quoque temporis infinite parvo momento dt circum axem fixum se gyraisset, et in figura circulus maximus $x'y'$ orbitae situm pro tempore $t+dt$ repraesentare debet. Ideo abeunt χ , ψ , σ in differentialia sua, nempe in $d\chi$, $d\psi$, $d\sigma$, nec non i' in $i+di$, θ' in $\theta+d\theta$ et ω' in $\omega+d\omega$. Jam triangulus sphaericus AEF praebet

$$\cos(\omega'+Ax') = \cos(\omega+Ax) \cos(\theta'-\theta) + \sin(\omega+Ax) \sin(\theta'-\theta) \cos i$$

vel neglectis quantitibus infinite parvis secundi ordinis

$$2 \sin(\omega+Ax) \sin \frac{1}{2}(Ax - Ax' + \omega - \omega') = \sin(\omega+Ax) \cos i d\theta$$

sive

$$d\chi = d\omega + \cos i d\theta$$

porro, denotato angulo inter z' et z , qui inclinatio planorum xy et $x'y'$, sive angulus in A est, per (zz') idem triangulus praebet

$$\sin(zz') \sin(\omega+Ax) = \sin i' \sin(\theta'-\theta)$$

$$\sin(zz') \cos(\omega+Ax) = -\cos i' \sin i + \sin i' \cos i \cos(\theta'-\theta)$$

hinc neglectis quantitibus secundi ordinis, emergit

$$(zz') \sin \omega \cos Ax + (zz') \cos \omega \sin Ax = \sin i d\theta$$

$$(zz') \cos \omega \cos Ax - (zz') \sin \omega \sin Ax = di$$

atque hinc, eliminando

$$(zz') \cos Ax = \sin \omega \sin i d\theta + \cos \omega di$$

$$(zz') \sin Ax = \cos \omega \sin i d\theta - \sin \omega di$$

Quum vero arcus χ , ψ , σ infinite parvi accipiantur, differentia $Bz - Bz'$ usque ad quantitates secundi ordinis arcui xH et differentia $Cy - Cy'$ arcui yK aequalis est, quare

$$d\psi = xH, \quad d\sigma = yK$$

sed

$$xH = (zz') \sin Ax$$

$$yK = (zz') \cos Ax$$

aequationes igitur praecedentes praebent

$$\left. \begin{aligned} d\chi &= d\omega + \cos i d\theta \\ d\psi &= \cos \omega \sin i d\theta - \sin \omega di \\ d\sigma &= \sin \omega \sin i d\theta + \cos \omega di \end{aligned} \right\} \dots\dots(16)$$

quae formulae cum iis, quas theoria nota rotationis corporis cujuscunque suppeditat, congruunt.

19.

Elementa igitur a , c , e , ω , i , θ functiones sunt elementorum φ , $l\varphi$, φ , χ , ψ , σ , et reciproce elementa haec functiones illorum. Coordinatae x , y , z nec non Ω , quae revera functiones illorum elementorum sunt, functiones horum elementorum quoque considerari possunt. Quum vero quantitates α , β , α , etc. e quibus coordinatae compositae sunt, modo ω , i , θ , contineant, illae quantitates functiones modo ipsarum χ , ψ , σ erunt, et ξ atque η , e quibus praeterea x , y , z pendent, quum a , c et e tantum contineant, functiones modo ipsarum φ , $l\varphi$ et φ , erunt, quibus considerationibus calculus ipsarum $(\varphi, l\varphi)$, (φ, φ) etc. secundum formulas (5) notabiliter contrahetur. Quum porro in expressionibus

$$x = \alpha \xi + \beta \eta$$

$$y = \alpha_1 \xi + \beta_1 \eta$$

$$z = \alpha_2 \xi + \beta_2 \eta$$

α , β , α_1 , etc. tempus non contineant, habetur statim

$$x_i = \alpha \xi_i + \beta \eta_i$$

$$y_i = \alpha_1 \xi_i + \beta_1 \eta_i$$

$$z_i = \alpha_2 \xi_i + \beta_2 \eta_i$$

ubi

$$\xi_i = \frac{d\xi}{dt}, \quad \eta_i = \frac{d\eta}{dt}$$

Praeterea habetur

$$\left(\frac{dx}{d\varphi}\right) = \alpha \left(\frac{d\xi}{d\varphi}\right) + \beta \left(\frac{d\eta}{d\varphi}\right), \quad \left(\frac{dx}{d\varphi}\right) = \alpha \left(\frac{d\xi_i}{d\varphi}\right) + \beta \left(\frac{d\eta_i}{d\varphi}\right)$$

$$\left(\frac{dy}{d\varphi}\right) = \alpha_1 \left(\frac{d\xi}{d\varphi}\right) + \beta_1 \left(\frac{d\eta}{d\varphi}\right), \quad \left(\frac{dy}{d\varphi}\right) = \alpha_1 \left(\frac{d\xi_i}{d\varphi}\right) + \beta_1 \left(\frac{d\eta_i}{d\varphi}\right)$$

$$\left(\frac{dz}{d\varphi}\right) = \alpha_2 \left(\frac{d\xi}{d\varphi}\right) + \beta_2 \left(\frac{d\eta}{d\varphi}\right), \quad \left(\frac{dz}{d\varphi}\right) = \alpha_2 \left(\frac{d\xi_i}{d\varphi}\right) + \beta_2 \left(\frac{d\eta_i}{d\varphi}\right)$$

quibus similes sunt quotientes differentiales respectu ipsarum, $l\rho$ et φ . Substituendo has expressiones in aequationibus (5) obtinetur

$$(\varphi, l\rho) = (\alpha^2 + \alpha_i^2 + \alpha_{ii}^2) \left(\frac{d\xi}{d\varphi} \right) \left(\frac{d\xi_i}{dl\rho} \right) + (\alpha\beta + \alpha_i\beta_i + \alpha_{ii}\beta_{ii}) \left[\left(\frac{d\xi}{d\varphi} \right) \left(\frac{d\eta_i}{dl\rho} \right) + \left(\frac{d\eta}{d\varphi} \right) \left(\frac{d\xi_i}{dl\rho} \right) \right] \\ - (\alpha^2 + \alpha_i^2 + \alpha_{ii}^2) \left(\frac{d\xi_i}{d\varphi} \right) \left(\frac{d\xi}{dl\rho} \right) - (\alpha\beta + \alpha_i\beta_i + \alpha_{ii}\beta_{ii}) \left[\left(\frac{d\xi_i}{d\varphi} \right) \left(\frac{d\eta}{dl\rho} \right) + \left(\frac{d\eta_i}{d\varphi} \right) \left(\frac{d\xi}{dl\rho} \right) \right] \\ + (\beta^2 + \beta_i^2 + \beta_{ii}^2) \left(\frac{d\eta}{d\varphi} \right) \left(\frac{d\eta_i}{dl\rho} \right) - (\beta^2 + \beta_i^2 + \beta_{ii}^2) \left(\frac{d\eta_i}{d\varphi} \right) \left(\frac{d\eta}{dl\rho} \right)$$

Sed facile reperitur, inter quantitates α, β, α_i , etc. quales per aequationes (15) dantur, existere aequationes conditionales has

$$\begin{cases} \alpha^2 + \alpha_i^2 + \alpha_{ii}^2 = 1 \\ \beta^2 + \beta_i^2 + \beta_{ii}^2 = 1 \\ \alpha\beta + \alpha_i\beta_i + \alpha_{ii}\beta_{ii} = 0 \end{cases}$$

quibus adjuvantibus, expressio praecedens pro $(\varphi, l\rho)$ abit in

$$(\varphi, l\rho) = \left(\frac{d\xi}{d\varphi} \right) \left(\frac{d\xi_i}{dl\rho} \right) + \left(\frac{d\eta}{d\varphi} \right) \left(\frac{d\eta_i}{dl\rho} \right) - \left(\frac{d\xi_i}{d\varphi} \right) \left(\frac{d\xi}{dl\rho} \right) - \left(\frac{d\eta_i}{d\varphi} \right) \left(\frac{d\eta}{dl\rho} \right)$$

mutatis mutandis ex hac aequatione inveniuntur

$$(\varphi, \varphi_i) = \left(\frac{d\xi}{d\varphi} \right) \left(\frac{d\xi_i}{d\varphi_i} \right) + \left(\frac{d\eta}{d\varphi} \right) \left(\frac{d\eta_i}{d\varphi_i} \right) - \left(\frac{d\xi_i}{d\varphi} \right) \left(\frac{d\xi}{d\varphi_i} \right) - \left(\frac{d\eta_i}{d\varphi} \right) \left(\frac{d\eta}{d\varphi_i} \right) \\ (l\rho, \varphi_i) = \left(\frac{d\xi}{dl\rho} \right) \left(\frac{d\xi_i}{d\varphi_i} \right) + \left(\frac{d\eta}{dl\rho} \right) \left(\frac{d\eta_i}{d\varphi_i} \right) - \left(\frac{d\xi_i}{dl\rho} \right) \left(\frac{d\xi}{d\varphi_i} \right) - \left(\frac{d\eta_i}{dl\rho} \right) \left(\frac{d\eta}{d\varphi_i} \right)$$

Porro habetur

$$\begin{aligned} \left(\frac{dx}{d\chi} \right) &= \left(\frac{d\alpha}{d\chi} \right) \xi + \left(\frac{d\beta}{d\chi} \right) \eta, & \left(\frac{dx_i}{d\chi} \right) &= \left(\frac{d\alpha_i}{d\chi} \right) \xi_i + \left(\frac{d\beta_i}{d\chi} \right) \eta_i \\ \left(\frac{dy}{d\chi} \right) &= \left(\frac{d\alpha_i}{d\chi} \right) \xi + \left(\frac{d\beta_i}{d\chi} \right) \eta, & \left(\frac{dy_i}{d\chi} \right) &= \left(\frac{d\alpha_i}{d\chi} \right) \xi_i + \left(\frac{d\beta_i}{d\chi} \right) \eta_i \\ \left(\frac{dz}{d\chi} \right) &= \left(\frac{d\alpha_{ii}}{d\chi} \right) \xi + \left(\frac{d\beta_{ii}}{d\chi} \right) \eta, & \left(\frac{dz_i}{d\chi} \right) &= \left(\frac{d\alpha_{ii}}{d\chi} \right) \xi_i + \left(\frac{d\beta_{ii}}{d\chi} \right) \eta_i \end{aligned}$$

itaque aequationes (5) praebent

$$(\varphi, \chi) = \left[\alpha \left(\frac{d\alpha}{d\chi} \right) + \alpha_i \left(\frac{d\alpha_i}{d\chi} \right) + \alpha_{ii} \left(\frac{d\alpha_{ii}}{d\chi} \right) \right] \xi_i \left(\frac{d\xi}{d\varphi} \right) + \left[\alpha \left(\frac{d\beta}{d\chi} \right) + \alpha_i \left(\frac{d\beta_i}{d\chi} \right) + \alpha_{ii} \left(\frac{d\beta_{ii}}{d\chi} \right) \right] \eta_i \left(\frac{d\xi}{d\varphi} \right) \\ + \left[\beta \left(\frac{d\beta}{d\chi} \right) + \beta_i \left(\frac{d\beta_i}{d\chi} \right) + \beta_{ii} \left(\frac{d\beta_{ii}}{d\chi} \right) \right] \eta_i \left(\frac{d\eta}{d\varphi} \right) + \left[\beta \left(\frac{d\alpha}{d\chi} \right) + \beta_i \left(\frac{d\alpha_i}{d\chi} \right) + \beta_{ii} \left(\frac{d\alpha_{ii}}{d\chi} \right) \right] \xi_i \left(\frac{d\eta}{d\varphi} \right) \\ - \left[\alpha \left(\frac{d\alpha}{d\chi} \right) + \alpha_i \left(\frac{d\alpha_i}{d\chi} \right) + \alpha_{ii} \left(\frac{d\alpha_{ii}}{d\chi} \right) \right] \xi \left(\frac{d\xi}{d\varphi} \right) - \left[\alpha \left(\frac{d\beta}{d\chi} \right) + \alpha_i \left(\frac{d\beta_i}{d\chi} \right) + \alpha_{ii} \left(\frac{d\beta_{ii}}{d\chi} \right) \right] \eta \left(\frac{d\xi}{d\varphi} \right) \\ - \left[\beta \left(\frac{d\beta}{d\chi} \right) + \beta_i \left(\frac{d\beta_i}{d\chi} \right) + \beta_{ii} \left(\frac{d\beta_{ii}}{d\chi} \right) \right] \eta \left(\frac{d\eta}{d\varphi} \right) - \left[\beta \left(\frac{d\alpha}{d\chi} \right) + \beta_i \left(\frac{d\alpha_i}{d\chi} \right) + \beta_{ii} \left(\frac{d\alpha_{ii}}{d\chi} \right) \right] \xi \left(\frac{d\eta}{d\varphi} \right)$$

sed aequationes conditionales supra allatae subministrant

$$\begin{aligned} \alpha d\alpha + \alpha_i d\alpha_i + \alpha_{ii} d\alpha_{ii} &= 0, & \beta d\beta + \beta_i d\beta_i + \beta_{ii} d\beta_{ii} &= 0 \\ \alpha d\beta + \alpha_i d\beta_i + \alpha_{ii} d\beta_{ii} &= -\beta d\alpha - \beta_i d\alpha_i - \beta_{ii} d\alpha_{ii} \end{aligned}$$

quare

$$(\varphi, \chi) = \left[\alpha \left(\frac{d\beta}{d\chi} \right) + \alpha_i \left(\frac{d\beta_i}{d\chi} \right) + \alpha_{ii} \left(\frac{d\beta_{ii}}{d\chi} \right) \right] \left\{ \eta_i \left(\frac{d\xi}{d\varphi} \right) - \xi_i \left(\frac{d\eta}{d\varphi} \right) - \eta \left(\frac{d\xi_i}{d\varphi} \right) + \xi \left(\frac{d\eta_i}{d\varphi} \right) \right\}$$

sive

$$(\varphi, \chi) = \left\{ \alpha \left(\frac{d\beta}{d\chi} \right) + \alpha_i \left(\frac{d\beta_i}{d\chi} \right) + \alpha_{ii} \left(\frac{d\beta_{ii}}{d\chi} \right) \right\} \frac{d \cdot (\xi\eta_i - \xi_i\eta)}{d\varphi}$$

hinc mutatis mutandis inveniuntur

$$\begin{aligned}(\varphi, \psi) &= \left\{ \alpha \left(\frac{d\beta}{d\psi} \right) + \alpha_1 \left(\frac{d\beta_1}{d\psi} \right) + \alpha_{11} \left(\frac{d\beta_{11}}{d\psi} \right) \right\} \frac{d \cdot (\xi_{\eta_1} - \xi_{\eta})}{d\varphi} \\(\varphi, \sigma) &= \left\{ \alpha \left(\frac{d\beta}{d\sigma} \right) + \alpha_1 \left(\frac{d\beta_1}{d\sigma} \right) + \alpha_{11} \left(\frac{d\beta_{11}}{d\sigma} \right) \right\} \frac{d \cdot (\xi_{\eta_1} - \xi_{\eta})}{d\varphi} \\(l\rho, \chi) &= \left\{ \alpha \left(\frac{d\beta}{d\chi} \right) + \alpha_1 \left(\frac{d\beta_1}{d\chi} \right) + \alpha_{11} \left(\frac{d\beta_{11}}{d\chi} \right) \right\} \frac{d \cdot (\xi_{\eta_1} - \xi_{\eta})}{dl\rho} \\(l\rho, \psi) &= \left\{ \alpha \left(\frac{d\beta}{d\psi} \right) + \alpha_1 \left(\frac{d\beta_1}{d\psi} \right) + \alpha_{11} \left(\frac{d\beta_{11}}{d\psi} \right) \right\} \frac{d \cdot (\xi_{\eta_1} - \xi_{\eta})}{dl\rho}\end{aligned}$$

$$\begin{aligned}(\chi, \psi) &= (\xi_{\eta_1} - \xi_{\eta}) \left\{ \left(\frac{d\alpha}{d\chi} \right) \left(\frac{d\beta}{d\psi} \right) + \left(\frac{d\alpha_1}{d\chi} \right) \left(\frac{d\beta_1}{d\psi} \right) + \left(\frac{d\alpha_{11}}{d\chi} \right) \left(\frac{d\beta_{11}}{d\psi} \right) - \left(\frac{d\alpha}{d\psi} \right) \left(\frac{d\beta}{d\chi} \right) - \left(\frac{d\alpha_1}{d\psi} \right) \left(\frac{d\beta_1}{d\chi} \right) - \left(\frac{d\alpha_{11}}{d\psi} \right) \left(\frac{d\beta_{11}}{d\chi} \right) \right\} \\(\chi, \sigma) &= (\xi_{\eta_1} - \xi_{\eta}) \left\{ \left(\frac{d\alpha}{d\chi} \right) \left(\frac{d\beta}{d\sigma} \right) + \left(\frac{d\alpha_1}{d\chi} \right) \left(\frac{d\beta_1}{d\sigma} \right) + \left(\frac{d\alpha_{11}}{d\chi} \right) \left(\frac{d\beta_{11}}{d\sigma} \right) - \left(\frac{d\alpha}{d\sigma} \right) \left(\frac{d\beta}{d\chi} \right) - \left(\frac{d\alpha_1}{d\sigma} \right) \left(\frac{d\beta_1}{d\chi} \right) - \left(\frac{d\alpha_{11}}{d\sigma} \right) \left(\frac{d\beta_{11}}{d\chi} \right) \right\} \\(\psi, \sigma) &= (\xi_{\eta_1} - \xi_{\eta}) \left\{ \left(\frac{d\alpha}{d\psi} \right) \left(\frac{d\beta}{d\sigma} \right) + \left(\frac{d\alpha_1}{d\psi} \right) \left(\frac{d\beta_1}{d\sigma} \right) + \left(\frac{d\alpha_{11}}{d\psi} \right) \left(\frac{d\beta_{11}}{d\sigma} \right) - \left(\frac{d\alpha}{d\sigma} \right) \left(\frac{d\beta}{d\psi} \right) - \left(\frac{d\alpha_1}{d\sigma} \right) \left(\frac{d\beta_1}{d\psi} \right) - \left(\frac{d\alpha_{11}}{d\sigma} \right) \left(\frac{d\beta_{11}}{d\psi} \right) \right\}\end{aligned}$$

Quae quindecim quantitates omnes sunt quibus opus est, quoties numerus elementorum sex est.

20.

Ad quotientes differentiales quos formulae art. praec. requirunt determinandos, inserviunt formulae in artt. 16 et 17 exhibitae. Primo praebent

$$\xi = r \cos f, \quad \eta = r \sin f$$

has

$$(18) \dots \dots \dots \begin{cases} d\xi = \left(\frac{d\xi}{df} \right) df + \left(\frac{d\xi}{dlr} \right) dlr \\ d\eta = \left(\frac{d\eta}{df} \right) df + \left(\frac{d\eta}{dlr} \right) dlr \end{cases}$$

ubi

$$\begin{aligned} \left(\frac{d\xi}{df} \right) &= -r \sin f, & \left(\frac{d\xi}{dlr} \right) &= r \cos f \\ \left(\frac{d\eta}{df} \right) &= r \cos f, & \left(\frac{d\eta}{dlr} \right) &= r \sin f \end{aligned}$$

Sed quum in evolutis quantitatibus $(\varphi, l\rho)$, (φ, φ_1) , etc. tempus evanescere debeat, ad libitum pro t valorem quemlibet accipere nobis licet, quare, ut earum computatio tantum, quantum fieri possit, contrahatur, ponam $t = \tau$, unde emergunt

$$(19) \dots \dots \dots \begin{cases} \left(\frac{d\xi}{d\varphi} \right) = -\rho \sin \varphi, & \left(\frac{d\xi}{dlr} \right) = \rho \cos \varphi \\ \left(\frac{d\eta}{d\varphi} \right) = \rho \cos \varphi, & \left(\frac{d\eta}{dlr} \right) = \rho \sin \varphi \end{cases}$$

Jam considerando f , lr atque quotientem differentialem ipsius f respectu temporis, quem f , nominabo, tanquam functiones ipsarum φ , $l\rho$, et φ_1 , habemus

$$\begin{aligned} df &= \left(\frac{df}{d\varphi} \right) d\varphi + \left(\frac{df}{dl\rho} \right) dl\rho + \left(\frac{df}{d\varphi_1} \right) d\varphi_1 \\ dlr &= \left(\frac{dlr}{d\varphi} \right) d\varphi + \left(\frac{dlr}{dl\rho} \right) dl\rho + \left(\frac{dlr}{d\varphi_1} \right) d\varphi_1 \\ df_1 &= \left(\frac{df_1}{d\varphi} \right) d\varphi + \left(\frac{df_1}{dl\rho} \right) dl\rho + \left(\frac{df_1}{d\varphi_1} \right) d\varphi_1 \end{aligned}$$

$$\begin{aligned} (l\rho, \sigma) &= \left\{ \alpha \left(\frac{d\beta}{d\sigma} \right) + \alpha_1 \left(\frac{d\beta_1}{d\sigma} \right) + \alpha_{11} \left(\frac{d\beta_{11}}{d\sigma} \right) \right\} \frac{d \cdot (\xi_{\eta_1} - \xi_{\eta})}{dl\rho} \\ (\varphi_1, \chi) &= \left\{ \alpha \left(\frac{d\beta}{d\chi} \right) + \alpha_1 \left(\frac{d\beta_1}{d\chi} \right) + \alpha_{11} \left(\frac{d\beta_{11}}{d\chi} \right) \right\} \frac{d \cdot (\xi_{\eta_1} - \xi_{\eta})}{d\varphi_1} \\ (\varphi_1, \psi) &= \left\{ \alpha \left(\frac{d\beta}{d\psi} \right) + \alpha_1 \left(\frac{d\beta_1}{d\psi} \right) + \alpha_{11} \left(\frac{d\beta_{11}}{d\psi} \right) \right\} \frac{d \cdot (\xi_{\eta_1} - \xi_{\eta})}{d\varphi_1} \\ (\varphi_1, \sigma) &= \left\{ \alpha \left(\frac{d\beta}{d\sigma} \right) + \alpha_1 \left(\frac{d\beta_1}{d\sigma} \right) + \alpha_{11} \left(\frac{d\beta_{11}}{d\sigma} \right) \right\} \frac{d \cdot (\xi_{\eta_1} - \xi_{\eta})}{d\varphi_1} \end{aligned}$$

Denique aequationes (5) et (17) immediate praebent

quae substitutae in (18) praebent

$$\begin{aligned} \left(\frac{d\xi}{d\varphi} \right) &= \left(\frac{d\xi}{df} \right) \left(\frac{df}{d\varphi} \right) + \left(\frac{d\xi}{dlr} \right) \left(\frac{dlr}{d\varphi} \right) \\ \left(\frac{d\xi}{dl\rho} \right) &= \left(\frac{d\xi}{df} \right) \left(\frac{df}{dl\rho} \right) + \left(\frac{d\xi}{dlr} \right) \left(\frac{dlr}{dl\rho} \right) \\ \left(\frac{d\xi}{d\varphi_1} \right) &= \left(\frac{d\xi}{df} \right) \left(\frac{df}{d\varphi_1} \right) + \left(\frac{d\xi}{dlr} \right) \left(\frac{dlr}{d\varphi_1} \right) \\ \left(\frac{d\eta}{d\varphi} \right) &= \left(\frac{d\eta}{df} \right) \left(\frac{df}{d\varphi} \right) + \left(\frac{d\eta}{dlr} \right) \left(\frac{dlr}{d\varphi} \right) \\ \left(\frac{d\eta}{dl\rho} \right) &= \left(\frac{d\eta}{df} \right) \left(\frac{df}{dl\rho} \right) + \left(\frac{d\eta}{dlr} \right) \left(\frac{dlr}{dl\rho} \right) \\ \left(\frac{d\eta}{d\varphi_1} \right) &= \left(\frac{d\eta}{df} \right) \left(\frac{df}{d\varphi_1} \right) + \left(\frac{d\eta}{dlr} \right) \left(\frac{dlr}{d\varphi_1} \right) \end{aligned} \dots \dots (20)$$

Sed aequationes

$$\begin{aligned} r \cos f &= a \cos u - ae \\ r \sin f &= a \sqrt{1-e^2} \cdot \sin u \\ f_1 &= \frac{a^2 n}{r^2} \sqrt{1-e^2} \\ nt + c &= u - e \sin u \\ a^2 n^2 &= k \mu \end{aligned} \dots \dots (21)$$

exhibent r , f et f_1 tanquam functiones ipsarum a , c et e , et praebent differentiando

$$\begin{aligned} df &= \left(\frac{df}{da} \right) da + \left(\frac{df}{dc} \right) dc + \left(\frac{df}{de} \right) de \\ dlr &= \left(\frac{dlr}{da} \right) da + \left(\frac{dlr}{dc} \right) dc + \left(\frac{dlr}{de} \right) de \\ df_1 &= \left(\frac{df_1}{da} \right) da + \left(\frac{df_1}{dc} \right) dc + \left(\frac{df_1}{de} \right) de \end{aligned} \dots \dots (22)$$

ex aequationibus vero

$$(23) \dots \left\{ \begin{array}{l} \rho \cos \varphi = a \cos v - a e \\ \rho \sin \varphi = a \sqrt{1-e^2} \cdot \sin v \\ \varphi = \frac{a^2 n}{\rho^2} \sqrt{1-e^2} \\ n \tau + c = v - e \sin v \\ a^3 n^2 = k \mu \end{array} \right.$$

prodibunt a , c et e tanquam functiones ipsarum φ , ρ et φ , unde differentiando et eliminando evadent

$$(24) \dots \left\{ \begin{array}{l} df = \left\{ \left(\frac{df}{da} \right) \left(\frac{da}{d\varphi} \right) + \left(\frac{df}{dc} \right) \left(\frac{dc}{d\varphi} \right) + \left(\frac{df}{de} \right) \left(\frac{de}{d\varphi} \right) \right\} d\varphi + \left\{ \left(\frac{df}{da} \right) \left(\frac{da}{dl\rho} \right) + \left(\frac{df}{dc} \right) \left(\frac{dc}{dl\rho} \right) + \left(\frac{df}{de} \right) \left(\frac{de}{dl\rho} \right) \right\} dl\rho \\ + \left\{ \left(\frac{df}{da} \right) \left(\frac{da}{d\varphi} \right) + \left(\frac{df}{dc} \right) \left(\frac{dc}{d\varphi} \right) + \left(\frac{df}{de} \right) \left(\frac{de}{d\varphi} \right) \right\} d\varphi, \end{array} \right.$$

et similes aequationes pro $d\tau$ et df .

Quum vero, posito $t = \tau$, aequationes (21), e quibus quotientes $\left(\frac{df}{da} \right)$, $\left(\frac{df}{dc} \right)$ etc. derivandi sunt, cum aequationibus (23) quae quotientes $\left(\frac{da}{d\varphi} \right)$, $\left(\frac{dc}{d\varphi} \right)$ etc. praebeant, plane congruant, necesse est habere aequationes identicas, si ad laevam in (24) resp. $d\varphi$, $dl\rho$ et $d\varphi$, loco df , $d\tau$ et df , posuerimus, quare

$$\begin{aligned} \left(\frac{df}{da} \right) \left(\frac{da}{d\varphi} \right) + \left(\frac{df}{dc} \right) \left(\frac{dc}{d\varphi} \right) + \left(\frac{df}{de} \right) \left(\frac{de}{d\varphi} \right) &= 1 \\ \left(\frac{dl\rho}{da} \right) \left(\frac{da}{dl\rho} \right) + \left(\frac{dl\rho}{dc} \right) \left(\frac{dc}{dl\rho} \right) + \left(\frac{dl\rho}{de} \right) \left(\frac{de}{dl\rho} \right) &= 1 \end{aligned}$$

$$(26) \dots \left\{ \begin{array}{l} \left(\frac{d\xi}{d\varphi} \right) = -\rho \sin \varphi; \quad \left(\frac{d\xi}{dl\rho} \right) = \rho \cos \varphi; \quad \left(\frac{d\xi}{d\varphi} \right) = 0 \\ \left(\frac{d\eta}{d\varphi} \right) = \rho \cos \varphi; \quad \left(\frac{d\eta}{dl\rho} \right) = \rho \sin \varphi; \quad \left(\frac{d\eta}{d\varphi} \right) = 0 \end{array} \right.$$

21.

Differentiatis aequationibus

$$\xi = r \cos f, \quad \eta = r \sin f$$

secundum tempus, obtinetur

$$\xi = \left(\frac{dr}{dt} \right) \cos f - \left(\frac{df}{dt} \right) r \sin f$$

$$\eta = \left(\frac{dr}{dt} \right) \sin f + \left(\frac{df}{dt} \right) r \cos f$$

quae, quum sint

$$\left(\frac{dr}{dt} \right) = a n e \frac{\sin f}{\sqrt{1-e^2}}; \quad \left(\frac{df}{dt} \right) = \frac{a^2 n}{r^2} \sqrt{1-e^2}$$

abeunt in

$$\begin{aligned} \xi &= -\frac{a n \sin f}{\sqrt{1-e^2}} \\ \eta &= \frac{a n e}{\sqrt{1-e^2}} + \frac{a n \cos f}{\sqrt{1-e^2}} \end{aligned}$$

$$(27) \dots d\xi = -\frac{a n}{\sqrt{1-e^2}} \cos \varphi df + 2 \frac{a n}{\sqrt{1-e^2}} \sin \varphi dl\rho + \frac{\rho^2 \sin \varphi}{a(1-e^2)} df,$$

$$da = \left(\frac{da}{d\varphi} \right) d\varphi + \left(\frac{da}{dl\rho} \right) dl\rho + \left(\frac{da}{d\varphi} \right) d\varphi,$$

$$dc = \left(\frac{dc}{d\varphi} \right) d\varphi + \left(\frac{dc}{dl\rho} \right) dl\rho + \left(\frac{dc}{d\varphi} \right) d\varphi,$$

$$de = \left(\frac{de}{d\varphi} \right) d\varphi + \left(\frac{de}{dl\rho} \right) dl\rho + \left(\frac{de}{d\varphi} \right) d\varphi,$$

quibus in (22) substitutis, elicitur

$$\begin{aligned} \left(\frac{df}{da} \right) \left(\frac{da}{d\varphi} \right) + \left(\frac{df}{dc} \right) \left(\frac{dc}{d\varphi} \right) + \left(\frac{df}{de} \right) \left(\frac{de}{d\varphi} \right) &= 1 \\ \left(\frac{df}{da} \right) \left(\frac{da}{dl\rho} \right) + \left(\frac{df}{dc} \right) \left(\frac{dc}{dl\rho} \right) + \left(\frac{df}{de} \right) \left(\frac{de}{dl\rho} \right) &= 0, \text{ etc.} \end{aligned}$$

unde sequitur esse, facto $t = \tau$,

$$\left. \begin{aligned} \left(\frac{df}{d\varphi} \right) &= 1; \quad \left(\frac{df}{dl\rho} \right) = 0; \quad \left(\frac{df}{d\varphi} \right) = 0 \\ \left(\frac{d\tau}{d\varphi} \right) &= 0; \quad \left(\frac{d\tau}{dl\rho} \right) = 1; \quad \left(\frac{d\tau}{d\varphi} \right) = 0 \\ \left(\frac{df_i}{d\varphi} \right) &= 0; \quad \left(\frac{df_i}{dl\rho} \right) = 0; \quad \left(\frac{df_i}{d\varphi} \right) = 1 \end{aligned} \right\} \dots (25)$$

Itaque his valoribus nec non aequationibus (19) in (20) substitutis, emergunt

aequatio vero

$$f_i = \frac{a^2 n}{r^2} \sqrt{1-e^2}$$

praebet

$$\frac{1}{\sqrt{1-e^2}} = \frac{a^2 n}{r^2 f_i}$$

unde

$$\xi_i = -\frac{a^3 n^2 \sin f}{r^2 f_i} = -\frac{k \mu \sin f}{r^2 f_i}$$

Quantitas igitur ξ_i est functio ipsarum f , r et f_i , absque elementis a , c et e , quare statim reperitur

$$d\xi_i = -\frac{k \mu \cos f}{r^2 f_i} df + 2 \frac{k \mu \sin f}{r^2 f_i} dl\rho + \frac{k \mu \sin f}{r^2 f_i^2} df,$$

hinc, eliminatis $k \mu$ et f_i , scriptaque τ loco t , evadit

quae aequatio, ratione habita aequationum (25), differentialia partialia ipsius ξ , respectu φ , $l\rho$ et φ , immediate praebet. Quum in valore ipsius η , supra allato elementa a et e haud aequae facile eliminare liceat, ad ejus differentialia partialia respectu φ , $l\rho$ et φ , obtinenda aliam viam ingrediamur necesse est. Si ex valoribus supra datis ipsarum ξ , ξ , η et η , computatur quantitas $\xi\eta - \xi\xi$, qua praeterea opus est, facile invenitur

$$\xi\eta - \xi\xi = a^2 n \sqrt{1-e^2} = r^2 f,$$

$$(28) \dots \dots \dots d\eta = \frac{an \sin \varphi (e - \cos \varphi)}{\cos \varphi \sqrt{1-e^2}} df + \frac{an(3+e \cos \varphi - 2 \cos^2 \varphi)}{\cos \varphi \sqrt{1-e^2}} dl\rho + \frac{\rho^2(2+e \cos \varphi - \cos^2 \varphi)}{a \cos \varphi (1-e^2)} df,$$

Ope aequationum igitur (25) adipiscimur ex (27) et (28)

$$(29) \dots \dots \dots \begin{cases} \left(\frac{d\xi}{d\varphi}\right) = -\frac{an}{\sqrt{1-e^2}} \cos \varphi, & \left(\frac{d\eta}{d\varphi}\right) = \frac{an}{\sqrt{1-e^2}} \cdot \frac{\sin \varphi (e - \cos \varphi)}{\cos \varphi} \\ \left(\frac{d\xi}{dl\rho}\right) = 2 \frac{an}{\sqrt{1-e^2}} \sin \varphi, & \left(\frac{d\eta}{dl\rho}\right) = \frac{an}{\sqrt{1-e^2}} \cdot \frac{3+e \cos \varphi - 2 \cos^2 \varphi}{\cos \varphi} \\ \left(\frac{d\xi}{d\varphi}\right) = \frac{\rho^2 \sin \varphi}{a(1-e^2)} & \left(\frac{d\eta}{d\varphi}\right) = \frac{\rho^2(2+e \cos \varphi - \cos^2 \varphi)}{a \cos \varphi (1-e^2)} \end{cases}$$

Jam omnia praesto sunt, quae ad computationem ipsarum $(\varphi, l\rho)$, (φ, φ) et $(l\rho, \varphi)$ requiruntur, atque substitutis expressionibus (26) et (29) in formulis art. 19, post facilem reductionem, inveniuntur

$$\begin{aligned} (\varphi, l\rho) &= 2a^2 n \sqrt{1-e^2} - \frac{an}{\sqrt{1-e^2}} \cdot \frac{e\rho}{\cos \varphi} \\ (\varphi, \varphi) &= \rho^2 \\ (l\rho, \varphi) &= \frac{\rho^3 \sin \varphi (2+e \cos \varphi)}{a \cos \varphi (1-e^2)} \end{aligned}$$

22.

Ad quantitates (φ, χ) , etc. obtinendas differentientur expressiones (15) unde

$$\begin{aligned} d\alpha &= \beta d\omega - \alpha d\theta + \alpha_{,,} \sin \theta di \\ d\beta &= -\alpha d\omega - \beta d\theta + \beta_{,,} \sin \theta di \\ d\alpha_{,,} &= \beta d\omega + \alpha d\theta - \alpha_{,,} \cos \theta di \\ d\beta_{,,} &= -\alpha d\omega + \beta d\theta - \beta_{,,} \cos \theta di \\ d\alpha_{,,} &= \beta_{,,} d\omega + \sin \omega \cos i di \\ d\beta_{,,} &= -\alpha_{,,} d\omega + \cos \omega \cos i di \end{aligned}$$

Sed aequationes (16) subministrant

$$\begin{aligned} d\omega &= d\chi - \cos i \frac{\cos \omega}{\sin i} d\psi - \cos i \frac{\sin \omega}{\sin i} d\sigma \\ d\theta &= \frac{\cos \omega}{\sin i} d\psi + \frac{\sin \omega}{\sin i} d\sigma \\ di &= -\sin \omega d\psi + \cos \omega d\sigma \end{aligned}$$

quae quantitas igitur functionem quoque ipsarum r et f , absque elementis a , c et e se praestat. Ideo differentiata praebet

$$d\eta = \frac{2r^2 f}{\xi} dl\rho + \frac{r^2}{\xi} df - \frac{\eta}{\xi} d\xi + \frac{\eta}{\xi} d\xi + \frac{\xi}{\xi} d\eta$$

quae, substitutis pro ξ , η , ξ , η , $d\xi$, $d\eta$ et $d\xi$, valoribus earum supra datis, mutatoque t in τ , facili opera *) abit in hanc

unde

$$\begin{aligned} \left(\frac{d\alpha}{d\chi}\right) &= \beta, & \left(\frac{d\alpha}{d\psi}\right) &= -\sin \theta \sin i, & \left(\frac{d\alpha}{d\sigma}\right) &= 0 \\ \left(\frac{d\beta}{d\chi}\right) &= -\alpha, & \left(\frac{d\beta}{d\psi}\right) &= 0, & \left(\frac{d\beta}{d\sigma}\right) &= \sin \theta \sin i \\ \left(\frac{d\alpha_{,,}}{d\chi}\right) &= \beta_{,,}, & \left(\frac{d\alpha_{,,}}{d\psi}\right) &= \cos \theta \sin i, & \left(\frac{d\alpha_{,,}}{d\sigma}\right) &= 0 \\ \left(\frac{d\beta_{,,}}{d\chi}\right) &= -\alpha_{,,}, & \left(\frac{d\beta_{,,}}{d\psi}\right) &= 0, & \left(\frac{d\beta_{,,}}{d\sigma}\right) &= -\cos \theta \sin i \\ \left(\frac{d\alpha_{,,}}{d\chi}\right) &= \beta_{,,}, & \left(\frac{d\alpha_{,,}}{d\psi}\right) &= -\cos i, & \left(\frac{d\alpha_{,,}}{d\sigma}\right) &= 0 \\ \left(\frac{d\beta_{,,}}{d\chi}\right) &= -\alpha_{,,}, & \left(\frac{d\beta_{,,}}{d\psi}\right) &= 0, & \left(\frac{d\beta_{,,}}{d\sigma}\right) &= \cos i \end{aligned}$$

quare

$$\begin{aligned} \alpha \left(\frac{d\beta}{d\chi}\right) + \alpha_{,,} \left(\frac{d\beta_{,,}}{d\chi}\right) + \alpha_{,,} \left(\frac{d\beta_{,,}}{d\chi}\right) &= -1 \\ \alpha \left(\frac{d\beta}{d\psi}\right) + \alpha_{,,} \left(\frac{d\beta_{,,}}{d\psi}\right) + \alpha_{,,} \left(\frac{d\beta_{,,}}{d\psi}\right) &= 0 \\ \alpha \left(\frac{d\beta}{d\sigma}\right) + \alpha_{,,} \left(\frac{d\beta_{,,}}{d\sigma}\right) + \alpha_{,,} \left(\frac{d\beta_{,,}}{d\sigma}\right) &= 0 \end{aligned}$$

*) Semel animadverto, reductiones algebraicas omnes quibus in hac commentatione opus est, ope formularum harum

$$\begin{aligned} 1 &= \frac{\rho}{a} \cdot \frac{1}{1-e^2} + \frac{\rho}{a} \cdot \frac{e \cos \varphi}{1-e^2} \\ 1 &= \frac{r}{a} \cdot \frac{1}{1-e^2} + \frac{r}{a} \cdot \frac{e \cos f}{1-e^2} \end{aligned}$$

quae ex notis expressionibus ipsarum ρ et r facile derivantur, percommode absolvi posse.

In art. praeced. invenimus

$$\xi_{\eta} - \xi_{\eta} = r^2 f,$$

unde

$$\frac{d(\xi_{\eta} - \xi_{\eta})}{d\varphi} = 0$$

$$\frac{d(\xi_{\eta} - \xi_{\eta})}{d\varphi} = 2\rho^2 \varphi = 2a^2 n \sqrt{1-e^2}$$

$$\frac{d(\xi_{\eta} - \xi_{\eta})}{d\varphi} = \rho^2$$

Aequationes igitur praecedentes una cum formulis art. 19 praebent

$$a^3 n^2 \left(\frac{d\Omega}{d\varphi} \right) dt = \left(2a^2 n \sqrt{1-e^2} - \frac{an}{\sqrt{1-e^2}} \cdot \frac{e\rho}{\cos\varphi} \right) d\varphi + \rho^2 d\varphi,$$

$$a^3 n^2 \left(\frac{d\Omega}{d\varphi} \right) dt = - \left(2a^2 n \sqrt{1-e^2} - \frac{an}{\sqrt{1-e^2}} \cdot \frac{e\rho}{\cos\varphi} \right) d\varphi + \frac{\rho^3 \sin\varphi (2+e \cos\varphi)}{a \cos\varphi (1-e^2)} d\varphi - 2a^2 n \sqrt{1-e^2} d\chi$$

$$a^3 n^2 \left(\frac{d\Omega}{d\varphi} \right) dt = -\rho^2 d\varphi - \frac{\rho^3 \sin\varphi (2+e \cos\varphi)}{a \cos\varphi (1-e^2)} d\varphi - \rho^2 d\chi$$

$$a^3 n^2 \left(\frac{d\Omega}{d\chi} \right) dt = 2a^2 n \sqrt{1-e^2} d\varphi + \rho^2 d\varphi,$$

$$a^3 n^2 \left(\frac{d\Omega}{d\psi} \right) dt = -a^2 n \sqrt{1-e^2} d\sigma$$

$$a^3 n^2 \left(\frac{d\Omega}{d\sigma} \right) dt = a^2 n \sqrt{1-e^2} d\psi$$

Hinc eliminatione perfacili, et ponendo $\rho \left(\frac{d\Omega}{d\varphi} \right)$ loco $\left(\frac{d\Omega}{d\varphi} \right)$ obtinentur

$$d\varphi = \frac{a^2 n \cos\varphi \sqrt{1-e^2}}{e} \left(\frac{d\Omega}{d\varphi} \right) dt - \frac{an \sin\varphi (2+e \cos\varphi)}{e \sqrt{1-e^2}} \left(\frac{d\Omega}{d\chi} \right) dt - 2 \frac{a^4 n^2 (1-e^2) \cos\varphi}{e \rho^3} \left(\frac{d\Omega}{d\varphi} \right) dt$$

$$d\varphi = \frac{a^2 n \cos\varphi \sqrt{1-e^2}}{e \rho} \left\{ \left(\frac{d\Omega}{d\chi} \right) - \left(\frac{d\Omega}{d\varphi} \right) \right\} dt$$

$$d\varphi = 2 \frac{a^4 n^2 (1-e^2) \cos\varphi}{e \rho^3} \left(\frac{d\Omega}{d\varphi} \right) dt + \left\{ \frac{a^3 n^2}{\rho^2} - 2 \frac{a^4 n^2 (1-e^2) \cos\varphi}{e \rho^3} \right\} \left(\frac{d\Omega}{d\chi} \right) dt$$

$$d\chi = \frac{an \sin\varphi (2+e \cos\varphi)}{e \sqrt{1-e^2}} \left(\frac{d\Omega}{d\varphi} \right) dt - \frac{a^2 n \cos\varphi \sqrt{1-e^2}}{e} \left(\frac{d\Omega}{d\varphi} \right) dt + \left\{ 2 \frac{a^4 n^2 (1-e^2) \cos\varphi}{e \rho^3} - \frac{a^3 n^2}{\rho^2} \right\} \left(\frac{d\Omega}{d\varphi} \right) dt$$

$$d\psi = \frac{an}{\sqrt{1-e^2}} \left(\frac{d\Omega}{d\sigma} \right) dt$$

$$d\sigma = - \frac{an}{\sqrt{1-e^2}} \left(\frac{d\Omega}{d\psi} \right) dt$$

Quibus aequationibus integratis, secundum theorema art. 9 valor verus perturbatus anomaliae verae obtinetur, si in valore ipsius φ , et valor verus perturbatus logarithmi hyperbolici radii vectoris, si in valore ipsius φ mutatur τ in ϵ . Valores ipsarum χ , ψ et σ ex posterioribus tribus harum aequationum prodituri situm orbitae in spatio determinant, aequatio vero pro φ , ad efficiendas integrationes ceterarum aequationum tantum inservit.

24.

Ad locum planetae in orbita determinandum apta est quantitas λ , quae ita cum quantitativis φ et χ juncta sit, ut habeatur

$$(\varphi, \chi) = 0, (\varphi, \psi) = 0, (\varphi, \sigma) = 0, (\varphi, \chi) = -2a^2 n \sqrt{1-e^2}$$

$$(\varphi, \psi) = 0, (\varphi, \sigma) = 0, (\varphi, \chi) = -\rho^2$$

$$(\varphi, \psi) = 0, (\varphi, \sigma) = 0$$

denique formulae art. 19, si quotientes differentiales in hoc articulo evoluti substituti fuerint, subministrabunt

$$(\chi, \psi) = 0, (\chi, \sigma) = 0, (\psi, \sigma) = -a^2 n \sqrt{1-e^2}.$$

23.

Ope valorum ipsarum (φ, φ) , (φ, φ) etc. etc. in artt. praec. inventorum, aequationes (6) praebent, si insuper pro $k\mu$ ponitur valor ejus $a^3 n^2$,

$$\lambda = \varphi + \chi \quad \text{unde} \quad \varphi = \lambda,$$

si per λ , designatur differentiale ipsius λ respectu τ per $d\tau$ divisum. Considerando vero Ω tum tanquam functionem ipsarum φ , φ , et χ , tum tanquam functionem ipsarum λ , λ , et χ , habetur

$$d\Omega = \left(\frac{d\Omega}{d\varphi} \right) d\varphi + \left(\frac{d\Omega}{d\varphi} \right) d\varphi + \left(\frac{d\Omega}{d\chi} \right) d\chi$$

$$= \left(\frac{d\Omega}{d\lambda} \right) d\lambda + \left(\frac{d\Omega}{d\lambda} \right) d\lambda + \left(\frac{d\Omega}{d\chi} \right) d\chi$$

e qua ope aequationum

$$d\lambda = d\varphi + d\chi, \quad d\varphi = d\lambda,$$

emergunt

$$\begin{aligned}\left(\frac{d\Omega}{d\varphi}\right) &= \left(\frac{d\Omega}{d\lambda}\right) \\ \left(\frac{d\Omega}{d\varphi}\right) &= \left(\frac{d\Omega}{d\lambda}\right) \\ \left(\frac{d\Omega}{d\chi}\right) &= \left(\frac{d\Omega}{d\chi}\right) + \left(\frac{d\Omega}{d\lambda}\right)\end{aligned}$$

$$(30) \dots \dots \dots \begin{cases} d\lambda = -\frac{an \sin \varphi (2+e \cos \varphi)}{e\sqrt{(1-e^2)}} \left(\frac{d\Omega}{d\chi}\right) dt - \frac{a^3 n^2}{\rho^2} \left(\frac{d\Omega}{d\lambda}\right) dt \\ d\lambda = \left\{ \frac{a^3 n^2}{\rho^2} - 2 \frac{a^4 n^2 (1-e^2) \cos \varphi}{e \rho^3} \right\} \left(\frac{d\Omega}{d\chi}\right) dt + \frac{a^3 n^2}{\rho^2} \left(\frac{d\Omega}{d\lambda}\right) dt \\ d\lambda = \frac{a^2 n \cos \varphi \sqrt{(1-e^2)}}{e \rho} \left(\frac{d\Omega}{d\chi}\right) dt \end{cases}$$

Denotantibus ν longitudinem veram in orbita, s sinum latitudinis supra planum fixum ipsarum xy et l longitudinem veram ad idem planum reductam, ex art. 16 habemus

$$\begin{aligned}s &= \sin i \sin (\nu - \theta) \\ \operatorname{tg} (l - \theta) &= \operatorname{tg} (\nu - \theta) \cos i\end{aligned}$$

Sed posito

$$\nu = f d\lambda$$

quoties in dextro hujus aequationis membro post integrationem τ in t mutatur, erit

$$\nu = f + \chi$$

$$\begin{aligned}s &= q \cos \nu - p \sin \nu, \\ l &= \nu + \omega - \chi + \theta + \operatorname{arc.} \operatorname{tg} \left\{ \frac{2pq \cos 2\nu + (p^2 - q^2) \sin 2\nu}{[1 + \sqrt{(1 - p^2 - q^2)}]^2 + 2pq \sin 2\nu - (p^2 - q^2) \cos 2\nu} \right\}\end{aligned}$$

25.

Quantitates p et q ope ψ et σ sequenti modo exprimuntur. Differentiatae aequationes (31) praebent

$$\begin{aligned}dp &= \cos i \sin (\chi - \omega) di + \sin i \cos (\chi - \omega) (d\chi - d\omega) \\ dq &= \cos i \cos (\chi - \omega) di - \sin i \sin (\chi - \omega) (d\chi - d\omega)\end{aligned}$$

Sed ex aequationibus (16) emergunt

$$(32) \dots \dots \begin{cases} d\chi - d\omega = \cos i \frac{\cos \omega}{\sin i} d\psi + \cos i \frac{\sin \omega}{\sin i} d\sigma \\ di = -\sin \omega d\psi + \cos \omega d\sigma \end{cases}$$

$$\begin{aligned}dp &= \frac{an}{\sqrt{(1-e^2)}} \left\{ \cos i \cos \chi \left(\frac{d\Omega}{d\sigma}\right) - \cos i \sin \chi \left(\frac{d\Omega}{d\psi}\right) \right\} dt \\ dq &= -\frac{an}{\sqrt{(1-e^2)}} \left\{ \cos i \sin \chi \left(\frac{d\Omega}{d\sigma}\right) + \cos i \cos \chi \left(\frac{d\Omega}{d\psi}\right) \right\} dt\end{aligned} \dots \dots \dots (34)$$

Sed habita Ω tum pro functione ipsarum p et q , tum pro functione ipsarum ψ et σ , erit

$$d\Omega = \left(\frac{d\Omega}{dp}\right) dp + \left(\frac{d\Omega}{dq}\right) dq = \left(\frac{d\Omega}{d\psi}\right) d\psi + \left(\frac{d\Omega}{d\sigma}\right) d\sigma$$

quae aequatio, habita ratione aequationum (33), praebet

$$\begin{aligned}\left(\frac{d\Omega}{d\psi}\right) &= \cos i \cos \chi \left(\frac{d\Omega}{dp}\right) - \cos i \sin \chi \left(\frac{d\Omega}{dq}\right) \\ \left(\frac{d\Omega}{d\sigma}\right) &= \cos i \sin \chi \left(\frac{d\Omega}{dp}\right) + \cos i \cos \chi \left(\frac{d\Omega}{dq}\right)\end{aligned}$$

quarum sinistrae partes pro functionibus ipsarum φ , Φ , et χ , dextrae vero pro functionibus ipsarum λ , λ , et χ habentur. Loco aequationum art. praec. pro $d\varphi$, $d\lambda$ et $d\Phi$, igitur nanciscimur has

Quum vero sit

$$\nu = f + \omega + \theta$$

aequationes hae praebent

$$\nu = \nu + \omega + \theta - \chi$$

et aequationes supra allatae transeunt in has

$$\begin{aligned}s &= \sin i \sin (\nu - (\chi - \omega)) \\ \operatorname{tg} (l - \theta) &= \operatorname{tg} (\nu - (\chi - \omega)) \cos i\end{aligned}$$

quae, introductis quantitatibus p et q per has aequationes determinandis

$$\begin{aligned}p &= \sin i \sin (\chi - \omega) \\ q &= \sin i \cos (\chi - \omega)\end{aligned} \dots \dots \dots (31)$$

facile mutantur in has

quibus praecedentes aequationes abeunt in

$$\begin{aligned}dp &= \cos i \cos \chi d\psi + \cos i \sin \chi d\sigma \\ dq &= -\cos i \sin \chi d\psi + \cos i \cos \chi d\sigma\end{aligned} \dots \dots (33)$$

Quantitates dp et dq commode quoque per vim perturbantem ipsam exprimuntur. Quem in finem substituantur loco $d\psi$ et $d\sigma$ valores earum ex art. 23, unde prodit

substitutis his valoribus in aequationibus (34), emergunt

$$\begin{aligned}dp &= \frac{an}{\sqrt{(1-e^2)}} \cos^2 i \left(\frac{d\Omega}{dq}\right) dt \\ dq &= -\frac{an}{\sqrt{(1-e^2)}} \cos^2 i \left(\frac{d\Omega}{dp}\right) dt\end{aligned}$$

Quantitas $\omega - \chi + \theta$ in ultima formula art. praec. occurrens per p et q exprimi potest. Prima aequatio (32) una cum

$$d\theta = \frac{\cos \omega}{\sin i} d\psi + \frac{\sin \omega}{\sin i} d\sigma$$

præbet

$$d(\omega - \chi + \theta) = (1 - \cos i) \left\{ \frac{\cos \omega}{\sin i} d\psi + \frac{\sin \omega}{\sin i} d\sigma \right\}$$

sed aequationes (33) subministrant

$$d\psi = \frac{\cos \chi}{\cos i} dp - \frac{\sin \chi}{\cos i} dq$$

$$d\sigma = \frac{\sin \chi}{\cos i} dp + \frac{\cos \chi}{\cos i} dq$$

quibus praecedens abit in hanc

$$d(\omega - \chi + \theta) = \frac{\sin i \cos(\chi - \omega) dp - \sin i \sin(\chi - \omega) dq}{(1 + \cos i) \cos i}$$

unde fit, ope aequationum (31)

$$d(\omega - \chi + \theta) = \frac{q dp - p dq}{[1 + \sqrt{(1 - p^2 - q^2)}] \sqrt{(1 - p^2 - q^2)}}$$

Quae aequationes omnes cum iis quas in commentatione: Disquisitiones continuatae etc. inscripta inveneram plane congruunt.

26.

Licet per aequationes hic evolutas, quae λ et $l\rho$ praebent, situs planetae in orbita sua semper accuratissime determinari possit, transformatio tamen harum coordinatarum ad aequationes perducit, quibus non modo computatio perturbationum commodius absolvitur, sed quae insuper ad novam aptioremque tabularum motum planetarum heliocentricum exhibentium formam viam muniunt.

Qualescunque sunt coordinatae novae, quas z atque w denotabo, certo pro functionibus ipsarum ν , atque r , sive, id quod idem est, z atque w pro functionibus ipsarum ν , atque $l\rho$ habendae sunt. Ideo vice versa ν , atque $l\rho$ functiones ipsarum z atque w erunt. Quare accipiam esse

$$\begin{aligned} \left(\frac{d\lambda}{dt} \right) \left(\frac{d^2}{d\tau} \right) - \left(\frac{d\lambda}{d\tau} \right) \left(\frac{d^2}{dt} \right) &= \Pi, (\zeta, \beta) \left\{ \left(\frac{d\beta}{dt} \right) \left(\frac{d^2}{d\tau} \right) - \left(\frac{d\beta}{d\tau} \right) \left(\frac{d^2}{dt} \right) \right\} \\ \left(\frac{dl\rho}{dt} \right) \left(\frac{d^2}{d\tau} \right) - \left(\frac{dl\rho}{d\tau} \right) \left(\frac{d^2}{dt} \right) &= \Gamma, (\zeta, \beta) \left\{ \left(\frac{d\beta}{dt} \right) \left(\frac{d^2}{d\tau} \right) - \left(\frac{d\beta}{d\tau} \right) \left(\frac{d^2}{dt} \right) \right\} \end{aligned}$$

quae igitur formulae, in transformatione coordinatarum casus omnes qui accidere possunt amplectenti, generaliter valent. Ut vero formulas adipiscamur, quae a functionibus arbitrariis liberae sunt, casus quem adhuc consideravimus restringatur, oportet. Sint itaque $\Pi, (\zeta, \beta) = 0$ atque $\Gamma, (\zeta, \beta) = 1$, unde praecedentes aequationes abeunt in has

$$(35) \dots \left\{ \begin{aligned} \frac{\left(\frac{d^2}{dt} \right)}{\left(\frac{d^2}{d\tau} \right)} &= \frac{\left(\frac{d\lambda}{dt} \right)}{\left(\frac{d\lambda}{d\tau} \right)} \\ \left(\frac{d\beta}{dt} \right) - \left(\frac{d\beta}{d\tau} \right) \frac{\left(\frac{d^2}{dt} \right)}{\left(\frac{d^2}{d\tau} \right)} &= \frac{\left(\frac{dl\rho}{dt} \right) \left(\frac{d\lambda}{d\tau} \right) - \left(\frac{dl\rho}{d\tau} \right) \left(\frac{d\lambda}{dt} \right)}{\left(\frac{d\lambda}{d\tau} \right)} \end{aligned} \right.$$

$$\nu = \Pi(z, w)$$

$$l\rho = \Gamma(z, w)$$

denotantibus Π et Γ functiones quasdam. Considerando vero loco ipsarum z atque w functiones ζ atque β , quae illis plane analogae eo tantum discrepant, quod in valoribus earum pure ellipticis τ loco t inest, secundum praecedentia manifestum est, easdem expressiones, quibus z atque w cum ipsis ν , atque $l\rho$ junguntur, praebituras esse ζ atque β per λ atque $l\rho$ et vice versa, nec non his ipsis expressionibus, quoties λ , $l\rho$ et cetera elementa variabilia spectantur et post integrationes peractas τ in t mutatum fuerit, valores veros perturbatos ipsarum z et w compertum iri tales, quales ex valoribus perturbatis ipsarum ν , et $l\rho$ immediate essent prodituri. Sunt igitur

$$\lambda = \Pi(\zeta, \beta)$$

$$l\rho = \Gamma(\zeta, \beta)$$

Differentiatis his aequationibus tum respectu τ tum respectu t , elicitur

$$\begin{aligned} \left(\frac{d\lambda}{dt} \right) &= \Pi'(\zeta, \beta) \left(\frac{d^2}{dt} \right) + \Pi, (\zeta, \beta) \left(\frac{d\beta}{dt} \right) \\ \left(\frac{d\lambda}{d\tau} \right) &= \Pi'(\zeta, \beta) \left(\frac{d^2}{d\tau} \right) + \Pi, (\zeta, \beta) \left(\frac{d\beta}{d\tau} \right) \\ \left(\frac{dl\rho}{dt} \right) &= \Gamma'(\zeta, \beta) \left(\frac{d^2}{dt} \right) + \Gamma, (\zeta, \beta) \left(\frac{d\beta}{dt} \right) \\ \left(\frac{dl\rho}{d\tau} \right) &= \Gamma'(\zeta, \beta) \left(\frac{d^2}{d\tau} \right) + \Gamma, (\zeta, \beta) \left(\frac{d\beta}{d\tau} \right) \end{aligned}$$

ubi Π' et Π , quotientes differentiales ipsius Π , et Γ' atque Γ , quotientes differentiales ipsius Γ , resp. respectu ζ atque β denotant. Eliminatis Π' et Γ' inter has aequationes, emergunt

$$\begin{aligned} \left(\frac{d\lambda}{dt} \right) \left(\frac{d^2}{d\tau} \right) - \left(\frac{d\lambda}{d\tau} \right) \left(\frac{d^2}{dt} \right) &= 0 \\ \left(\frac{dl\rho}{dt} \right) \left(\frac{d^2}{d\tau} \right) - \left(\frac{dl\rho}{d\tau} \right) \left(\frac{d^2}{dt} \right) &= \left(\frac{d\beta}{dt} \right) \left(\frac{d^2}{d\tau} \right) - \left(\frac{d\beta}{d\tau} \right) \left(\frac{d^2}{dt} \right) \end{aligned}$$

quae facili opera transformantur in has

quarum dextrae partes functiones ipsarum λ et $l\rho$ sunt, quae quantitates datae spectantur, sinistrae vero partes incognitas ζ et β tantum continent. Aequationes vero $\Pi, (\zeta, \beta) = 0$ atque $\Gamma, (\zeta, \beta) = 1$, probant haberi

$$\begin{aligned}\lambda &= \Pi(\zeta) \\ l\rho &= \Gamma(\zeta) + \beta\end{aligned}$$

quae vice versa praebent

$$\begin{aligned}\zeta &= P(\lambda) \\ \beta &= G(\lambda) + l\rho\end{aligned}$$

denotantibus P et G functiones inversas ipsarum Π et Γ . Quoties igitur transformatio coordinatarum ita perficitur, ut formulis praecedentibus, ubi $P(\lambda)$ et $G(\lambda)$ sunt functiones quaecunque ipsius λ , satisfaciatur, aequationes (35) semper locum habent. Si loco primae harum aequationum ejus differentiale respectu τ in calculum vocatur, et ponantur

$$\begin{aligned}T &= \frac{\left(\frac{d^2\lambda}{d\tau^2}\right)\left(\frac{d\lambda}{d\tau}\right) - \left(\frac{d^2\lambda}{d\tau}\right)\left(\frac{d\lambda}{dt}\right)}{\left(\frac{d\lambda}{d\tau}\right)^2} \\ R &= \frac{\left(\frac{d l\rho}{dt}\right)\left(\frac{d\lambda}{d\tau}\right) - \left(\frac{d l\rho}{d\tau}\right)\left(\frac{d\lambda}{dt}\right)}{\left(\frac{d\lambda}{d\tau}\right)}\end{aligned}$$

sunt T atque R functiones ipsarum λ atque $l\rho$, quarum altera alteri fere similis est, quas ante omnia evolvi oportet.

27.

Quantitates $\left(\frac{d^2\lambda}{d\tau^2}\right)$ et $\left(\frac{d l\rho}{d\tau}\right)$ quas expressiones ipsarum T et R continent, duplici modo in calculum vocari possunt. Aut aequationes illis art. 23 plane similes pro dif-

$$\begin{aligned}T &= \frac{an}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{d\chi}\right) + \left(\frac{d\Omega}{d\lambda}\right) \right\} - 2 \frac{n\rho(2e + \cos\varphi + e^2\cos\varphi)}{e(1-e^2)^{\frac{3}{2}}} \left(\frac{d\Omega}{d\chi}\right) - 2 \frac{a^2n^2e\sin\varphi}{\rho(1-e^2)} \left(\frac{d\Omega}{d\lambda_r}\right) \\ R &= \frac{n\rho(2e + \cos\varphi + e^2\cos\varphi)}{e(1-e^2)^{\frac{3}{2}}} \left(\frac{d\Omega}{d\chi}\right) + \frac{a^2n^2e\sin\varphi}{\rho(1-e^2)} \left(\frac{d\Omega}{d\lambda_r}\right)\end{aligned}$$

Quae expressiones jam insignem monstrant relationem inter T et R quam alibi dedi. Nam posita aequatione hac

$$(36) \dots \frac{an}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{d\chi}\right) + \left(\frac{d\Omega}{d\lambda}\right) \right\} = d\lambda$$

aequationes praecedentes praebent hanc

$$(37) \dots 2R + T = d\lambda$$

quae aequatio, quum sit

$$T = \frac{d \cdot \left(\frac{d\zeta}{dt}\right)}{d\tau}$$

ferentialibus earum respectu temporis evolvi possunt, quae per integrationem quantitates ipsas praebebunt, aut quantitates eae nec non $\left(\frac{d\lambda}{d\tau}\right)$ tamquam functiones ipsarum

Φ, ρ, φ , etc. spectari possunt, quarum valores perturbatos nancisceris, si in earum expressionibus pure ellipticis loco Φ, ρ, φ , etc. valores perturbatos harum quantitatum substitueris. Methodus posterior in finem eum quem hic persequor illa aptior est, et optimo jure applicari potest, licet

expressio ipsius T quantitatem $\left(\frac{d^2\lambda}{d\tau^2}\right)$ contineat, cujus significatio in motu perturbato secundum art. 10 a significatione in motu imperturbato differt; sed quoniam expressio ipsius T antequam ζ ex ea deducitur, respectu ipsius τ integranda est, ea quae in fine art. 10 de $\frac{d^2\Phi}{d\tau^2}$ exposui huc referri possunt, et significationes ejus in utroque motus genere diversae hoc loco nullam vim habent. Quibus positis, juxta formulas notas ad motum ellipticum pertinentes in expressionibus pro T et R ponenda sunt

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right) &= \frac{a^2n}{\rho^2} \sqrt{1-e^2} \\ \left(\frac{d^2\lambda}{d\tau^2}\right) &= -2 \frac{a^2n^2}{\rho^3} e \sin\varphi \\ \left(\frac{d l\rho}{d\tau}\right) &= \frac{an}{\rho} \frac{e \sin\varphi}{\sqrt{1-e^2}}\end{aligned}$$

Jam quum formulae (30) quantitates $\left(\frac{d\lambda}{dt}\right)$, $\left(\frac{d l\rho}{dt}\right)$ et $\left(\frac{d^2\lambda}{d\tau^2}\right)$ sive $\left(\frac{d\lambda_r}{dt}\right)$ immediate praebeant, omnia quae ad computationem ipsarum T et R requiruntur praesto sunt et facili opera inveniuntur

$$R = \left(\frac{d\beta}{dt}\right) - \left(\frac{d\beta}{d\tau}\right) \left(\frac{d\zeta}{dt}\right) \left(\frac{d\zeta}{d\tau}\right)$$

abit in

$$2 \left(\frac{d\beta}{dt}\right) - 2 \left(\frac{d\beta}{d\tau}\right) \left(\frac{d\zeta}{dt}\right) \left(\frac{d\zeta}{d\tau}\right) + d \cdot \frac{\left(\frac{d\zeta}{dt}\right)}{d\tau} = d\lambda$$

cujus integrale est

$$2\beta + l\left(\frac{d\zeta}{d\tau}\right) - \lambda = \psi(\zeta) \dots \dots \dots (38)$$

designante $\psi(\zeta)$ functionem arbitrariam ipsius ζ , sicut in theoria Jovis atque Saturni jam demonstravi. Quod igitur integrale non modo in casu eo valet, ubi ζ longitudo media perturbata repraesentat, sed generaliter valet, manentibus Π et Γ functiones quascunque ipsius ζ . Functio vero ψ e functionibus Π et Γ pendet et vice versa assumpto ad libitum valore ipsius ψ , functiones Π et Γ penitus arbitrarie spectari nequeunt.

Expressiones pro T et R supra datae adjuncto aequationum art. 23 semper integrari possunt, ut vero ex T et R quantitates ζ et β eliciantur, $\left(\frac{d\zeta}{d\tau}\right)$ et $\left(\frac{d\beta}{d\tau}\right)$ requiruntur,

$$(39) \dots \dots \dots \begin{cases} d\Omega = \left(\frac{d\Omega}{d\nu}\right) d\nu + \left(\frac{d\Omega}{dr}\right) dr + \left(\frac{d\Omega}{d\theta}\right) d\theta + \left(\frac{d\Omega}{di}\right) di \\ d\Omega = \left(\frac{d\Omega}{d\lambda}\right) d\lambda + \left(\frac{d\Omega}{d\rho}\right) d\rho + \left(\frac{d\Omega}{d\lambda_1}\right) d\lambda_1 + \left(\frac{d\Omega}{d\chi}\right) d\chi + \left(\frac{d\Omega}{dp}\right) dp + \left(\frac{d\Omega}{dq}\right) dq \end{cases}$$

sed aequationes (23) differentiatæ præbent

$$\begin{aligned} d\varphi &= \left(\frac{d\varphi}{da}\right) da + \frac{a^2}{\rho^2} \sqrt{1-e^2} dc + \left(\frac{1}{1-e^2} + \frac{a}{\rho}\right) \sin \varphi de \\ d\lambda &= \left(\frac{d\lambda}{da}\right) da + \frac{a^2}{\rho^2} \sqrt{1-e^2} dc + \left(\frac{1}{1-e^2} + \frac{a}{\rho}\right) \sin \varphi de + dw + \cos i d\theta \\ d\rho &= \left(\frac{d\rho}{da}\right) da + ae \frac{\sin \varphi}{\sqrt{1-e^2}} dc - a \cos \varphi de \\ d\lambda_1 &= \left(\frac{d\lambda_1}{da}\right) da - 2 \frac{a^3 n}{\rho^3} e \sin \varphi dc + \left\{ 2 \frac{a^3 n}{\rho^3} \cos \varphi \sqrt{1-e^2} - \frac{a^2 n e}{\rho^2 \sqrt{1-e^2}} \right\} de \end{aligned}$$

ex quibus mutato τ in t evadunt

$$\begin{aligned} d\nu &= \left(\frac{d\nu}{da}\right) da + \frac{a^2}{r^2} \sqrt{1-e^2} dc + \left(\frac{1}{1-e^2} + \frac{a}{r}\right) \sin f de + dw + \cos i d\theta \\ dr &= \left(\frac{dr}{da}\right) da + ae \frac{\sin f}{\sqrt{1-e^2}} dc - a \cos f de \end{aligned}$$

porro ex aequationibus (16) et ex initio art. 25 habetur

$$\begin{aligned} d\chi &= dw + \cos i d\theta \\ dp &= \cos i \sin(\chi - \omega) di + \sin i \cos i \cos(\chi - \omega) d\theta \\ dq &= \cos i \cos(\chi - \omega) di - \sin i \cos i \sin(\chi - \omega) d\theta \end{aligned}$$

quibus differentialium valoribus in (39) substitutis, inveniuntur, si termini per eadem differentialia multiplicati aequati erunt, aequationes hae

$$(40) \dots \dots \dots \left\{ \begin{aligned} \left(\frac{d\Omega}{d\nu}\right) \left(\frac{d\nu}{da}\right) + \left(\frac{d\Omega}{dr}\right) \left(\frac{dr}{da}\right) &= \left(\frac{d\Omega}{d\lambda}\right) \left(\frac{d\lambda}{da}\right) + \left(\frac{d\Omega}{d\rho}\right) \left(\frac{d\rho}{da}\right) + \left(\frac{d\Omega}{d\lambda_1}\right) \left(\frac{d\lambda_1}{da}\right) \\ \left(\frac{d\Omega}{d\nu}\right) \frac{a^2}{r^2} \sqrt{1-e^2} + \left(\frac{d\Omega}{dr}\right) ae \frac{\sin f}{\sqrt{1-e^2}} &= \left(\frac{d\Omega}{d\lambda}\right) \frac{a^2}{\rho^2} \sqrt{1-e^2} + \left(\frac{d\Omega}{d\rho}\right) \frac{ae \sin \varphi}{\sqrt{1-e^2}} - \left(\frac{d\Omega}{d\lambda_1}\right) 2 \frac{a^3 n}{\rho^3} e \sin \varphi \\ \left(\frac{d\Omega}{d\nu}\right) \left(\frac{1}{1-e^2} + \frac{a}{r}\right) \sin f - \left(\frac{d\Omega}{dr}\right) a \cos f &= \left(\frac{d\Omega}{d\lambda}\right) \left(\frac{1}{1-e^2} + \frac{a}{\rho}\right) \sin \varphi - \left(\frac{d\Omega}{d\rho}\right) a \cos \varphi + \left(\frac{d\Omega}{d\lambda_1}\right) \left\{ 2 \frac{a^3 n}{\rho^3} \cos \varphi \sqrt{1-e^2} - \frac{a^2 n e}{\rho^2 \sqrt{1-e^2}} \right\} \\ \left(\frac{d\Omega}{d\nu}\right) &= \left(\frac{d\Omega}{d\lambda}\right) + \left(\frac{d\Omega}{d\chi}\right) \\ \left(\frac{d\Omega}{d\nu}\right) \cos i + \left(\frac{d\Omega}{d\theta}\right) &= \left(\frac{d\Omega}{d\lambda}\right) \cos i + \left(\frac{d\Omega}{d\chi}\right) \cos i + \left(\frac{d\Omega}{dp}\right) \sin i \cos i \cos(\chi - \omega) - \left(\frac{d\Omega}{dq}\right) \sin i \cos i \sin(\chi - \omega) \\ \left(\frac{d\Omega}{di}\right) &= \left(\frac{d\Omega}{dp}\right) \cos i \sin(\chi - \omega) + \left(\frac{d\Omega}{dq}\right) \cos i \cos(\chi - \omega) \end{aligned} \right.$$

quae computari nequeunt, nisi functiones Π et Γ sint datae.

28.

Jam sit nobis negotium primum expressiones pro T et R in art. praec. inventas ita exhibere, ut functiones ipsius τ quas cohibent explicite adstant, id quod fit introducendo quotientes differentiales ipsius Ω respectu ν , et r loco quotientium respectu χ , λ et λ_1 . Habita Ω tum pro functione ipsarum ν , r , θ et i , tum pro functione ipsarum λ , ρ , λ_1 , χ , p et q erit

Multiplicata secunda harum aequationum per $\cos \varphi$, tertia per $\frac{e \sin \varphi}{\sqrt{1-e^2}}$, additisque productis, facili opera invenitur

$$\left(\frac{d\Omega}{d\nu}\right)\left\{\frac{\cos \varphi}{(1-e^2)^{\frac{3}{2}}} + \frac{e \cos(f-\varphi)}{(1-e^2)^{\frac{3}{2}}} + \frac{ae \cos(f-\varphi)}{r\sqrt{1-e^2}}\right\} + r\left(\frac{d\Omega}{dr}\right)\frac{ae \sin(f-\varphi)}{r\sqrt{1-e^2}} = \left(\frac{d\Omega}{d\lambda}\right)\frac{2e + \cos \varphi + e^2 \cos \varphi}{(1-e^2)^{\frac{3}{2}}} - \left(\frac{d\Omega}{d\lambda}\right)\frac{a^2 n e^2 \sin \varphi}{\rho^2 (1-e^2)}$$

sed quarta aequatio praebet

$$\left(\frac{d\Omega}{d\lambda}\right) = \left(\frac{d\Omega}{d\nu}\right) - \left(\frac{d\Omega}{d\chi}\right)$$

Quo valore ipsius $\left(\frac{d\Omega}{d\lambda}\right)$ in aequatione praecedenti sub-

$$(41) \dots R = -\left\{\frac{\rho}{r} \cos(\nu, -\lambda) - 1 + \frac{\rho}{a(1-e^2)} [\cos(\nu, -\lambda) - 1]\right\} \frac{an}{\sqrt{1-e^2}} \left(\frac{d\Omega}{d\nu}\right) - \frac{\rho}{r} \sin(\nu, -\lambda) \frac{an}{\sqrt{1-e^2}} r \left(\frac{d\Omega}{dr}\right)$$

hinc, et quum sit $\left(\frac{d\Omega}{d\lambda}\right) + \left(\frac{d\Omega}{d\chi}\right) = \left(\frac{d\Omega}{d\nu}\right)$ aequationes (36) et (37) subministrant statim

$$(42) \dots T = \frac{an}{\sqrt{1-e^2}} \left(\frac{d\Omega}{d\nu}\right) + 2\left\{\frac{\rho}{r} \cos(\nu, -\lambda) - 1 + \frac{\rho}{a(1-e^2)} [\cos(\nu, -\lambda) - 1]\right\} \frac{an}{\sqrt{1-e^2}} \left(\frac{d\Omega}{d\nu}\right) + 2\frac{\rho}{r} \sin(\nu, -\lambda) \frac{an}{\sqrt{1-e^2}} r \left(\frac{d\Omega}{dr}\right)$$

quae aequationes cum illis congruunt, quas alibi ad determinandas perturbationes longitudinis mediae et logarithmi radii vectoris dedi, sed hoc loco demonstratum est, easdem semper locum habere, quomocunque functiones $\Pi(\zeta)$ et $\Gamma(\zeta)$ compositae sunt.

Ad solutionem problematis ζ et β in genere determinandi complendam restat, ut constans, quae integrali $\int T d\tau$ adjicienda est, definiatur. Mutato τ in t , secunda vel tertia et quarta aequationum (40) solo intuitu probant fieri

$$\left(\frac{d\Omega}{d\chi}\right) = 0, \quad \left(\frac{d\Omega}{d\lambda}\right) = 0$$

Itaque prima aequationum (30) monstrat, fieri $\frac{d\lambda}{dt} = 0$ quoties τ in t mutatur, sed

$$\int T d\tau = \frac{\left(\frac{d\zeta}{dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)} = \frac{\left(\frac{d\lambda}{dt}\right)}{\left(\frac{d\lambda}{d\tau}\right)}$$

ergo necesse est habeamus

$$\int T d\tau = 0$$

quoties in hoc integrale τ in t mutatur, quae conditio constantem huic integrationi adjiciendam, quae functio ipsius t esse potest, plane determinat.

29.

Perscrutemur quid quantitas in art. 27 h denotata et per aequationem hanc

$$dh = \frac{an}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{d\lambda}\right) + \left(\frac{d\Omega}{d\chi}\right) \right\} dt = \frac{an}{\sqrt{1-e^2}} \left(\frac{d\Omega}{d\nu}\right) dt$$

determinanda, designet. Multiplicata secunda aequationum (30) per ρ^2 , tertia per $\{2a^2 n \sqrt{1-e^2}\}$, additisque productis, invenitur, propter $a^2 n \sqrt{1-e^2} = \rho^2 \lambda$,

stituto, dextrum ejus membrum abit in

$$-R \frac{e}{\rho n} + \left(\frac{d\Omega}{d\nu}\right) \frac{2e + \cos \varphi + e^2 \cos \varphi}{(1-e^2)^{\frac{3}{2}}}$$

itaque, quum $1 + e \cos \varphi = \frac{a(1-e^2)}{\rho}$ et $f - \varphi = \nu, -\lambda$, fit

$$a^3 n^2 \left\{ \left(\frac{d\Omega}{d\chi}\right) + \left(\frac{d\Omega}{d\lambda}\right) \right\} dt = \rho^2 d\lambda + 2\rho^2 \lambda, d\lambda \rho = d\lambda \rho^2 = d\lambda a^2 n \sqrt{1-e^2}$$

sive

$$\frac{an}{\sqrt{1-e^2}} \left(\frac{d\Omega}{d\nu}\right) dt = d\lambda a^2 n \sqrt{1-e^2}$$

itaque

$$\text{const.} \times h = a^2 n \sqrt{1-e^2}$$

et si constans haec aequalis $\sqrt{k\mu}$ ponatur, h radicem quadraticam ex semiparametro denotat.

Quantitate ζ inventa, aequatio (38) ad computationem quantitatis ω absque cognitione ipsius h inservire potest. Nam quum h sit quantitas ab ipsa τ libera, aequatio (38) respectu ipsius τ differentiatia praebet

$$\frac{d\beta}{d\tau} = \frac{1}{2} \frac{d\psi(\zeta)}{d\zeta} \cdot \frac{d\zeta}{d\tau} - \frac{1}{2} \frac{\frac{d^2 \zeta}{d\tau^2}}{\left(\frac{d\zeta}{d\tau}\right)}$$

sed quum secundum art. 11 habeatur

$$\left(\frac{d\beta}{d\tau}\right) = \frac{d\omega}{dt} \text{ et } \left(\frac{d\zeta}{d\tau}\right) = \frac{dz}{dt}$$

ubi linea superscripta denotat τ in t mutandum esse, aequatio praecedens fit

$$\frac{d\omega}{dt} = \frac{1}{2} \frac{d\psi(z)}{dz} \cdot \frac{dz}{dt} - \frac{1}{2} \frac{\left(\frac{d^2 \zeta}{d\tau^2}\right)}{\frac{dz}{dt}}$$

e qua integrata emergit

$$\omega = \text{const.} + \frac{1}{2} \psi(z) - \frac{1}{2} \int \frac{\left(\frac{d^2 \zeta}{d\tau^2}\right)}{\frac{dz}{dt}} dt$$

ubi constans adjecta vera constans est.

30.

In commentatione pluries memorata expressiones illas (41) et (42) pro T et R ex consideratione perturbationum longitudinis mediae derivaveram, hoc vero loco hic casus specialis et sine dubio simplicissimus ex generali casu, quem adhuc consideravimus, deducendus est. Quum habeamus

$$(43) \dots \dots \dots \begin{cases} \lambda = \Pi(\zeta) \\ l\rho = \Gamma(\zeta) + \beta \end{cases}$$

ubi functiones Π et Γ penitus arbitrariae sunt, ad libitum aequationes inter ζ atque $\Pi(\zeta)$ et $\Gamma(\zeta)$ statuere nobis licet. Ideo accipio esse

$$(44) \dots \dots \dots \begin{cases} (n)\zeta + (c) = \bar{v} - (e) \sin \bar{v} \\ \bar{\rho} \cos \bar{\varphi} = (a) \cos \bar{v} - (a)(e) \\ \bar{\rho} \sin \bar{\varphi} = (a) \sqrt{1-(e)^2} \cdot \sin \bar{v} \end{cases}$$

ubi (a) , (n) , (c) et (e) constantes atque $\bar{v}$ angulus auxiliaris est. Sunt igitur $\bar{\rho}$ atque $\bar{\varphi}$ functiones solius variabilis ζ , et aequationes praecedentes non minus quam innumerae aliae aequationes conditioni requisitae respondent. Praeterea nihil refert, quosnam valores constantibus (a) , (n) , (c) et (e) tribuamus, adeo ad libitum sive $\Pi(\zeta) = \bar{\varphi}$ et $\Gamma(\zeta) = \bar{\rho}$, sive $\Pi(\zeta) = \bar{\rho}$ et $\Gamma(\zeta) = \bar{\varphi}$ statuere licet, modo quantitates $\left(\frac{d\zeta}{d\tau}\right)$ et $\left(\frac{d\beta}{d\tau}\right)$ ita computentur, ut his hypothesis respondant. Ut vero solutio simplicissima investigetur, ponam

$$(45) \dots \dots \dots \begin{cases} \Pi(\zeta) = \bar{\varphi} + (\chi) \\ \Gamma(\zeta) = \bar{\rho} + \delta \end{cases}$$

denotantibus (χ) et δ constantes quasdam. Jam aequationes (43) una cum praecedentibus praebent

$$\frac{d\zeta}{d\tau} = \frac{\frac{d\lambda}{d\tau}}{\frac{d\Pi(\zeta)}{d\zeta}} = \frac{\frac{d\lambda}{d\tau}}{\frac{d\bar{\varphi}}{d\zeta}}$$

$$\frac{d\beta}{d\tau} = \frac{dl\rho}{d\tau} - \frac{d\Gamma(\zeta)}{d\zeta} \cdot \frac{d\zeta}{d\tau} = \frac{dl\rho}{d\tau} - \frac{d\bar{\rho}}{d\zeta} \cdot \frac{d\zeta}{d\tau}$$

Sunt vero notae aequationes hae

$$\frac{d\lambda}{d\tau} = \frac{a^2 n}{\rho^2} \sqrt{1-e^2}, \quad \frac{dl\rho}{d\tau} = \frac{an}{\rho} \cdot \frac{e \sin \bar{\varphi}}{\sqrt{1-e^2}}$$

et simili modo aequationes (44) suppeditabunt

$$\frac{d\bar{\varphi}}{d\zeta} = \frac{(a)^2 (n)}{\bar{\rho}^2} \sqrt{1-(e)^2}, \quad \frac{d\bar{\rho}}{d\zeta} = \frac{(a)(n)}{\bar{\rho}} \cdot \frac{(e) \sin \bar{\varphi}}{\sqrt{1-(e)^2}}$$

unde

$$(46) \dots \dots \dots \begin{cases} \frac{d\zeta}{d\tau} = \frac{\bar{\rho}^2 a^2 n \sqrt{1-e^2}}{\rho^2 (a)^2 (n) \sqrt{1-(e)^2}} \\ \frac{d\beta}{d\tau} = \frac{an e \sin \bar{\varphi}}{\rho \sqrt{1-e^2}} - \frac{\bar{\rho}^2 a^2 n (e) \sin \bar{\varphi} \sqrt{1-e^2}}{\rho^2 (a) (1-(e)^2)} \end{cases}$$

31.

Constantes (a) , (n) , (c) , (e) nec non constantes quae alio modo ipsi ζ inesse possent in art. praec. quaecunque fuerunt. Nunc vero accipio esse (a) , (n) , (c) , (e) resp. valores ipsarum a , n , c , e , si vires perturbantes evanescent, hoc est si massae m' , m'' , etc. cifrae aequales statuuntur, aequationes igitur (46) praebent sub hac conditione

$$\frac{d\zeta}{d\tau} = \frac{\bar{\rho}^2}{(\rho)^2}$$

$$\frac{d\beta}{d\tau} = \frac{(a)(n)(e) \sin(\bar{\varphi})}{(\rho) \sqrt{1-(e)^2}} - \frac{\bar{\rho}^2 (a)(n)(e) \sin \bar{\varphi}}{(\rho)^2 \sqrt{1-(e)^2}}$$

denotantibus (ρ) et (φ) valores ipsarum ρ et φ si vires perturbantes evanescent. In his aequationibus constantes quae ipsi ζ inesse possent si vires perturbantes evanescent adhuc indeterminatae quidem sunt, sed non penitus arbitrariae. Quae enim constantes ita determinari debent, ut aequationi praecedenti pro $\frac{d\zeta}{d\tau}$ satisfaciant. Statuta vero, si vires perturbantes desunt, aequatione hac $\zeta = \tau$, habemus $\left(\frac{d\zeta}{d\tau}\right) = 1$, et quum tum fiat

$$\bar{\rho} = (\rho)$$

aequatio praecedens quoque praebet

$$\left(\frac{d\zeta}{d\tau}\right) = 1$$

aequatio igitur

$$\zeta = \tau$$

si vires perturbantes evanescent, optimo jure statui potest. Quum praeterea hoc in casu sit $\bar{\varphi} = (\varphi)$, aequatio praecedens pro $\left(\frac{d\beta}{d\tau}\right)$ praebet

$$\left(\frac{d\beta}{d\tau}\right) = 0.$$

Ergo, his omnibus hypothesis valentibus in approximatione prima ad integralia expressionum (41) et (42) poni debet

$$\left(\frac{d\zeta}{d\tau}\right) = 1, \quad \left(\frac{d\beta}{d\tau}\right) = 0$$

unde

$$T = \frac{d^2 \zeta}{d\tau dt}, \quad R = \frac{d\beta}{dt}$$

Ad approximationes secundam et ultiores perficiendas quantitatibus $\left(\frac{d\zeta}{d\tau}\right)$ et $\left(\frac{d\beta}{d\tau}\right)$ valores ii attribuendi sunt, qui ex aequationibus (46) emergunt, sed his ipsis aequationibus opus non est. Nam quum approximatio prima ipsas ζ et β per t et τ immediate expressas praebitura sit, hae expressiones postquam respectu ipsius τ differentiatas fuerint,

valores ipsarum $\left(\frac{d\zeta}{d\tau}\right)$ et $\left(\frac{d\beta}{d\tau}\right)$ quales in approximatione secunda adhibendi erunt, statim suppeditabunt. Eodem modo approximatō secunda valores eos harum quantitatum, quibus ad approximationem tertiam absolvendam opus erit, praebunt, et sic porro.

Restat ut constantes (χ) et δ determinantur. Quum aequationes (43), (44) et (45) valores inperturbatos ipsarum λ et $l\rho$ suppeditare debeant, si vires perturbantes evanescent, quumque in hoc casu sit $\zeta = \tau$, facile concluditur esse (χ) valori inperturbato ipsius χ , et δ cifrae aequalem, siquidem constans arbitraria integrali $\int d\beta$ addenda cifrae aequalis ponatur. Quoties planetae orbita ad planum quoddam fixum refertur, est

$$(\chi) = (\omega) + (\theta)$$

nec non constantes perturbationibus ipsarum p , q et $\omega - \chi + \theta$, art. 24 addendae resp. sunt

$$(p) = \sin(i) \sin(\theta)$$

$$(q) = \sin(i) \cos(\theta)$$

$$(\omega) - (\chi) + (\theta) = 0$$

denotantibus (ω) , (i) et (θ) valores ipsarum ω , i et θ qui existunt, si vires perturbantes evanescent. Quomodo constantes hae se habeant, quoties orbita ad planum quoddam mobile referatur, in huius libri Nr. 244 seqq. copiose explicavi.

Mutato τ in t in aequationibus (43), (44) et (45), emergunt ad valores perturbatos ipsarum ν , et lr computandos aequationes hae

$$(a)^3(n)^2 = k\mu$$

$$(n)z + (c) = \bar{u} - (e) \sin \bar{u}$$

$$\bar{r} \cos \bar{f} = (a) \cos \bar{u} - (a)(e)$$

$$\bar{r} \sin \bar{f} = (a) \sqrt{1-(e)^2} \cdot \sin \bar{u}$$

$$\nu = \bar{f} + (\chi)$$

$$lr = \bar{r} + \omega$$

unde factum est, ut $\Pi(z)$ atque $\Gamma(z)$ ex z eodem modo pendeant, quo valores inperturbati ipsarum ν , et lr ex t . Itaque z tempus perturbatum sive $(n)z$ longitudo media perturbata appellari potest; praeterea loco istarum aequationum series notas, si placet, substituere licet.

32.

Perfacile demonstratur fieri hoc in casu $\psi(\zeta) = 0$. Etenim prior aequatio (46) et alterae aequationes (43) et (45) praebent

$$l\left(\frac{d\zeta}{d\tau}\right) = l \cdot a^2 n \sqrt{1-e^2} - l \cdot (a)^2 (n) \sqrt{1-(e)^2} + 2\bar{l}\bar{\rho} - 2l\rho$$

$$\beta = l\rho - \bar{l}\bar{\rho}$$

quae in (38) substitutae suppeditant

$$l \cdot a^2 n \sqrt{1-e^2} - l \cdot (a)^2 (n) \sqrt{1-(e)^2} - l\bar{h} = \psi(\zeta)$$

Quum vero secundum art. 29 $l\bar{h}$ perturbationes ipsius $l \cdot a^2 n \sqrt{1-e^2}$ denotet, et primi duo termini aequationis praecedentis easdem designent, habetur

$$l \cdot a^2 n \sqrt{1-e^2} - l \cdot (a)^2 (n) \sqrt{1-(e)^2} - l\bar{h} = 0.$$

itaque

$$\psi(\zeta) = 0.$$

Q. E. D.

33.

In hoc articulo considerabimus casum ubi in aequationibus (44) loco constantium (a) , (n) et (c) constantes aliae adsunt. Statuamus igitur esse

$$\left. \begin{aligned} (n)\zeta + (c) &= \bar{u} - \sin \bar{u} \\ \bar{\rho} \cos \bar{\varphi} &= (a) \cos \bar{u} - (a)(e) \\ \bar{\rho} \sin \bar{\varphi} &= (a) \sqrt{1-(e)^2} \sin \bar{u} \\ (a)^3 (n)^2 &= \mu k \end{aligned} \right\} \dots\dots (47)$$

denotantibus semper (a) , (n) , (c) et (e) resp. valores ipsarum a , n , c et e , si vires perturbantes evanescent. Hinc factum est, ut loco aequationum (46) habeamus has

$$\frac{d\zeta}{d\tau} = \frac{\bar{\rho}^2 a^2 n \sqrt{1-e^2}}{\rho^2 (a)^2 (n) \sqrt{1-e^2}}$$

$$\frac{d\beta}{d\tau} = \frac{a n e \sin \bar{\varphi}}{\rho \sqrt{1-e^2}} - \frac{\bar{\rho}^2 a^2 n (e) \sin \bar{\varphi} \sqrt{1-e^2}}{\rho^2 (a) (1-e)^2}$$

at, si supponimus, per substitutionem ipsarum (a) , (n) et (c) loco (a) , (n) et (c) nos valoribus veris perturbatis ipsarum ζ et β appropinquavisse, in approximatione prima ad integralia $\int d\zeta$ et $\int d\beta$ ubique (a) , (n) et (c) loco (a) , (n) et (c) substituere licebit. Quo facto habetur

$$\frac{d\zeta}{d\tau} = \frac{\bar{\rho}^2}{(\rho)^2}$$

$$\frac{d\beta}{d\tau} = \frac{(a)(n)(e) \sin \bar{\varphi}}{(\rho) \sqrt{1-(e)^2}} - \frac{\bar{\rho}^2 (a)(n)(e) \sin \bar{\varphi}}{(\rho)^2 \sqrt{1-(e)^2}}$$

quae, posita aequatione $\zeta = \tau$ si vires perturbantes evanescent, veluti in art. 31, pro approximatione prima praebent

$$\left(\frac{d\zeta}{d\tau}\right) = 1, \quad \left(\frac{d\beta}{d\tau}\right) = 0$$

$$T = \frac{d^2 \zeta}{d\tau dt}, \quad R = \frac{d\beta}{dt}$$

Itaque integratio aequationum (41) et (42) hoc in casu eodem modo peragenda est, uti in casu quem in art. 31 statueramus. Conditio vero, ut aequationes (47), (43) et (45) valores inperturbatos ipsarum λ et $l\rho$ praebant si vires

perturbantes evanescant, ad aequationes paullulum diversas perducit. Primum, quum tum fieri debeat

$$(n_r)\zeta^2 + (c_r) = (n)\tau + (c)$$

habetur, quoniam $\zeta = \tau$,

$$(c_r) = (c) + [(n) - (n_r)]\tau$$

sive ponendo

$$(n) = (n_r)(1-b)$$

$$(c_r) = (c) - (n_r)b\tau$$

unde generaliter evadit

$$(n_r)\zeta^2 + (c_r) = (n_r)(1-b)\tau + (c) + \text{perturb.}$$

Secundum, per hunc valorem ipsius $(n_r)\zeta^2 + (c_r)$ aequationes (47), subductis perturbationibus, valorem quidem ipsius $\bar{\varphi}$ praebeunt, qui ope constantis (χ) quemadmodum in art. 31 definita est, et ope aequationum (43) et (45) valorem imperturbatum ipsius λ suppeditabit, sed loco valoris ejusmodi ipsius ξ subministrabunt $\bar{\xi}$ talem, ut sit

$$\frac{\bar{\xi}}{(a_r)} = \frac{(\xi)}{(a)}$$

Aequatio igitur altera (45) haec

$$\Gamma(\zeta) = l\bar{\varphi} + \delta$$

quae, viribus perturbantibus evanescentibus, fieri debet

$$l(\xi) = l\bar{\xi} + \delta$$

per substitutionem valoris ipsius $\bar{\xi}$ modo dati, abit in

$$l(\xi) = l(\xi) + l(a_r) - l(a) + \delta$$

$$l\left(\frac{d\zeta}{d\tau}\right) = l.a^2n\sqrt{1-e^2} - l.(a_r)^2(n_r)\sqrt{1-(e)^2} + 2l\bar{\xi} - 2l\rho$$

$$\beta = l\xi - l\bar{\xi} + \frac{2}{3}l(-b)$$

$$l(a_r)^2(n_r) = l.(a)^2(n) + \frac{1}{3}l(1-b)$$

unde aequatio (38) abit in

$$l.a^2n\sqrt{1-e^2} - l.(a)^2(n)\sqrt{1-(e)^2} + l(1-b) - lh = \psi(\zeta)$$

quae, quum sit

$$l.a^2n\sqrt{1-e^2} - l.(a)^2(n)\sqrt{1-(e)^2} - lh = 0$$

praebet

$$\psi(\zeta) = l(1-b)$$

sive $\psi(\zeta)$ constantem. Q. E. D.

$$\left(\frac{d\Omega}{d\theta}\right) = \left(\frac{d\Omega}{dp}\right) \sin i \cos i \cos(\chi-\omega) - \left(\frac{d\Omega}{dq}\right) \sin i \cos i \sin(\chi-\omega)$$

$$\left(\frac{d\Omega}{di}\right) = \left(\frac{d\Omega}{dp}\right) \cos i \sin(\chi-\omega) + \left(\frac{d\Omega}{dq}\right) \cos i \cos(\chi-\omega)$$

quibus per eliminationem emergunt

$$\left(\frac{d\Omega}{dp}\right) = \left(\frac{d\Omega}{d\theta}\right) \frac{\cos(\chi-\omega)}{\sin i \cos i} + \left(\frac{d\Omega}{di}\right) \frac{\sin(\chi-\omega)}{\cos i}$$

$$\left(\frac{d\Omega}{dq}\right) = -\left(\frac{d\Omega}{d\theta}\right) \frac{\sin(\chi-\omega)}{\sin i \cos i} + \left(\frac{d\Omega}{di}\right) \frac{\cos(\chi-\omega)}{\cos i}$$

Sed in commentatione in huius libri Nr. 244 seqq. inserta

unde

$$\delta = l(a) - l(a_r)$$

quae aequatio ope aequationum

$$(a_r)^3 (n_r)^2 = k\mu$$

$$(a)^3 (n)^2 = k\mu$$

$$(n) = (n_r)(1-b)$$

facile transmutatur in

$$\delta = -\frac{2}{3}l(1-b)$$

Itaque mutato τ in t nanciscimur ad ν , et lr determinandos aequationes has

$$(n_r)z + (c_r) = \bar{u} - (e) \sin \bar{u}$$

$$\bar{r} \cos \bar{f} = (a_r) \cos \bar{u} - (a_r)(e)$$

$$\bar{r} \sin \bar{f} = (a_r) \sqrt{1-(e)^2} \cdot \sin \bar{u}$$

$$\nu_r = \bar{f} + (\chi)$$

$$lr = l\bar{r} - \frac{2}{3}l(1-b) + \omega$$

ubi

$$(n_r)z + (c_r) = (n_r)(1-b)t + (c) + \text{perturb.}$$

In theoria Jovis atque Saturni quantitatem b , ibi c denotatam, ita determinaveram, ut evaderet

$$(n_r)z + (c_r) = (n_r)t + (c) + \text{perturb. periodicis.}$$

34.

Adhibitis aequationibus (47) loco aequationum (44) $\psi(\zeta)$ constanti aequalis est. Habetur enim secundum art. praec.

35.

His ita absolutis, de quantitibus $\left(\frac{d\Omega}{dp}\right)$ et $\left(\frac{d\Omega}{dq}\right)$ pauca dicenda sunt. Quarta, quinta et sexta aequationum (40) praebent

demonstravi esse

$$\left(\frac{d\Omega}{d\theta}\right) = -U \sin i \cos(\nu-\theta)$$

$$\left(\frac{d\Omega}{di}\right) = U \sin(\nu-\theta)$$

designante U functionem quandam ipsarum ν, r, i, θ ,

ν' , r' , etc. His igitur valoribus ipsarum $\left(\frac{d\Omega}{d\theta}\right)$ et $\left(\frac{d\Omega}{di}\right)$ in aequationibus praecedentibus substitutis, elicitur

$$\left(\frac{d\Omega}{dp}\right) = -\frac{U}{\cos i} \cos(\nu - \omega + \chi - \theta)$$

$$\left(\frac{d\Omega}{dq}\right) = \frac{U}{\cos i} \sin(\nu - \omega + \chi - \theta)$$

In art. 24 vero inventa est aequatio haec

$$\nu = \nu' + \omega - \chi + \theta$$

qua aequationes praecedentes abeunt in has

$$\left(\frac{d\Omega}{dp}\right) = -\frac{U}{\cos i} \cos \nu,$$

$$\left(\frac{d\Omega}{dq}\right) = \frac{U}{\cos i} \sin \nu,$$

quae cum valoribus harum quantitatum, quos in commutatione modo citata, inveneram, plane congruunt.

(Sequetur paragrahus secunda.)

Schreiben des Herrn M. Koller an den Herausgeber.

Kremsmünster 1834. Jänner 19.

Ihrer gütigen Aufforderung durch Circular vom 8. Febr. v. J. mit Vergnügen entsprechend übersende ich Ihnen hie- mit die Beobachtungen der Mondculminationen, welche im verflossenen Jahre am Meridiankreise der hiesigen Stern-

warte angestellt wurden. Sie sind sämmtlich nach *Bessel's* Anleitung in tab. regiom. LII seqq. und mit Benutzung seiner Hülf- und Refractionstabeln nach folgenden Formeln reduzirt:

$$\alpha = \mu \pm \frac{h}{(1-\lambda) \cos \delta} + fF + \left\{ \frac{1-p \sin \pi \cos \Phi' \sec \delta}{1-\lambda} \right\} (am + bn + c \sec \delta')$$

$$\delta = \delta' + p \mp h \mp \left(\frac{p \mp h}{2} \right) \sin p \sin h - \left(\frac{15}{2} \right)^2 \sin 1'' \sin 2\delta, (1-\lambda)^2 (\mu' - \mu)^2 + (\mu' - \mu) \lambda'.$$

Die Columnen $d\alpha$ und $d\delta$ enthalten die Ergebnisse der Vergleichung mit Ihrer Ephemeride, und geben, an die Beobachtungen algebraisch angebracht, die aus der Ephemeride abgeleiteten Positionen. Die geographische Breite der hiesigen Sternwarte setzte ich bei diesen und allen übrigen Berechnungen $48^\circ 3' 24''$, indem ich nach neueren, mit besseren Hülfsmitteln angestellten Bestimmungen guten Grund habe, diese Angabe für die der Wahrheit nächste zu halten. Jedoch kann ich diese Untersuchung noch nicht als geendigt betrachten, und werde bei einer andern Gelegenheit mehreres darüber mittheilen. Für $\Phi = 48^\circ 3' 24''$ und die Abplattung der Erde $\frac{1}{297,648}$, nach Dr. *Eduard Schmidt*

in Astr. Nachr. Nr. 209, folgt aber die geocentr. Polhöhe der Sternwarte $\Phi' = 47^\circ 51' 53''$ und $\log p = 9,9991951$. Die Meridiendifferenz zwischen Ihrer Sternwarte und der hiesigen wurde, wie sie sich aus den Mondsternen ergab, $16' 52''$ angenommen. Ich füge auch die im vorigen Jahre gemachten Planetenbeobachtungen bei. Sie sind mit dem Berliner Jahrbuche verglichen, und $d\alpha$ und $d\delta$ in demselben Sinne, wie oben genommen. Juno ertrug keine Beleuchtung, daher mußten die Beobachtungen derselben am Meridiankreise aufgegeben werden. Die Beobachtungen des Jupiter und der Mondsterne werde ich später übersenden.

M. Koller.

Beobachtungen des Mondes auf der Sternwarte zu Kremsmünster. 1833.

Tag.	AR. des Mond- Centr. im Merid.	Decl. des Mond- Centr. im Merid.	$d\alpha$	$d\delta$
Jan. 4	5h 36' 39" 16	— — —	+ 0" 17	—
5	6 43 38,03	— — —	— 0,27	—
6	7 51 45,60	— — —	— 0,72	—
März 29	7 22 51,51	+ 21° 56' 33" 3	0,00	+ 1" 6
Mai 2	13 50 42,22	— 6 8 23,9	+ 0,22	+ 4,4
3	14 45 11,03	— 11 10 48,7	— 0,28	— 5,2
4	15 40 18,29	— — —	— 0,25	—
28	12 37 58,60	+ 1 29 9,5	+ 0,14	— 8,7
Jun. 2	17 8 15,34	— — —	— 0,15	—
28	15 53 20,38	— 17 25 3,5	— 0,06	+ 2,8
29	16 47 55,07	— 19 35 27,0	— 0,37	+ 8,4
Aug. 24	18 3 30,67	— 22 15 4,9	+ 0,47	+ 0,5
27	20 44 52,27	— 20 14 31,8	+ 0,87	+ 5,0
28	21 35 29,46	— — —	+ 0,66	—

Tag.	AR. des Mond- Centr. im Merid.	Decl. des Mond- centr. im Merid.	$d\alpha$	$d\delta$
Aug 29	22h 24' 13" 64	— 14° 6' 13" 3	— 0" 40	+ 1" 2
Sept. 24	21 19 11,30	— 18 38 55,3	+ 0,07	+ 1,0
25	22 8 24,65	— 15 23 43,1	— 0,33	— 0,2
26	22 55 54,61	— 11 29 29,6	+ 0,18	+ 3,5
27	23 42 10,99	— 7 6 10,8	+ 0,25	+ 1,3
28	0 27 50,58	— — —	+ 0,13	—
29	1 13 35,37	+ 2 26 47,5	+ 0,51	— 5,2
Oct. 25	0 11 34,62	— 5 12 25,9	+ 0,75	— 1,8
26	0 57 18,68	+ 0 41 22,8	+ 0,40	— 2,6
28	2 31 54,12	+ 10 23 31,4	— 0,14	— 1,2
29	3 22 18,14	— — —	— 0,10	—
Nov. 27	4 51 0,31	+ 20 31 16,8	— 0,08	— 3,6
Dec. 22	2 39 8,37	+ 10 57 40,6	+ 0,17	— 1,5
26	6 25 42,89	— — —	— 0,40	—