on a variety of series in games of chance and on biostatistical data,—a small change in a high moment makes a large change in n. Accordingly we are liable to form quite erroneous impressions of the nature of the hypergeometrical series, and even to reach impossible values for p, q, and r, which are determined through n. Thus the problem, which is practically an important one, as enabling us to test the sufficiency of the usual hypothesis, $n=\infty$, of the theory of errors, *i. e.* to test the "independence or interdependence of contributory causes," is seen to admit of a solution, but one which is hardly likely to be of much service unless in the case to which it is applied a very large amount of data is available.

XVI. *On James Bernouilli's Theorem in Probabilities.* *By* Lord Rayleigh, *F.R.S.**

If p denote the probability of an event, then the probability that in μ trials the event will happen m times and fail n times is equal to a certain term in the expansion of $(p+q)^\mu$, namely,

$$\frac{\mu\,!}{m\,!\,n\,!}\,p^m q^n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $p+q=1$, $m+n=\mu$.

"Now it is known from Algebra that if m and n vary subject to the condition that $m+n$ is constant, the greatest value of the above term is when m/n is as nearly as possible equal to p/q, so that m and n are as nearly as possible equal to μp and μq respectively. We say *as nearly as possible*, because μp is not necessarily an integer, while m is. We may denote the value of m by $\mu p+z$, where z is some proper fraction, positive or negative; and then $n=\mu q-z$."

The rth term, counting onwards, in the expansion of $(p+q)^\mu$ after (1) is

$$\frac{\mu\,!}{\overline{m-r}\,!\,\overline{n+r}\,!}\,p^{m-r}q^{n+r}. \quad . \quad . \quad . \quad (2)$$

The approximate value of (2) when m and n are large numbers may be obtained with the aid of Stirling's theorem, viz.

$$\mu\,!=\mu^{\mu+\frac{1}{2}}e^{-\mu}\sqrt{(2\pi)}\left\{1+\frac{1}{12\mu}+\ldots\right\}. \quad . \quad (3)$$

* Communicated by the Author.

The process is given in detail after Laplace in Todhunter's 'History of the Theory of Probability,' p. 549, from which the above paragraph is quoted. The expression for the rth term after the greatest is

$$\frac{e^{-\frac{\mu r^2}{2mn}}\sqrt{\mu}}{\sqrt{(2\pi mn)}}\left\{1+\frac{\mu rz}{mn}+\frac{r(n-m)}{2mn}-\frac{r^3}{6m^2}+\frac{r^3}{6n^2}\right\}; \quad . \quad (4)$$

and that for the rth term before the greatest may be deduced by changing the sign of r in (4).

It is assumed that r^2 does not surpass μ in order of magnitude, and fractions of the order $1/\mu$ are neglected.

There is an important case in which the circumstances are simpler than in general. It arises when $p=q=\frac{1}{2}$, and μ is an even number, so that $m=n=\frac{1}{2}\mu$. Here z disappears *ab initio*, and (4) reduces to

$$\frac{2e^{-2r^2/\mu}}{\sqrt{(2\pi\mu)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

representing (2), which now becomes

$$\frac{\mu\,!}{2^{\mu}\,.\,\frac{1}{2}\mu-r\,!\,\frac{1}{2}\mu+r\,!}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

An important application of (5) is to the theory of random vibrations. If μ vibrations are combined, each of the same phase but of amplitudes which are at random either $+1$ or -1, (5) represents the probability of $\frac{1}{2}\mu+r$ of them being positive vibrations, and accordingly $\frac{1}{2}\mu-r$ being negative. In this case, and in this case only, is the resultant $+2r$. Hence if x represent the resultant, the chance of x, which is necessarily an *even* integer, is

$$\frac{2e^{-x^2/2\mu}}{\sqrt{(2\pi\mu)}}.$$

The next greater resultant is $(x+2)$; so that when x is great the above expression may be supposed to correspond to a range for x equal to 2. If we represent the range by dx, the chance of a resultant lying between x and $x+dx$ is given by

$$\frac{e^{-x^2/2\mu}dx}{\sqrt{(2\pi\mu)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)^*$$

Another view of this matter, leading to (5) or (7) without the aid of Stirling's theorem, or even of formula (1), is given

* Phil. Mag. vol. x. p. 75 (1880).

(somewhat imperfectly) in 'Theory of Sound,' 2nd ed. § 42*a*. It depends upon a transition from an equation in finite differences to the well-known equation for the conduction of heat and the use of one of Fourier's solutions of the latter. Let $f(\mu, r)$ denote the chance that the number of events occurring (in the special application *positive* vibrations) is $\frac{1}{2}\mu + r$, so that the *excess* is r. Suppose that each random combination of μ receives *two* more random contributions—*two* in order that the whole number may remain even,—and inquire into the chance of a subsequent excess r, denoted by $f(\mu+2, r)$. The excess after the addition can only be r if previously it were $r-1$, r, or $r+1$. In the first case the excess becomes r by the occurrence of both of the two new events, of which the chance is $\frac{1}{4}$. In the second case the excess remains r in consequence of one event happening and the other failing, of which the chance is $\frac{1}{2}$; and in the third case the excess becomes r in consequence of the failure of both the new events, of which the chance is $\frac{1}{4}$. Thus

$$f(\mu+2, r) = \tfrac{1}{4}f(\mu, r-1) + \tfrac{1}{2}f(\mu, r) + \tfrac{1}{4}f(\mu, r+1). \quad . \quad (8)$$

According to the present method the limiting form of f is to be derived from (8). We know, however, that f has actually the value given in (6), by means of which (8) may be verified.

Writing (8) in the form

$$f(\mu+2, r) - f(\mu, r) = \tfrac{1}{4}f(\mu, r-1) - \tfrac{1}{2}f(\mu, r) + \tfrac{1}{4}f(\mu, r+1), \quad (9)$$

we see that when μ and r are infinite the left-hand member becomes $2\,df/d\mu$, and the right-hand member becomes $\frac{1}{4}d^2f/dr^2$, so that (9) passes into the differential equation

$$\frac{df}{d\mu} = \frac{1}{8}\frac{d^2f}{dr^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In (9), (10) r is the excess of the actual occurrences over $\frac{1}{2}\mu$. If we take x to represent the difference between the number of occurrences and the number of failures, $x = 2r$ and (10) becomes

$$\frac{df}{d\mu} = \frac{1}{2}\frac{d^2f}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In the application to vibrations $f(\mu, x)$ then denotes the chance of a resultant $+x$ from a combination of μ unit vibrations which are positive or negative at random.

In the formation of (10) we have supposed for simplicity that the addition to μ is 2, the lowest possible consistently with the total number remaining even. But if we please we may suppose the addition to be any even number μ'. The analogue of (8) is then

$$2^{\mu'}.f(\mu+\mu',r)=f(\mu,r-\tfrac{1}{2}\mu')+\mu'f(\mu,r-\tfrac{1}{2}\mu'+1) + \frac{\mu'(\mu'-1)}{1.2}f(\mu,r-\tfrac{1}{2}\mu'+2)+\ldots+f(\mu,r+\tfrac{1}{2}\mu');$$

and when μ is treated as very great the right-hand member becomes

$$f(\mu,r)\left\{1+\mu'+\frac{\mu'(\mu'-1)}{1.2}+\ldots+\mu'+1\right\} + \frac{1}{8}\frac{d^2f}{dr^2}\left\{1.\mu'^2+\mu'(\mu'-2)^2+\frac{\mu'(\mu'-1)}{1.2}(\mu'-4)^2 + \ldots\ldots+\mu'(\mu'-2)^2+1.\mu'^2\right\}$$

The series which multiplies f is $(1+1)^{\mu'}$, or $2^{\mu'}$. The second series is equal to $\mu'.2^{\mu'}$, as may be seen by comparison of coefficients of x^2 in the equivalent forms

$$(e^x+e^{-x})^n=2^n(1+\tfrac{1}{2}x^2+\ldots)^n = e^{nx}+ne^{(n-2)x}+\frac{n(n-1)}{1.2}e^{(n-4)x}+\ldots.$$

The value of the left-hand member becomes simultaneously

$$2^{\mu'}\{f+\mu' df/d\mu\};$$

so that we arrive at the same differential equation (10) as before.

This is the well-known equation for the conduction of heat, and the solution developed by Fourier is at once applicable. The symbol μ corresponds to time and r to a linear coordinate. The special condition is that initially—that is when μ is relatively small—f must vanish for all values of r that are not small. We take therefore

$$f(\mu,r)=\frac{A}{\sqrt{\mu}}e^{-2r^2/\mu}, \quad \ldots\ldots \quad (12)$$

which may be verified by differentiation.

The constant A may be determined by the understanding that $f(\mu,r)\,dr$ is to represent the chance of an excess lying between r and $r+dr$, and that accordingly

$$\int_{-\infty}^{+\infty} f(\mu,r)\,dr=1. \quad \ldots\ldots \quad (13)$$

Since $\int_{-\infty}^{+\infty}e^{-z^2}dz=\sqrt{\pi}$, we have

$$\frac{A}{\sqrt{\mu}}=\sqrt{\left(\frac{2}{\pi\mu}\right)}; \quad \ldots\ldots \quad (14)$$

and, finally, as the chance that the excess lies between r and $r+dr$,

$$\sqrt{\left(\frac{2}{\pi\mu}\right)} e^{-2r^2/\mu} dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Another method by which A in (12) might be determined would be by comparison with (6) in the case of $r=0$. In this way we find

$$\frac{A}{\sqrt{\mu}} = \frac{\mu\,!}{2^{\mu}.\,\frac{1}{2}\mu\,!\,\frac{1}{2}\mu\,!} = \frac{1\,.\,3\,.\,5\,\ldots\,\mu-1}{2\,.\,4\,.\,6\,\ldots\,\mu}$$

$$= \sqrt{\left(\frac{2}{\pi\mu}\right)} \text{ by Wallis' theorem.}$$

If, as is natural in the problem of random vibrations, we replace r by x, denoting the difference between the number of occurrences and the number of failures, we have as the chance that x lies between x and $x+dx$

$$\frac{e^{-x^2/2\mu}dx}{\sqrt{(2\pi\mu)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

identical with (7).

In the general case when p and q are not limited to the values $\frac{1}{2}$, it is more difficult to exhibit the argument in a satisfactory form, because the most probable numbers of occurrences and failures are no longer definite, or at any rate simple, fractions of μ. But the general idea is substantially the same. The excess of occurrences over the most probable number is still denoted by r, and its probability by $f(\mu, r)$. We regard r as continuous, and we then suppose that μ increases by unity. If the event occurs, of which the chance is p, the total number of occurrences is increased by unity. But since the most probable number of occurrences is increased by p, r undergoes only an increase measured by $1-p$ or q. In like manner if the event fails, r undergoes a *decrease* measured by p. Accordingly

$$f(\mu+1, r) = pf(\mu, r-q) + qf(\mu, r+p). \quad . \quad . \quad . \quad (17)$$

On the right of (17) we expand $f(\mu, r-q)$, $f(\mu, r+p)$ in powers of p and q. Thus

$$f(\mu, r+p) = f + \frac{df}{dr}p + \tfrac{1}{2}\frac{d^2f}{dr^2}p^2,$$

$$f(\mu, r-q) = f - \frac{df}{dr}q + \tfrac{1}{2}\frac{d^2f}{dr^2}q^2;$$

so that the right-hand member is

$$(p+q)f+\tfrac{1}{2}\frac{d^2f}{dr^2}(p^2q+pq^2), \qquad \text{or} \qquad f+\tfrac{1}{2}pq\,\frac{d^2f}{dr^2}.$$

The left-hand member may be represented by $f+df/d\mu$, so that ultimately

$$\frac{df}{d\mu}=\tfrac{1}{2}pq\frac{d^2f}{dr^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Accordingly by the same argument as before the chance of an excess r lying between r and $r+dr$ is given by

$$\frac{1}{\surd(2\pi pq\mu)}e^{-r^2/2pq\mu}dr. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

We have already considered the case of $p=q=\frac{1}{2}$. Another particular case of importance arises when p is very small, and accordingly q is nearly equal to unity. The whole number μ is supposed to be so large that $p\mu$, or m, representing the most probable number of occurrences, is also large. The general formula now reduces to

$$\frac{1}{\surd(2\pi m)}e^{-r^2/2m}dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

which gives the probability that the number of occurrences shall lie between $m+r$ and $m+r+dr$. It is a function of m and r only.

The probability of the deviation from m lying between $\pm r$

$$=\frac{2}{\surd(2\pi m)}\int_0^r e^{-r^2/2m}dr=\frac{2}{\surd\pi}\int_0^\tau e^{-r^2}d\tau, \quad . \quad . \quad . \quad (21)$$

where $\tau=r/\surd(2m)$. This is equal to $\cdot84$ when $\tau=1\cdot0$, or $r=\surd(2m)$; so that the chance is comparatively small of a deviation from m exceeding $\pm\surd(2m)$. For example, if m is 50, there is a rather strong probability that the actual number of occurrences will lie between 40 and 60.

The formula (20) has a direct application to many kinds of statistics.

XVII. *Notices respecting New Books.*

Textbook of Algebra with exercises for Secondary Schools and Colleges. By G. E. Fisher, *M.A., Ph.D., and* I. J. Schwatt, *Ph.D.* Part I. (pp. xiv + 683: Philadelphia, Fisher & Schwatt, 1898).

THIS is a big book for the comparatively small extent of ground it covers. The usual elementary parts are discussed up to and including simultaneous Quadratic equations, and then, in the remaining 80 pages, we have an account of Ratio, Proportion, Variation, Exponents, and Progressions. The Binomial Theorem for a positive Integral Exponent occupies about a dozen pages, the treatment by Combinations being reserved, we presume, for Part II.