272. Geometrical Proof That the Polar Circle of a Triangle Is Orthogonal to the Director Circle of the Inscribed Circle

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intersection of the curves. Moreover, in rectangular coordinates, the equation of the Limaçon is of degree 4, and that of the Quatrefoil of degree 6; therefore there should be in all 24 points of intersection.

This apparent discrepancy is to be accounted for.

Eliminating y from the rectangular equations of the two curves, an equation may be obtained of degree 16 in x , which contains x^8 as a factor.

Hence there are 8 points at ∞ , 8 at the origin and 8 others.

Of these last, 4 are determined by r, θ in one curve, and $-r, \pi + \theta$ in the other; therefore they do not appear in the treatment by polar coordinates.

The 8 points at the origin, arising from the intersections of the 2 branches of the Limaçon with the 4 branches of the Quatrefoil, shew as 1 in the figure; the values of θ that make $r=0$ are not the same in both curves, so that these points also do not appear in the treatment by polar coordinates.

The 8 points at ∞ are the imaginary circular points, each counted 4 times.

In order to exclude the 4 superfluous points which, besides the origin, appear in the figure and are not the representation of the solution of the biquadratic, it is well to mark with dots, or in some other way, the portions of the curves for which a negative value of r corresponds to a positive value of θ ; the points where a dotted portion intersects one not dotted are excluded.

W. H. H. HUDSON.

271. [K. 2. c.] *The polar circle of a triangle is cut orthogonally by the director circle of any inscribed conic.*

Let TP, TP' be tangents to a conic whose foci are S, S' , centre O , axes $2\alpha, 2\beta$. On SP cut off $SZ=2\alpha$. Then we know that $TZ=TS', STZ=P'TP$.

$$\begin{aligned} \text{Hence} \quad ST \cdot S'T \cos PTP' &= \frac{1}{2}(ST^2 + S'T^2 - 4\alpha^2) \\ &= OT^2 + OS^2 - 2\alpha^2 = OT^2 - \alpha^2 - \beta^2 \\ &= \text{sq. of tangent from } T \text{ to the director circle.} \end{aligned}$$

$$\begin{aligned} \text{Also} \quad \frac{1}{2}ST \cdot S'T \sin PTP' &= \triangle STZ \\ &= STP + S'TP' \\ &= 2OTP \\ &= \text{area of quadrilateral } OP'TP. \end{aligned}$$

Thus if ABC is a circumscribed triangle, K its orthocentre and ρ the radius of its polar circle,

$$\Sigma SA \cdot S'A \sin A = 2\triangle ABC = 4R^2 \sin A \sin B \sin C,$$

and since K is the mean centre of A, B, C for multipliers $\cot C \cot B, \cot C \cot A, \cot A \cot B$, whose sum is 1,

Sq. of tangent from K to director circle

$$\begin{aligned} &= \Sigma SA \cdot S'A \cos A \cdot \cot B \cot C + \text{sq. of tangent from } K \text{ to circumcircle} \\ &= \cot A \cot B \cot C \cdot \Sigma SA \cdot S'A \sin A + 2\rho^2 \\ &= 4R^2 \cos A \cos B \cos C + 2\rho^2 = \rho^2. \end{aligned}$$

Q.E.D.

A. C. DIXON.

272. [K. 2. c.] *Geometrical proof that the polar circle of a triangle is orthogonal to the director circle of the inscribed circle.*

Produce AX to T so that $AT=FE$; then since $AH=2OD$ therefore $HT=2DE$.

Since AET is a right angle

$$AT^2 - IT^2 = AE^2 - IE^2 = AE^2 - CE^2 = FE(NE - DE) = FE \cdot AX;$$

therefore $IT^2 = AT(AT - AX) = AT \cdot XT$.

Again, completing the rectangle $EDPQ$, since $IE^2 = AE \cdot LE$; therefore

$$IQ^2 = GQ \cdot PQ.$$

Thus T and Q are points on the radical axis of a point-circle at I and the circle upon AP as diameter.

Now $HT = 2DE = 2PQ$; consequently if HP be produced to K so that $HP = PK$ then K is also a point on the radical axis TQ .

Drawing AU perpendicular to HPK , $IK^2 = PK \cdot UK$.

But
$$IH^2 + IK^2 = 2IP^2 + 2HP^2;$$
therefore
$$2IP^2 - IH^2 = PK \cdot UK - 2HP^2$$

$$= HP(UH + 2HP) - 2HP^2$$

$$= HP \cdot UH$$

$$= AH \cdot HX.$$

R. F. DAVIS.

273. [K. 1. c.] *Note on Euclid I. 32. Corollary.*

If one vertex of a polygon of n sides be joined to the others the number of triangles formed is $n - 2$.

Each triangle contains angles whose sum is 2 right angles.

$\therefore$ the sum of all the interior angles of the polygon $= 2(n - 2)$ right angles.

A. A. BOURNÆ.

274. [A. 1. b.] *Note on the Power Inequality.* [Cf. Mr. V. R. Aiyar, p. 322.]

In proving the fundamental inequality that if $p > q$ and $x > y$, then

$$\frac{p}{q} x^{p-q} > \frac{x^p - y^p}{x^q - y^q} > \frac{p}{q} y^{p-q}$$

for the simple case when p and q are integers, it is well to notice that the statement can be written

$$\frac{x^{p-q}(x^q - y^q)}{(x-y)q} > \frac{x^p - y^p}{(x-y)p} > \frac{y^{p-q}(x^q - y^q)}{q(x-y)},$$

which is equivalent to the following truism:

In a Geometrical Progression $x^{p-1} + x^{p-2} + \dots + y^{p-1}$ the average of all the terms is intermediate between the average of the first q and the average of the last q .

F. J. W. WHIPPLE.