et de la relation entre les discriminants on obtient, à cause de cette dernière: (7) ...... $g_3^3\phi^2 = \frac{5^5}{4^3 . 3^8}\left[5(12y-A)^2+27(A^2-144B)\right]$.

Enfin de la valeur de L on déduit

$$(8) \ldots\ldots g_3^2 h^3 = -\frac{5^{10}}{4^6 . 3^7}\left[870y^3+165Ay^2-40A^2y-4050By+\tfrac{16}{9}A^3 + 216AB-4^3 . 3^7 . C\right].$$

Au moyen de ces quatre relations on arrive à exprimer les (4), (5) en fonction de y, A, B, C. Mais auparavant il importe d'observer qu'en multipliant les deux dernières (7), (8) entre elles, on a, à cause des (6), que le premier membre est une fonction de y et de A ; une très-simple calculation conduit à l'équation :

$$y^5-10By^3-40Cy^2+5\left(5B^2+\tfrac{4}{3}AC\right)y-\left(\tfrac{16}{9}A^2C+\tfrac{4}{3}AB^2-216BC\right)=0,$$

transformée en y de l'équation $f(x) = 0$.

On trouve pour le covariant (4) :

$$(10) \ldots\ldots \frac{54r+l\beta}{f'(x)} = 9y^2+4Ay-36B,$$

et analoguement on aura pour les covariants (5) deux polynomes du troisième et du quatrième degré en y.

En multipliant y et les trois polynomes en y des degrés 2, 3, 4 par des indeterminées, et en nommant avec z leur somme, par l'expression en z, analogue à celle de Tchirnauss, on transformera l'équation $f(x) = 0$ dans une autre, pour laquelle le coefficient du second terme est égal à zéro et les coefficients suivants seront fonctions de A, B, C et des quatre indéterminées.

Je démontrérais dans une prochaine occasion l'application de la méthode exposée à la transformation des équations du septième degré.

On Projective Cyclic Concomitants, or Surface Differential Invariants. By E. B. Elliott, M.A.

[*Read Feb. 14th, 1889.*]

Contents.

I.

1. In analogy with Professor Sylvester's nomenclature in connection with Reciprocants, I propose to give the name Projective or Principiant Cyclicants to those Pure Cyclicants* which are Differential, Invariants for all homographic transformations of the three variables of the second and higher derivatives of one of which with regard to the other two they are functions, *i.e.*, to those which have the property of persistence in form, but for a factor not involving second and higher derivatives, when the variables undergo any such transformation as

$$\frac{x}{a_1x'+b_1y'+c_1z'+d_1} = \frac{y}{a_2x'+b_2y'+c_2z'+d_2} = \frac{z}{a_3x'+b_3y'+c_3z'+d_3}$$
$$= \frac{1}{Ax'+By'+Cz'+D} \quad \text{..........................} \quad (1).$$

It will be remembered that all pure cyclicants persist for the included most general *linear* transformation, in which the A, B, C of (1) are zeroes.

Projective cyclicants obey of course all the laws of pure cyclicants in general; *i.e.*, they are homogeneous, doubly isobaric, and subject to annihilation by the four operators (see *Proceedings*, Vol. xviii., pp. 142, 164).

$$\Omega_1 \equiv \Sigma \left\{ (m+1)\, x_{m+1,\,n-1} \frac{d}{dx_{mn}} \right\} \quad \text{.....................} \quad (2),$$

* *Cf. Proceedings*, Vol. XIX., pp. 377, &c.

$$\Omega_2 \equiv \Sigma \left\{ (n+1)\, x_{m-1,\, n+1} \frac{d}{dx_{mn}} \right\} \quad\text{.....................}\quad (3),$$

$$V_1 \equiv \Sigma \left\{ \Sigma\, (r x_{rs} x_{m+1-r,\, n-s}) \frac{d}{dx_{mn}} \right\} \quad\text{..................}\quad (4),$$

$$V_2 \equiv \Sigma \left\{ \Sigma\, (s x_{rs} x_{m-r,\, n+1-s}) \frac{d}{dx_{mn}} \right\} \quad\text{..................}\quad (5),$$

in which x_{mn} denotes $\dfrac{1}{m!\ n!}\,\dfrac{d^{m+n}x}{dy^m\, dz^n}$ for all values of m, n, and in which the limits of the summations are adequately expressed by saying that in every derivative x_{pq} which occurs, whether in a coefficient or in an operating symbol, p and q must be positive integers, or a positive integer and a zero, whose sum is not less than 2. We shall presently see that the further conditions, necessary and sufficient, for a pure cyclicant to be projective are, that it have the two additional annihilators

$$\omega_1 \equiv \Sigma \left\{ (m+n-2)\, x_{m,\, n-1} \frac{d}{dx_{mn}} \right\} \quad\text{..................}\quad (6),$$

$$\omega_2 \equiv \Sigma \left\{ (m+n-2)\, x_{m-1,\, n} \frac{d}{dx_{mn}} \right\} \quad\text{..................}\quad (7).$$

It will be noticed that I am, for subsequent convenience, using a notation which regards y and z as the independent variables, and x as the dependent. In passages where this is not the case, Ω_1, Ω_2, V_1, V_2, ω_1, ω_2 will still be used to denote the operators (2) to (7) with the dependent variable, whichever it may be, written in them for x.

2. The formulæ of transformation (1) may be replaced by the succession of the three following substitutions

$$\left.\begin{aligned} Cx &= (Ca_1 - c_1 A)\, X + (Cb_1 - c_1 B)\, Y + (Cd_1 - c_1 D)\, Z + Cc_1 \\ Cy &= (Ca_2 - c_2 A)\, X + (Cb_2 - c_2 B)\, Y + (Cd_2 - c_2 D)\, Z + Cc_2 \\ Cz &= (Ca_3 - c_3 A)\, X + (Cb_3 - c_3 B)\, Y + (Cd_3 - c_3 D)\, Z + Cc_3 \end{aligned}\right\} \ldots\ (8),$$

$$\frac{X}{X'} = \frac{Y}{Y'} = \frac{Z}{1} = \frac{1}{Z'} \quad\text{......................}\quad (9),$$

$$\left.\begin{aligned} X' &= x' \\ Y' &= \qquad y' \\ CZ' &= Ax' + By' + Cz' + D \end{aligned}\right\} \quad\text{.....................}\quad (10),$$

the only case of failure being when $C = 0$. In this special case A and B cannot be both also zero without the transformation (1) degenerating into a merely linear one. Suppose then that B, for instance, is different from zero. The transformation (1) may now be effected by the series of transformations (8), (9), (10), altered only by the interchange of C, c_1, c_2, c_3, and B, b_1, b_2, b_3, preceded by the particular linear substitution of x for y and y for x, and followed by that of y' for x' and x' for y'. In all cases, therefore, the homographic transformation (1) may be replaced by a succession of linear transformations and a transformation like (9).

II.

3. We have, accordingly, to study the transformation (9), or say

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{1} = \frac{1}{z'} \quad \text{..................... (11)},$$

with a view to determine differential expressions which persist in form after the transformation, and in particular cyclicants which have this property of persistence. There are advantages of simplicity, as the sequel will make sufficiently clear, in regarding y and z as the independent variables, so that the relation, of course perfectly unrestricted in form, which is supposed to connect x, y, z, is regarded as one expressing the first in terms of the second and third of these variables. Similarly, of x', y', z', the two last are taken as the independent variables. In this and the following nine articles, forms of persistent expressions for the transformation (11) are investigated without any special reference to the theory of cyclicants.

A reason for the greater simplicity gained by regarding x and x' as the dependent variables in the two sets is, that the formulæ for the transformation of the independent variables

$$y = \frac{y'}{z'}, \quad z = \frac{1}{z'} \quad \text{..................... (12)},$$

or

$$y' = \frac{y}{z}, \quad z' = \frac{1}{z} \quad \text{..................... (13)},$$

are thus quite unencumbered by any presence of the dependent. Thus we obtain from them at once the equivalences of operators

$$\frac{d}{dy} = \frac{1}{z}\frac{d}{dy'} = z'\frac{d}{dy'} \quad \text{..................... (14)},$$

$$\frac{d}{dz} = -\frac{y}{z^2}\frac{d}{dy'} - \frac{1}{z^2}\frac{d}{dz'} = -y'z'\frac{d}{dy'} - z'^2\frac{d}{dz'} \quad \text{......... (15)},$$

from which also

$$-yz\frac{d}{dy} - z^2\frac{d}{dz} = \frac{d}{dz'} \quad \text{..................... (16)}.$$

The completely reciprocal relations between the accented and unaccented letters in (11), and its consequences, are of fundamental importance.

It is to be remarked that the operative symbols in (14), (15), (16) are symbols of total and not partial differentiation, so that, for instance, if the function operated on involve the dependent variable x or x' explicitly, $\frac{d}{dy}$ stands for $\left[\frac{d}{dy}\right]+x_{10}\left(\frac{d}{dx}\right)$ in which $\left(\frac{d}{dx}\right)$ is the symbol of differentiation with regard to x in so far as it is explicitly involved, and $\left[\frac{d}{dy}\right]$ that for all those parts of the operation $\frac{d}{dy}$ which ignore the explicit presence of x. We shall be in little danger of confusion in this matter, for it is only at the outset that we shall have occasion to operate on functions in which the dependent variable explicitly appears.

It is easy from (14)—(16) to obtain two independent linear differential operators which persist in form after the transformation. From (16), by aid of (12),

$$z^4\frac{d}{dy}=z'^4\frac{d}{dy'}, \qquad\qquad (17),$$

and by subtraction of (16) from z^2 times (15), and use of (12),

$$y\frac{d}{dy}+2z\frac{d}{dz}=-\left\{y'\frac{d}{dy'}+2z'\frac{d}{dz'}\right\} \qquad\qquad (18).$$

It is clear, then, that we are to expect two classes of persistent functions—a class which persist absolutely, and a skew class which persist but for a change of sign. They may be called persistents of positive and negative character respectively, and the operators (17) and (18) may be called, respectively, positively and negatively persistent operators.

Other persistent operators, equivalent of course in the aggregate to these two only, may be with ease written down. Thus the sum and difference of (15) and (16) give us, respectively, the positively and negatively persistent operators

$$yz\frac{d}{dy}+(z^2-1)\frac{d}{dz}=y'z'\frac{d}{dy'}+(z'^2-1)\frac{d}{dz'} \qquad\qquad (19),$$

$$yz\frac{d}{dy}+(z^2+1)\frac{d}{dz}=-\left\{y'z'\frac{d}{dy'}+(z'^2+1)\frac{d}{dz'}\right\} \qquad\qquad (20);$$

and, again, the sum and difference of (16) and z times (15) give us

$$z^{\frac{1}{2}}\left\{y\frac{d}{dy}+(z-1)\frac{d}{dz}\right\}=z'^{\frac{1}{2}}\left\{y'\frac{d}{dy'}+(z'-1)\frac{d}{dz'}\right\} \quad \text{......(21)},$$

and

$$z^{\frac{1}{2}}\left\{y\frac{d}{dy}+(z+1)\frac{d}{dz}\right\}=-z'^{\frac{1}{2}}\left\{y'\frac{d}{dy'}+(z'+1)\frac{d}{dz'}\right\} \quad \text{......(22)}.$$

It should be remarked that in (17), and either (19) or (21), we have two independent positively persistent operators.

4. The persistent operators arrived at in the last article enable us at once to write down any number of persistent functions of the variables and derivatives, when we notice that the formulæ of transformation (11) may themselves be written in persistent form. Thus, in the independent variable z only, we have the persistents, not of course independent,

$$z+\frac{1}{z}=z+\frac{1}{z'} \quad \text{..........................(23)},$$

$$z^{-\frac{1}{2}}(z+1)=z'^{-\frac{1}{2}}(z'+1) \quad \text{......................(24)},$$

$$z^{-\frac{1}{2}}(z-1)=-z'^{-\frac{1}{2}}(z'-1) \quad \text{......................(25)},$$

$$\log z=-\log z'=w, \text{ say} \quad \text{.....................(26)},$$

&c., &c. The two first of these are of positive, and the third and fourth of negative, character.

Again, in both independent variables y and z, we have the persistent

$$z^{-\frac{1}{2}}y=z'^{-\frac{1}{2}}y'=v, \text{ say} \quad \text{.....................(27)},$$

and in the dependent and independent variables

$$z^{-\frac{1}{2}}x=z'^{-\frac{1}{2}}x'=u, \text{ say} \quad \text{.......................(28)},$$

or, again,

$$y^{-1}x=y'^{-1}x' \quad \text{.............................(29)}.$$

In (26), (27), and (28), we have in persistent form the exact equivalents of (11).

By operation on (28) or (29) with any one of the persistent operators (17) to (22) we get a linear persistent function in x, x_{10}, x_{01}, or some of them, the coefficients involving one or both of y and z. By a second operation with the same or another of the operators, we get a new persistent linear in x, x_{10}, x_{01}, x_{20}, x_{11}, x_{02}, or some of them. In fact, we get a linear persistent after any number of repetitions of such operations. Let us choose two independent persistent operators, (17) and (18) say. By use of them as thus indicated, noticing that the effect of the compound operation

$$\left(y\frac{d}{dy}+2z\frac{d}{dz}\right)\left(z^{\frac{1}{2}}\frac{d}{dy}\right)$$

is not altered by reversing the order of its component simple operations, we obtain a perfectly complete system of linear persistent functions by assigning to m and n, in

$$\left(z^{\frac{1}{2}}\frac{d}{dy}\right)^m\left(y\frac{d}{dy}+2z\frac{d}{dz}\right)^n(z^{-\frac{1}{2}}x)$$
$$=(-1)^n\left(z'^{\frac{1}{2}}\frac{d}{dy'}\right)^m\left(y'\frac{d}{dy'}+2z'\frac{d}{dz'}\right)^n(z'^{-\frac{1}{2}}x')\;\ldots\ldots(30),$$

all zero and positive integral values in succession. The functions are positively or negatively persistent as n is even or odd.

For the explanation of (30) we have not far to seek. The relation, whatever it may be, which connects x, y, z, may be expressed in terms of the elementary persistent functions u, v, w, of (28), (27), and (26). Thus we have

$$\left.\begin{aligned} z &= e^{w} \\ y &= v e^{\frac{1}{2}w} \\ x &= u e^{\frac{1}{2}w} \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31),$$

and the companion formulæ

$$\left.\begin{aligned} z &= e^{-w} \\ y' &= v e^{-\frac{1}{2}w} \\ x' &= u e^{-\frac{1}{2}w} \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32).$$

The first two of each of these groups give us

$$\frac{d}{dv}=e^{\frac{1}{2}w}\frac{d}{dy}=e^{-\frac{1}{2}w}\frac{d}{dy'}$$
$$=z^{\frac{1}{2}}\frac{d}{dy}=z'^{\frac{1}{2}}\frac{d}{dy'}\;\ldots\ldots\ldots\ldots\ldots\ldots(33),$$

and

$$\frac{d}{dw}=e^{w}\frac{d}{dz}+\tfrac{1}{2}ve^{\frac{1}{2}w}\frac{d}{dy}=-e^{-w}\frac{d}{dz'}-\tfrac{1}{2}ve^{-\frac{1}{2}w}\frac{d}{dy'}$$
$$=\tfrac{1}{2}\left(y\frac{d}{dy}+2z\frac{d}{dz}\right)=-\tfrac{1}{2}\left(y'\frac{d}{dy'}+2z'\frac{d}{dz'}\right)\;\ldots\ldots\ldots(34).$$

Thus the two sides of (30) are merely equivalent expressions for

$$2^n \frac{d^{m+n} u}{dv^m dw^n} \quad \text{..........................(35)},$$

and the completeness of the series of linear persistent expressions given by (30) or (35) lies in the fact that these are constant multiples of the entire system of coefficients in the expansion of an increment of u or $z^{-\frac{1}{2}}x$ in terms of increments of v or $z^{-\frac{1}{2}}y$ and w or $\log z$.

It is at once clear that a complete system of persistent functions, with y instead of x taken for dependent variable, is afforded in like manner by the series of derivatives

$$\frac{d^{m+n} v}{dw^m du^n} \quad \text{..............................(36)}.$$

The case when z is taken as the dependent variable has less simplicity, for u, v, and w all involve z, while only one of them involves x or y.

5. It is not to be assumed that the complete system of linear persistents for the transformation (11) given by (30) or (35) is the simplest in form of all complete systems when written explicitly. This statement may be illustrated by writing down the complete system for the first two orders.

Operating on (28) with (17) and (18) in turn, we get the complete system of the first order

$$\frac{du}{dv} = x_{10} = x'_{10} \text{..........................(37)},$$

$$2\frac{du}{dw} = z^{-\frac{1}{2}}(yx_{10} + 2zx_{01} - x) = -z'^{-\frac{1}{2}}(y'x'_{10} + 2z'x'_{01} - x') \quad \text{...(38)}.$$

An equivalent pair, obtained by operating on (28) with (21) and (22), and remembering (24) and (25), is

$$yx_{10} + (z-1)\,x_{01} - x = y'x'_{10} + (z'-1)\,x'_{01} - x' \quad \text{...........(39)},$$

$$yx_{10} + (z+1)\,x_{01} - x = -\{y'x'_{10} + (z'+1)\,x'_{01} - x'\} \quad \text{......(40)}.$$

Note that the formulæ giving x_{10} and x_{01} in terms of accented letters are (37), and

$$x_{01} = x' - y'x'_{10} - z'x'_{01} \quad \text{......................(41)}.$$

The complete set of linear persistents of the second order are

$$\tfrac{1}{2}\frac{d^2u}{dv^2} = z^{\frac12}x_{20} = z'^{\frac12}x'_{20} \quad\text{.....................}(42),$$

$$\frac{d^2u}{dv\,dw} = yx_{20}+zx_{11} = -(y'x'_{20}+z'x'_{11}) \quad\text{..............}(43),$$

and

$$2^2\frac{d^2u}{dw^2} = 2z^{-\frac12}y^2x_{20}+4z^{\frac12}yx_{11}+8z^{\frac32}x_{02}-z^{-\frac12}yx_{10}+z^{-\frac12}x$$

$$= +\,(\text{the same expression in accented letters})\ldots(44).$$

Now, the last two terms in (44) are themselves positively persistent functions, by (27), (37), and (28). So too is its first term, by (42) and (27). Thus our simplest complete system of the second order consists of (42), (43), and the remaining terms of (44), *i.e.*,

$$z^{\frac12}(yx_{11}+2zx_{02}) = z'^{\frac12}(y'x'_{11}+2z'x'_{02}) \quad\text{..............}(45).$$

There are advantages, however, as will be seen presently, in not omitting the first term of (44), but in taking, rather than (45) the somewhat less simple persistent

$$z^{-\frac12}(y^2x_{20}+2zyx_{11}+4z^2x_{02}) = z'^{-\frac12}(y'^2x'_{20}+2z'y'x'_{11}+4z'^2x'_{02}) \ldots(46).$$

Again, note that the formulæ for x_{20}, x_{11}, x_{02} in terms of accented letters are

$$\left.\begin{aligned} x_{20} &= z'x'_{20} \\ x_{11} &= -z'(2y'x'_{20}+z'x'_{11}) \\ x_{02} &= z'(y'^2x'_{20}+y'z'x'_{11}+z'^2x'_{02}) \end{aligned}\right\} \quad\text{.................}(47).$$

6. In the last article we have found a complete system of three linear persistents of the second order, which do not involve the dependent variable x nor the first derivatives x_{10}, x_{01}. Now the persistent operators (33) and (34) cannot produce functions involving x, x_{10}, x_{01} from functions which are free from them. It follows that, from the second order onwards, a complete system exists which only involves the independent variables y, z, and second and higher derivatives. It will now be proved that the type of such a complete system is

$$u_{mn} = z^{\frac12(2n+m-1)}\,e^{\frac12 z^{-1}(\omega_1+y\Omega_1)}\,x_{mn} \quad\text{....................}(48),$$

where $m+n \nless 2$, and ω_1, Ω_1 denote the operators (6) and (2).

We have already a complete system of the second order which accords with the type in (42), (43), (46), *i.e.*, in

$$u_{20} = z^{\frac{1}{2}} x_{20} \quad \ldots\ldots\ldots\ldots (42),$$

$$u_{11} = z x_{11} + y x_{20}$$

$$= z \left\{ 1 + \frac{y}{2z}\, 2x_{20} \frac{d}{dx_{11}} \right\} x_{11} \quad \ldots\ldots\ldots\ldots (43a),$$

$$u_{02} = z^{-\frac{1}{2}} \left\{ z^2 x_{02} + \tfrac{1}{2} z y x_{11} + \tfrac{1}{4} y^2 x_{20} \right\}$$

$$= z^{\frac{3}{2}} \left\{ 1 + \frac{y}{2z} x_{11} \frac{d}{dx_{02}} + \frac{1}{1.2} \left(\frac{y}{2z}\, 2x_{20} \frac{d}{dx_{11}} \right) \left(\frac{y}{2z} x_{11} \frac{d}{dx_{02}} \right) \right\} x_{02} \ldots (46a),$$

of which the first two are $\frac{1}{2!}\frac{d^2u}{dv^2}$, $\frac{1}{1!\,1!}\frac{d^2u}{dv\,dw}$, and the third differs from $\frac{1}{2!}\frac{d^2u}{dw^2}$ by a persistent of the same character. The general law will be proved to be that u_{m0} and u_{m1} are $\frac{1}{m!}\frac{d^m u}{dv^m}$ and $\frac{1}{m!\,1!}\frac{d^{m+1}u}{dv^m dw}$, while, in general, u_{mn} differs from $\frac{1}{m!\,n!}\frac{d^{m+n}u}{dv^m dw^n}$ by a persistent of the same character not involving x_{mn}.

In the first place, that u_{m0} is the persistent $\frac{1}{m!}\frac{d^m u}{dv^m}$ is proved instantaneously as follows

$$\frac{1}{m!}\frac{d^m u}{dv^m} = \frac{1}{m!}\left(z^{\frac{1}{2}} \frac{d}{dy} \right)^m (z^{-\frac{1}{2}} x) = z^{\frac{1}{2}(m-1)} x_{m0}$$

$$= z^{\frac{1}{2}(m-1)} e^{\frac{1}{2}z^{-1}(\omega_1 + y\Omega_1)} x_{m0}$$

$$= u_{m0} \quad \ldots\ldots\ldots\ldots (49).$$

We proceed to ground a mathematical induction, proving that u_{mn} is in all cases a persistent function, upon the actual evaluation of $\frac{d}{dw} u_{mn}$.

7. The order of the operations of ω_1 and Ω_1 on any function of the derivatives is immaterial, for it is at once clear that the alternant identity

$$\omega_1 \Omega_1 - \Omega_1 \omega_1 = 0$$

is satisfied.

Thus (48) may be written

$$u_{mn} = z^{\frac{1}{2}(2n+m-1)} \Sigma \left(\frac{1}{r!\,s!} \frac{y^s}{(2z)^{r+s}} \omega_1^r \Omega_1 \right) x_{mn}$$

$$= \Sigma \left(y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{1}{2^{r+s} r!\,s!}\, \omega_1^r \Omega_1^s \right) x_{mn}$$

$$= \Sigma \left\{ y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} (m+n-2)(m+n-3)\dots \right.$$
$$\left. \dots(m+n-r-1)(m+1)(m+2)\dots(m+s)\, x_{m+s,\, n-r-s} \right\}$$

$$= \Sigma \left(y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{(m+n-2)!\,(m+s)!}{2^{r+s} r!\, s!\, m!\, (m+n-r-2)!} x_{m+s,\, n-r-s} \right) \dots(50),$$

the summation being with regard to r and s, and comprising all pairs of values (including zero) of those numbers whose sum does not exceed n.

It follows that

$$\frac{d}{dw} u_{mn} = \left(z \frac{d}{dz} + \tfrac{1}{2} y \frac{d}{dy} \right) u_{mn}$$

$$= \Sigma \left\{ y^s z^{\frac{1}{2}(2n+m+1-2r-2s)} \frac{(m+n-2)!\,(m+s)!\,(n-r-s+1)}{2^{r+s} r!\, s!\, m!\, (m+n-r-2)!} x_{m+s,\, n-r-s+1} \right\}$$

$$+ \Sigma \left\{ y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{(2n+m-1-2r-2s)(m+n-2)!\,(m+s)!}{2^{r+s+1} r!\, s!\, m!\, (m+n-r-2)!} \times x_{m+s,\, n-r-s} \right\}$$

$$+ \Sigma \left\{ y^{s+1} z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{(m+n-2)!\,(m+s+1)!}{2^{r+s+1} r!\, s!\, m!\, (m+n-r-2)!} x_{m+s+1,\, n-r-s} \right\}$$

$$+ \Sigma \left\{ y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{(m+n-2)!\,(m+s)!}{2^{r+s+1} r!\, (s-1)!\, m!\, (m+n-r-2)!} x_{m+s,\, n-r-s} \right\};$$

in which the first summation may, by putting $r+1$ for r, be written

$$\Sigma \left\{ y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{(m+n-2)!\,(m+s)!\,(n-r-s)}{2^{r+s+1} (r+1)!\, s!\, m!\, (m+n-r-3)!} x_{m+s,\, n-r-s} \right\},$$

and the third, by putting $r+1$ for r and $s-1$ for s,

$$\Sigma \left\{ y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{(m+n-2)!\,(m+s)!}{2^{r+s+1} (r+1)!\, (s-1)!\, m!\, (m+n-r-3)!} x_{m+s,\, n-r-s} \right\}.$$

In these two summations the value -1 of r must be regarded as admissible, and in the latter the value 0 of s is excluded. Subject to this remark we obtain, then,

$$\frac{d}{dw} u_{mn} = \Sigma \left\{ y^s z^{\frac{1}{2}(2n+m-1-2r-2s)} \frac{(m+n-2)!\,(m+s)!}{2^{r+s+1} (r+1)!\, s!\, m!\, (m+n-r-2)!} x_{m+s,\, n-r-s} \right.$$
$$\left. \Big[(n-r-s)(m+n-r-2) + (2n+m-1-2r-2s)(r+1) + s\,(m+n-r-2) + (r+1)\,s \Big] \right\} \dots\dots\dots(51),$$

the value -1 of r being admitted in the first and third products within the square brackets. It is also clear that the admission of the same value into the second and fourth products will introduce only zero terms to the summation, since $r+1$ is a factor of each of those terms, and since $(r+1)\,!$ in the denominator is taken as unity when $r+1$ vanishes. Again, that the value $s=0$ is excluded from the third product makes no difference, for a like reason.

Now, it is readily seen that the sum of products in the square brackets is

$$(n+1)(m+n-1)-(r+1)(r+s)$$

$$=(n+1)(m+n-1)-r\,(r+1)-s\,(r+1)\ldots\ldots\ldots\ldots(52).$$

It will be convenient to put r for $r+1$, since this may have all values. Doing so, we find that

$$\frac{d}{dw}u_{mn}=(n+1)\,\Sigma\left\{y^s z^{\frac{1}{2}(2n+m+1-2r-2s)}\times\frac{(m+n-1)\,!\,(m+s)\,!}{2^{r+s}\,r\,!\,s!\,m\,!\,(m+n-r-1)\,!}x_{m+s,\,n+1-r-s}\right\}$$

$$-\frac{(m+n-1)(m+n-2)}{2^2}\,\Sigma\left\{y^s z^{\frac{1}{2}(2n'+m-1-2r'-2s)}\times\frac{(m+n'-2)\,!\,(m+s)\,!}{2^{r'+s}\,r'\,!\,s\,!\,m\,!\,(m+n'-r'-2)\,!}\,x_{m+s,\,n'-r'-s}\right\}$$

$$-\frac{y}{z^{\frac{1}{2}}}\,\frac{(m+1)(m+n-1)}{2^2}\,\Sigma\left\{y^{s''}z^{\frac{1}{2}(2n''+m''-1-2r''-2s'')}\times\frac{(m''+n''-2)\,!\,(m''+s'')\,!}{2^{r''+s''}\,r''\,!\,s''\,!\,m''\,!\,(m''+n''-r''-2)\,!}\,x_{m''+s'',\,n''-r''-s''}\right\}\ldots(53),$$

where

$$n'=n-1,\quad r'=r-2,\quad m''=m+1,\quad n''=n-1,\quad r''=r-1,\quad s''=s-1.$$

Consequently, by (50),

$$\frac{d}{dw}u_{mn}=(n+1)\,z^{\frac{1}{2}(2\,\overline{n+1}+m-1)}e^{\frac{1}{2}z^{-1}(\omega_1+y\Omega_1)}\,x_{m,\,n+1}$$

$$-\frac{(m+n-1)(m+n-2)}{2^2}\,z^{\frac{1}{2}(2\,\overline{n-1}+m-1)}e^{\frac{1}{2}z^{-1}(\omega_1+y\Omega_1)}\,x_{m,\,n-1}$$

$$-\frac{y}{z^{\frac{1}{2}}}\,\frac{(m+1)(m+n-1)}{2^2}\,z^{\frac{1}{2}(2\,\overline{n-1}+\overline{m+1}-1)}e^{\frac{1}{2}z^{-1}(\omega_1+y\Omega_1)}\,x_{m+1,\,n-1}$$

$$=u_{m,\,n+1}-\tfrac{1}{4}\,(m+n-1)\,(m+n-2)\,u_{m,\,n-1}$$

$$-\tfrac{1}{4}yz^{-\frac{1}{2}}(m+1)\,(m+n-1)\,u_{m+1,\,n-1}\ldots\ldots\ldots(54).$$

It would, at first sight, appear as if the second and third members of the right-hand side of (54) are not complete, but need to be reinforced by the addition of terms corresponding to the values -2, -1 for r' in the second, and -1 for r'' and -1 for s'' in the third member on the right of (53). But this is not the case. Values -2, -1 of r' would mean values -1, 0 of r in (51) and (52), for which the term $r(r+1)$ in (52) vanishes. Again, -1 for r'' and -1 for s'' would mean 0 for s and -1 for r, respectively, in (51) and (52), values which make $s(r+1)$ vanish.

A special result of greater simplicity replaces (54) for all cases when $n=0$, the then meaningless symbols $u_{m,\,n-1}$ and $u_{m+1,\,n-1}$ having in such cases to be replaced by zeroes. For, when $n=0$, s and r can be only zero in (50), and consequently in (52) s can only be zero and r only zero or -1; so that the first of the three parts of the right-hand member of (53) or (54) is the only one that exists.

This is easy to see by actual operation on u_{m0} without introduction of the general notation. Thus

$$\frac{d}{dw}u_{m0} = \left(z\frac{d}{dz}+\tfrac{1}{2}y\frac{d}{dy}\right)\left(z^{\frac{1}{2}(m-1)}x_{m0}\right)$$

$$= z^{\frac{1}{2}(m+1)}x_{m1}+\tfrac{1}{2}(m-1)\,z^{\frac{1}{2}(m-1)}x_{m0}+\tfrac{1}{2}yz^{\frac{1}{2}(m-1)}(m+1)\,x_{m+1,0}$$

$$= z^{\frac{1}{2}(m+1)}\left\{1+\frac{1}{2z}\omega_1+\frac{y}{2z}\Omega_1\right\}x_m$$

$$= u_{m1} \qquad\qquad (55).$$

8. The materials for a mathematical induction proving u_{mn} a persistent for the transformation (11), whatever numbers (including zero) m and n be, provided that $m+n \not< 2$, are now ready. By (49) u_{m0} is always a persistent. By (55) so is u_{m1}. Now (54) tells us that, if u_{mn}, $u_{m,\,n-1}$ and $u_{m+1,\,n-1}$ are persistents, $u_{m,\,n+1}$ must be one. Thus, since u_{m1}, u_{m0}, $u_{m+1,\,0}$ are persistents, so is u_{m2}; since u_{m2}, u_{m1}, $u_{m+1,\,1}$ are, so is u_{m3}; since u_{m3}, u_{m2}, $u_{m+1,\,2}$ are, so is u_{m4}, &c. Thus, finally, u_{mn} is one for all values of n as well as for all of m.

That the general persistent u_{mn} is linear, is clear from the method of its formation, ω_1 and Ω_1 being lineo-linear operators. That all the linear persistents u_{mn} are independent is also evident, for in the order u_{20}, u_{11}, u_{02}, u_{30}, u_{21}, u_{12}, u_{03}, u_{40}, u_{31}, &c., each involves one derivative of x which does not appear in any of the preceding. That the system is complete from the second order onwards follows from the fact that up to any order it has just as many members as there are of second and higher derivatives $\dfrac{d^{m+n}u}{dv^m\,dw^n}$ or x_{mn}. With u, $\dfrac{du}{dv}$, and $\dfrac{du}{dw}$, [see (28), (37), and (38)], the system is absolutely complete.

9. The absolutely complete system of linear persistents for the transformation (11), which is now before us, suffices of course to produce by combinations of its members every persistent which exists, non-linear as well as linear. We now enter upon the theory of the formation of non-linear persistents.

It is well known that, if ϑ be any linear differential operator, and U, V any two functions of the arguments on which it operates, and if ϑ be called ϑ_1 when it and its repetitions act on U only, and ϑ_2 when upon V only, then

$$\vartheta^n(UV) = (\vartheta_1+\vartheta_2)^n(UV),$$

for all positive integral values of n, and consequently

$$\begin{aligned} e^{\vartheta}(uv) &= e^{\vartheta_1+\vartheta_2}(uv) \\ &= e^{\vartheta_1}.e^{\vartheta_2}(uv) \\ &= e^{\vartheta_1}u.e^{\vartheta_2}v \\ &= e^{\vartheta}u.e^{\vartheta}v \quad \text{..........................(56).} \end{aligned}$$

Now $\dfrac{1}{2z}(\omega_1+y\Omega_1)$ is such an operator ϑ upon functions of the second and higher derivatives x_{mn}. It follows, from (56) and (48), that

$$\begin{aligned} u_{mn}u_{m'n'} &= z^{\frac{1}{2}(2n+m-1)}e^{\frac{1}{2}z^{-1}(\omega_1+y\Omega_1)}x_{mn}\,z^{\frac{1}{2}(2n'+m'-1)}e^{\frac{1}{2}z^{-1}(\omega_1+y\Omega_1)}x_{m'n'} \\ &= z^{\frac{1}{2}(2n+2n'+m+m'-2)}e^{\frac{1}{2}z^{-1}(\omega_1+y\Omega_1)}(x_{mn}x_{m'n'}) \quad \text{......................(57).} \end{aligned}$$

The right-hand member of this equality is then a persistent.

The extensions to products of 3, 4, ..., and finally all numbers of elementary persistents u_{mn}, is effected in like manner, the general conclusion being that, $\Pi^{(i)}u_{mn}$ denoting a product of i factors u_{mn},

$$\Pi^{(i)}u_{mn} = z^{\frac{1}{2}(2\Sigma n+\Sigma m-i)}e^{\frac{1}{2}z^{-1}(\omega_1+y\Omega_1)}(\Pi^{(i)}x_{mn}) \quad \text{...........(58).}$$

Consequently, taking any sum of a number of such products of i factors for all of which Σm and Σn are constant numbers, w_1 and w_2 respectively, we deduce that, if $H^{(i)}_{w_1w_2}(u)$ denote a homogeneous (of degree i) and doubly isobaric (of partial weights w_1 and w_2) function of the second and higher linear persistents u_{mn}, and if $H^{(i)}_{w_1 w_2}(x)$ denote the same function of the corresponding derivatives x_{mn}, then

$$H^{(i)}_{w_1 w_2}(u) = z^{\frac{1}{2}(2w_2+w_1-i)}\, e^{\frac{1}{2}z^{-1}(\omega_1+y\Omega_1)}\, H^{(i)}_{w_1 w_2}(x) \ldots\ldots\ldots\ldots\ldots (59).$$

The right-hand member is then a persistent. Its sign character will be $+$ or $-$ according as w_2 is even or odd. Thus the identity expressive of its persistency is

$$z^{\frac{1}{2}(2w_2+w_1-i)}\, e^{\frac{1}{2}z^{-1}(\omega_1+y\Omega_1)}\, H^{(i)}_{w_1 w_2}(x) = (-1)^{w_2}\, z'^{\frac{1}{2}(2w_2+w_1-i)}\, e^{\frac{1}{2}z'^{-1}(\omega_1'+y'\Omega_1')}\, H^{(i)}_{w_1 w_2}(x')$$
$$\ldots\ldots\ldots\ldots (60).$$

10. The fundamental linear persistents u_{20}, u_{11}, u_{02}, ... u_{mn}, ... are free from the dependent variable x and the first derivatives x_{10} and x_{01}. The same will consequently be the case with the rational integral persistents of any degree given by (59). On the other hand, the independent variables y and z occur in all the linear persistents, except that y is absent when $n = 0$. As a rule, therefore, both y and z will occur in all the rational and integral persistents (59).

Now y and z occur in different manners in u_{mn}. This linear persistent contains terms free from y and terms involving $y, y^2 \ldots y^n$ respectively as factors. The leading term $z^{\frac{1}{2}(2n+m-1)}x_{mn}$ is among those which have no power of y as a factor. On the other hand, this leading term has for a factor the other independent variable z raised to a power which is never zero but always positive. There is in fact no term in the expression for u_{mn} which has not a positive power of z for a factor, except when $m = 0$, in which case a single term has for its z factor the negative power $z^{-\frac{1}{2}}$. Moreover, no other term involves so high a power of z as the leading one.

We cannot expect, then, to find rational integral persistents (59) which do not involve z. It is now to be seen that there are, however, a vast, and no doubt infinite, number which only involve a single power of z as a factor throughout, and which do not involve y at all. Such will be called rational integral *pure* persistents. By dividing one pure persistent by a suitable power of any other, the z factor may be made to disappear, thus yielding us an *absolute* pure persistent. Absolute pure persistents are of necessity fractional.

Now, in (59), the leading term $H^{(i)}_{w_1 w_2}(x)$ is the one on the right which certainly cannot be made to disappear; for, if it did, so would $H^{(i)}_{w_1 w_2}(u)$, and (59) would be a mere identity of zeroes. We seek then, in order to make H a pure persistent, necessary and sufficient conditions that all the other terms disappear. These conditions are at once seen to be

$$\omega_1 H^{(i)}_{w_1 w_2}(x) = 0, \text{ and } \Omega_1 H^{(i)}_{w_1 w_2}(x) = 0 \ldots\ldots\ldots\ldots (61).$$

The two are necessary, for the coefficients of the various powers and products of powers of the independent quantities z and y must vanish separately, and consequently, in particular, the coefficients of $z^{\frac{1}{2}(2w_2+w_1-i-2)}$ and $yz^{\frac{1}{2}(2w_2+w_1-i-2)}$ must vanish. They are also sufficient,

for, if $$\omega_1 H = 0, \text{ and } \Omega_1 H = 0,$$

it will be a consequence that

$$\omega_1^r H = 0, \quad \omega_1^r \Omega_1^s H = 0, \text{ and } \Omega_1^s H = 0 \ldots\ldots\ldots\ldots (62),$$

for all positive integral values of r and s.

Space will not now permit a systematic classification of the pure persistents for the transformation (11) to which we thus obtain the clue, nor a development of their interesting properties. It will be remembered that we have been only seeking them in order to discuss the selection from them of those which are also cyclicants, *i.e.*, persistents, but for a first derivative factor, for linear transformations of the variables. In the rest of this paper they will then be dealt with only in their bearing on the theory of cyclicants and kindred functions.

11. It is worth while to remark that I first arrived at the necessary and sufficient conditions (61), that a pure function H be, but for a power of z as factor, a persistent for the transformation (11), in a somewhat different manner, by proving that, since

$$x' = z^{-1} x,$$

$$\frac{d}{dy'} = z \frac{d}{dy},$$

and $$\frac{d}{dz'} = -yz \frac{d}{dy} - z^2 \frac{d}{dz},$$

$$(-1)^n x'_{mn} = z^{2n+m-1} e^{(1/z)(\omega_1 + y\Omega_1)} x_{mn} {}^{*} \ldots\ldots\ldots\ldots\ldots (63),$$

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* Comparison of (63) with (48) gives us the following rule for deducing u_{mn}, or rather $(-1)^n 2^{\frac{1}{2}(2n+m-1)} u_{mn}$, from the value of x'_{mn} in terms of variables and unaccented derivatives:—"Divide by the square root of the multiplier of x_{mn}, and in the quotient replace z by $2z$."

For the deduction of x'_{mn} from u_{mn} the rule is—

"Multiply by $(-1)^n$ and by $(4z)^{\frac{1}{2}(2n+m-1)}$, and then replace z by $\frac{1}{2}z$."

and consequently that

$$(-1)^{w_2} H^{(i)}_{w_1 w_2}(x') = z^{2w_2+w_1-i}\, e^{(1/z)(\varpi_1+y\Omega_1)}\, H^{(i)}_{w_1 w_2}(x) \text{ .... } (64).$$

The method is on the whole easier than that above detailed, for, in the work corresponding to that of Art. 7 above, no residual terms such as the $(r+1)(r+s)$ of (52) occur. The advantage of the method which I have here followed lies in its incidental investigation of complete systems of linear and rational integral persistents which are not pure or free from y.

Combination of the two results (60) and (64) leads to an interesting conclusion which might undoubtedly be proved independently and made the basis of a third method. By operating on each side of (64) with $e^{(-1/2z)(\varpi_1+y\Omega_1)}$, and multiplying by $z'^{\frac{1}{2}(2w_2+w_1-i)}$, we get

$$(-1)^{w_2} z'^{\frac{1}{2}(2w_2+w_1-i)}\, e^{(-1/2z)(\varpi_1+y\Omega_1)}\, \Pi^{(i)}_{w_1 w_2}(x') = z^{\frac{1}{2}(2w_2+w_1-i)}\, e^{(1/2z)(\varpi_1+y\Omega_1)}\, \Pi^{(i)}_{w_1 w_2}(x).$$

But, by (60), the right-hand member of this identity is also equal to

$$(-1)^{w_2} z'^{\frac{1}{2}(2w_2+w_1-i)}\, e^{(1/2z')(\varpi_1'+y'\Omega_1')}\, \Pi^{(i)}_{w_1 w_2}(x').$$

We have then the identity of operators

$$e^{(-1/2z)(\varpi_1+y\Omega_1)} = e^{(1/2z')(\varpi_1'+y'\Omega_1')} \text{ .... } (65),$$

the function operated on upon the left being any function of the second and higher derivatives of x with regard to y and z, or, more generally, of any function of those derivatives and of y and z, but not also of x, x_{10}, x_{01}. The transform of such a function is necessarily free from x', x'_{10}, x'_{01}.

12. The results as to the transformation (11) with y instead of x taken as dependent variable are, from the complete symmetry of the formulæ of transformation in x and y, at once seen to be deduced from those already obtained by mere interchange of the letters x and y and of first and second suffixes throughout.

For the purposes of the rest of this paper, it is best to state the equivalent of this fact in a somewhat different way. Companion to the transformation (11), *i.e.*,

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{1} = \frac{1}{z'} \text{ .... } (11),$$

we have the transformation

$$\frac{x}{x''} = \frac{y}{1} = \frac{1}{y''} = \frac{z}{z''} \text{ .... } (66).$$

For this transformation as for the other there is a complete system of linear persistents of which the type, for values of m and n whose sum exceeds unity, is

$$y^{\frac{1}{2}(2m+n-1)}\, e^{(1/2y)(\omega_2+z\Omega_2)}\, x_{mn} \quad\text{.....................}\quad (67),$$

and a complete system of rational integral persistents

$$y^{\frac{1}{2}(2w_1+w_2-i)}\, e^{(1/2y)(\omega_2+z\Omega_2)}\, H^{(i)}_{w_1 w_2}(x) \quad\text{...............}\quad (68),$$

in which ω_2, Ω_2 are the operators (7) and (3), obtained by interchanging first and second suffixes in ω_1 and Ω_1 respectively. The necessary and sufficient conditions that the homogeneous and doubly isobaric function H be a pure persistent for the transformation (66) are then

$$\omega_2 H = 0 \text{ and } \Omega_2 H = 0 \quad\text{.....................}\quad (69).$$

III.

13. We now proceed to the consideration of homogeneous and doubly isobaric functions of the second and higher derivatives, which, as well as being annihilated by one or both of the operators ω_1, ω_2, and one or both of Ω_1, Ω_2, have the further property of being annihilated by one or both of the operators V_1, V_2. Three classes of these functions will occupy our attention. The most restricted class is mentioned first.

A. *Projective or Principiant Cyclicants*, as defined in Art. 1. It is now seen that the necessary and sufficient conditions which they satisfy are that they have all six annihilators ω_1, ω_2, Ω_1, Ω_2, V_1, V_2. In the next article but one, it will be made clear that their annihilation by ω_2 and V_2 is a mere consequence of their annihilation by the other four.

B. *Projective Semicyclicants.*—These possess the three annihilators ω_1, Ω_1, V_1. Their property is that they persist in form, but for a factor involving the variables and first derivatives, after any transformation which can be produced by a succession of transformations like (11), and special linear transformations like

$$\left.\begin{aligned} x &= lX + mY + nZ + p \\ y &= l'X + m'Y + n'Z + p' \\ z &= \qquad\qquad\quad\ n''Z + p'' \end{aligned}\right\} \quad\text{.....................}\quad (70),$$

i.e., after any special homographic transformation, such as

$$\frac{x}{ax'+by'+cz'+d} = \frac{y}{a'x'+b'y'+c'z'+d'} = \frac{z}{c''z'+d''} = \frac{1}{Cz'+D} \ldots (71).$$

C. *Principiant Semicyclicants*, the specially interesting class of projective semicyclicants which have the fourth annihilator ω_2 as well as the three ω_1, Ω_1, V_1.

Some projective and principiant semicyclicants are annihilated by V_2 as well as by the operators, annihilation by which defines them, and have in consequence additional properties. It is probably unwise, however, to burden the subject with further nomenclature.

14. The following important alternant identities are easily verified:—

$$\omega_1\omega_2-\omega_2\omega_1 = 0 \quad\quad (72),$$

$$\omega_1\Omega_1-\Omega_1\omega_1 = 0 \quad\quad (73),$$

$$\omega_2\Omega_2-\Omega_2\omega_2 = 0 \quad\quad (74),$$

$$\omega_1\Omega_2-\Omega_2\omega_1 = \omega_2 \quad\quad (75),$$

$$\omega_2\Omega_1-\Omega_1\omega_2 = \omega_1 \quad\quad (76),$$

$$\omega_1\frac{d}{dy}-\frac{d}{dy}\omega_1 = \Omega_1 \quad\quad (77),$$

$$\omega_2\frac{d}{dz}-\frac{d}{dz}\omega_2 = \Omega_2 \quad\quad (78),$$

$$\omega_1\frac{d}{dz}-\frac{d}{dz}\omega_1 = w_1+2w_2-i \quad\quad (79),$$

$$\omega_2\frac{d}{dy}-\frac{d}{dy}\omega_2 = 2w_1+w_2-i \quad\quad (80),$$

$$\omega_1V_1-V_1\omega_1 = 0 \quad\quad (81),$$

$$\omega_2V_2-V_2\omega_2 = 0 \quad\quad (82),$$

$$\omega_1V_2-V_2\omega_1 = \Sigma\left\{\Sigma\left(\overline{r+s-1}\,x_{rs}\,x_{m-r,\,n-s}\right)\frac{d}{dx_{mn}}\right\}$$

$$= \omega_2V_1-V_1\omega_2 \quad\quad (83),$$

the function operated on being one of second and higher derivatives of x with regard to the independent variables y and z, and in the case of (79) and (80), one which is homogeneous and doubly isobaric. In case of all but the four (77) to (80), the function may involve also the variables.

A close analogy between the properties of ω_1, ω_2 and V_1, V_2 will be observed on comparing many of the above identities with corresponding ones obtained in my previous papers in Vols. XVII., XVIII., XIX. of the *Proceedings*. It may be well to rewrite these here, especially as where they were obtained it was convenient to use x

and y as the independent variables instead of y and z as at present, and as it is now desirable, in order to use the two sets of results in connection, to have them, too, in the present notation before us. They are

$$\Omega_1 V_1 - V_1 \Omega_1 = 0 \quad\quad (84),$$

$$\Omega_2 V_2 - V_2 \Omega_2 = 0 \quad\quad (85),$$

$$\Omega_1 V_2 - V_2 \Omega_1 = V_1 \quad\quad (86),$$

$$\Omega_2 V_1 - V_1 \Omega_2 = V_2 \quad\quad (87),$$

from Vol. xix., p. 9, and

$$V_1 \frac{d}{dy} - \frac{d}{dy} V_1 = 2x_{20}(i+w_1) + x_{11}\Omega_1 \quad\quad (88),$$

$$V_2 \frac{d}{dz} - \frac{d}{dz} V_2 = x_{11}\Omega_2 + 2x_{02}(i+w_2) \quad\quad (89),$$

$$V_1 \frac{d}{dz} - \frac{d}{dz} V_1 = x_{11}(i+w_1) + 2x_{02}\Omega_1 \quad\quad (90),$$

$$V_2 \frac{d}{dy} - \frac{d}{dy} V_2 = 2x_{20}\Omega_2 + x_{11}(i+w_2) \quad\quad (91),$$

$$\Omega_1 \frac{d}{dy} - \frac{d}{dy}\Omega_1 = 0 \quad\quad (92),$$

$$\Omega_2 \frac{d}{dz} - \frac{d}{dz}\Omega_2 = 0 \quad\quad (93),$$

$$\Omega_1 \frac{d}{dz} - \frac{d}{dz}\Omega_1 = \frac{d}{dy} \quad\quad (94),$$

$$\Omega_2 \frac{d}{dy} - \frac{d}{dy}\Omega_2 = \frac{d}{dz} \quad\quad (95),$$

from Vol. xix., p. 380. To these add

$$\Omega_1\Omega_2 - \Omega_2\Omega_1 = w_1 - w_2 \quad\quad (96),$$

and

$$V_1 V_2 - V_2 V_1 = 0 \quad\quad (97),$$

of which last, as it is far from obvious, a proof is appended. The symbolical notation of my paper in the *Proceedings*, Vol. xviii., pp. 142, &c., is used, so that $\eta^r\zeta^s$ in an expanded result means $\frac{d}{dx_{rs}}$.

Let c_{mn} denote the coefficient of $\eta^m\zeta^n$ in

$$(\xi - x_{10}\eta - x_{01}\zeta)^2 = \left\{\Sigma\,(x_{mn}\xi^m\eta^n)\right\}^2,$$

which, for shortness, call μ^2. Then

$$V_1 = \tfrac{1}{2}\frac{d}{d\eta}(\mu^2) \text{ and } V_2 = \tfrac{1}{2}\frac{d}{d\zeta}(\mu^2),$$

therefore $4V_1V_2 = \left\{\Sigma\left[(r+1)\,c_{r+1,s}\,\eta^r\zeta^s\right]\right\}\left\{\Sigma\left[(n+1)\,c_{m,n+1}\,\eta^m\zeta^n\right]\right\}$.

Therefore $\qquad$ co. $\eta^m\zeta^n$ in $4(V_1V_2 - V_2V_1)$

$$= \left\{(n+1)\,\Sigma\left[(r+1)\,c_{r+1,s}\,x_{m-r,n+1-s}\right]\right.$$

$$\left. -(m+1)\,\Sigma\left[(s+1)\,c_{r,s+1}\,x_{m+1-r,n-s}\right]\right\}$$

$$= (n+1) \text{ co. } \eta^m\zeta^{n+1} \text{ in } \frac{d}{d\eta}(\mu^2)\,\mu - (m+1) \text{ co. } \eta^{m+1}\zeta^n \text{ in } \frac{d}{d\eta}(\mu^2)\,\mu$$

$$= \text{co. } \eta^m\zeta^n \text{ in } \left\{\frac{d}{d\zeta}\left(\mu^2\frac{d}{d\eta}\mu\right) - \frac{d}{d\eta}\left(\mu^2\frac{d}{d\zeta}\mu\right)\right\}$$

$$= \text{co. } \eta^m\zeta^n \text{ in } \left(2\mu\frac{d\mu}{d\eta}\frac{d\mu}{d\zeta} + \mu^2\frac{d^2\mu}{d\eta\, d\zeta} - 2\mu\frac{d\mu}{d\zeta}\frac{d\mu}{d\eta} - \mu^2\frac{d^2\mu}{d\eta\, d\zeta}\right)$$

$= 0$, for all values of m and n.

Hence (97) follows. We have thus evaluated all the alternants of pairs of $\frac{d}{dx}$, $\frac{d}{dy}$, ω_1, ω_2, Ω_1, Ω_2, V_1, V_2, the only one not written down above being the elementary identity

$$\frac{d}{dy}\cdot\frac{d}{dz} - \frac{d}{dz}\cdot\frac{d}{dy} = 0.$$

15. A few only of the many conclusions which are involved in the alternant identities before us will be added to those detailed in my former papers.

(i.) *If a function P is annihilated by ω_1, so are $\Omega_1 P$, $V_1 P$, and $\omega_2 P$.*

(ii.) *If a function P is annihilated by ω_2, so are $\Omega_2 P$, $V_2 P$, and $\omega_1 P$.*

(iii.) *If a function P is annihilated by both ω_1 and ω_2, so are both $\Omega_1 P$ and $\Omega_2 P$.*

(iv.) *If a function P is annihilated by V_1, so are $\Omega_1 P$, $\omega_1 P$, and $V_2 P$.*

(v.) *If a function P is annihilated by V_2, so are $\Omega_2 P$, $\omega_2 P$, and $V_1 P$.*

(vi.) *If a function P is annihilated by both V_1 and V_2, so are both $\Omega_1 P$ and $\Omega_2 P$.*

This last is stated as the companion of (iii.), but I have given and applied it in an earlier paper.

(vii.) *If a function P is annihilated by* Ω_1, *so are* V_1P, $\frac{d}{dy}P$, *and* $\omega_1 P$.

(viii.) *If a function P is annihilated by* Ω_2, *so are* V_2P, $\frac{d}{dz}P$, *and* $\omega_2 P$.

(ix.) *If* ω_1 *and* Ω_2 *annihilate a function, so does* ω_2.

(x.) *If* ω_2 *and* Ω_1 *annihilate a function, so does* ω_1.

(xi.) *If* ω_1, V_1, Ω_1, Ω_2 *annihilate a function, so do* ω_2 *and* V_2.

The last fact tells us, as was stated in Article 12, that annihilation by ω_1, V_1, Ω_1, Ω_2 is enough to certify a projective cyclicant.

16. *Generation of seminvariants from other seminvariants and invariants.*

From (vii.) above we learn that, operating on any seminvariant or invariant of the forms

$$\left.\begin{array}{c}(z_{20}, z_{11}, z_{02}\between (u, v)^2 \\ (z_{30}, z_{21}, z_{12}, z_{03}\between (u, v)^3 \\ \&c. \qquad \&c.\end{array}\right\} \qquad (98),$$

ω_1 generates another seminvariant of the system. That the operators V_1 and $\frac{d}{dy}$ have the same property, I have mentioned in earlier papers.

It is to be noticed that the three generators produce from any given seminvariant other seminvariants of quite different types. Thus,

(α) V_1 generates from a given seminvariant another of higher degree, the same weight (second partial weight), and eventually (after a succession of operations) one of a diminished number of the forms.

(β) $\frac{d}{dy}$, y being the first of the two independent variables, generates one of the same degree, of the same weight (second partial weight), and of an increased number of forms.

(γ) ω_1 generates one of the same degree, *diminished* weight (second partial weight), and eventually (upon repeated application) one of a diminished number of forms.

I hope on some future occasion to deal with these and other simple generators of seminvariants, *e.g.*, separate parts of the above generators, at greater length.

17. *Generation of pure persistents for the transformation* (11) *from others.*

If ω_1 and Ω_1 annihilate a pure function P, then P is a pure persistent. Now, by (77) and (92), ω_1 and Ω_1 under these circumstances annihilate also $\frac{dP}{dy}$; and, by (72) and (76), they also annihilate $\omega_2 P$. Consequently $\frac{d}{dy}$ and ω_2, *i.e.*, ω_2 and

$$\frac{d}{dy} = \Sigma \left\{ (m+1)\, x_{m+1,n} \frac{d}{dx_{mn}} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (99),$$

are generators of pure persistents for the transformation (11) from other such persistents.

18. *Generators of projective semicyclicants from other projective semicyclicants or cyclicants.*

If ω_1, Ω_1, and V_1 annihilate a pure function P, it is a projective semicyclicant; or a projective cyclicant if it is also annihilated by Ω_2, *i.e.*, if its two partial weights are equal.

Now (77) and (92) tell us, as in the last article, that ω_1 and Ω_1 must also annihilate $\frac{dP}{dy}$; and (88) tells us that, if only $i+w_1 = 0$, V_1 will also annihilate $\frac{dP}{dy}$. Consequently, the operator $\frac{d}{dy}$, applied to a projective semicyclicant or cyclicant the sum of whose degree and first partial weight vanishes, generates another projective semicyclicant. That from any semicyclicant it generates a semicyclicant I have previously shown (*Proceedings*, Vol. XIX., p. 382).

Now, $P(i, w_1, w_2)$ being any projective cyclicant or semicyclicant of type i, w_1, w_2, since x_{20} is another, it follows that

$$\frac{P(i, w_1, w_2)}{x_{20}^{\frac{1}{3}(i+w_1)}}$$

is one whose degree and first partial weight have a vanishing sum. It follows that

$$\frac{d}{dy}\, \frac{P(i, w_1, w_2)}{x_{20}^{\frac{1}{3}(i+w_1)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (100)$$

is another projective semicyclicant, which when multiplied by $x_{20}^{\frac{1}{3}(i+w_1)+1}$ produces one that is integral, its type being $(i+1, w_1+3, w_2)$.

But

$$\frac{d}{dy}\left\{\frac{P(i, w_1, w_2)}{x_{20}^{\frac{1}{2}(i+w_1)}}\right\} = \frac{1}{x_{20}^{\frac{1}{2}(i+w_1)+1}}\left\{x_{20}\frac{d}{dy} - \tfrac{1}{2}(i+w_1)\,3x_{30}\right\} P(i, w_1, w_2)$$

$$= \frac{1}{x_{20}^{\frac{1}{2}(i+w_1)+1}}\left\{x_{20}\Sigma\left[(m+1)\,x_{m+1,n}\frac{d}{dx_{mn}}\right]\right.$$

$$\left. - x_{30}\Sigma\left[(m+1)\,x_{mn}\frac{d}{dx_{mn}}\right]\right\} P(i, w_1, w_2)$$

$$= \frac{1}{x_{20}^{\frac{1}{2}(i+w_1)+1}}\Sigma\left\{(m+1)(x_{20}x_{m+1,n} - x_{30}x_{mn})\frac{d}{dx_{mn}}\right\} P(i, w_1, w_2).$$

Thus the quadro-linear operator

$$G = \Sigma\left\{(m+1)(x_{20}x_{m+1,n} - x_{30}x_{mn})\frac{d}{dx_{mn}}\right\} \quad \ldots\ldots\ldots\ldots(101)$$

is a generator of *projective* semicyclicants from other *projective* semicyclicants and cyclicants. That it generates semicyclicants from semicyclicants, is practically shown in the passage above cited of my paper in Vol. XIX.

Two classes of projective semicyclicants have simpler generators from other semicyclicants of the same classes. These follow.

19. *Generator of projective semicyclicants having the additional property of being annihilated by V_2 from other semicyclicants of the same kind.*

If $\omega_1 P = 0$, $\Omega_1 P = 0$, $V_1 P = 0$, $V_2 P = 0$, it follows from (76) that $\Omega_1 . \omega_2 P = 0$, from (72) that $\omega_1 . \omega_2 P = 0$, from (83) that $V_1 . \omega_2 P = 0$, and from (82) that $V_2 . \omega_2 P = 0$. Thus ω_2 is a generator of semicyclicants of the kind described in the heading from other semicyclicants of the same kind.

If P be of type (i, w_1, w_2), $\omega_2 P$ is of type $(i, w_1 - 1, w_2)$. Thus, in particular, if w_1 exceeds w_2 by unity, we may expect a projective cyclicant to be generated.

20. *Generator of principiant semicyclicants from principiant semicyclicants or from cyclicants.*

Principiant semicyclicants have been defined as those projective semicyclicants which ω_2, as well as ω_1, Ω_1, V_1, annihilates.

V_2 is a generator of such from such or from projective cyclicants; for, as in the last article, we see, by means of (82), (86), (83), and (97), that if ω_1, ω_2, Ω_1, V_1 all annihilate a pure function P, they all annihilate $V_2 P$.

If P is of type (i, w_1, w_2), $V_2 P$ is of type $(i+1, w_1, w_2+1)$.

21. *Projective and principiant cocyclicants.*

A cocyclicant is, it will be remembered, a covariant of the cyclicogenitive forms

$$\left.\begin{aligned} E_2 &= (z_{20}, z_{11}, z_{02} \between (-z_{01}, z_{10})^2 \\ E_3 &= (z_{30}, z_{21}, z_{12}, z_{03} \between (-z_{01}, z_{10})^3 \\ E_4 &= (z_{40}, z_{31}, z_{22}, z_{13}, z_{04} \between (-z_{01}, z_{10})^4 \\ &\&c. \qquad\qquad \&c. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(102),$$

whose leading coefficient is a semicyclicant; and it has been shown (Vol. xix., pp. 379 and 20) that, $(S_0, S_1, \ldots S_m)(-z_{01}, z_{10})^m$ being a cocyclicant,

$$\frac{S_0(x, yz)}{x_{01}^{i+w_1}} = (-1)^m \frac{S_m(y, zx)}{y_{10}^{i+w_1}} = (-1)^{i+w_1} (S_0, S_1, \ldots S_m)(-z_{01}, z_{10})^m \quad \ldots\ldots\ldots\ldots\ldots(103),$$

$S_0(x, yz)$ being the corresponding semicyclicant (in x dependent).

A *projective* cocyclicant is now defined as one whose leading coefficient is a *projective* semicyclicant, and in particular a *principiant* cocyclicant as one whose leading coefficient is a *principiant* semicyclicant.

In proving that every projective, or in particular principiant, semicyclicant in x persists, but for a factor involving first derivatives and variables only, after the special homographic transformation (71), we have, in virtue of (103), proved the same property of persistence to belong to all projective, and in particular principiant, cocyclicants in z.

22. *Some instances of projective cyclicants, semicyclicants, &c.*

It is *à priori* clear, and has been noticed by Halphen, that in the criteria of developable and ruled surfaces we must have two projective or principiant pure cyclicants, persisting for all homographic transformations.

The two are
$$x_{20}x_{02} - \tfrac{1}{4}x_{11}^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(104),$$

and
$$\begin{vmatrix} x_{30}, & x_{21}, & x_{12}, & x_{03} & \\ & x_{30}, & x_{21}, & x_{12}, & x_{03} \\ x_{20}, & x_{11}, & x_{02} & & \\ & x_{20}, & x_{11}, & x_{02} & \\ & & x_{20}, & x_{11}, & x_{02} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(105).$$

Both have been seen in earlier papers to be annihilated by Ω_1, Ω_2, V_1, V_2. The further conditions, that ω_1 and (therefore also) ω_2 annihilate them, are easily verified. ω_1, for instance, produces the fourth row of (105) from the first, and the fifth from the second.

The operator G, (101), annihilates (104). By repeated operation on (105), it will however produce a succession of projective semicyclicants.

23. The simplest of all semicyclicants is x_{20}. This is annihilated by ω_1, ω_2, and V_2. It is, then, a principiant semicyclicant having the additional property of being annihilated by V_2. Consequently, the quadratic cyclicogenitive form

$$(z_{20}, z_{11}, z_{02}\between(-z_{01}, z_{10})^2 = -\frac{x_{20}}{x_{01}^3} \quad\text{..................(106)}$$

is a principiant cocyclicant with the additional V_2 property.

The other cyclicogenitive forms are not cocyclicants at all.

The generators G, ω_2, V_2 produce nothing from the principiant semicyclicant x_{20}.

24. Of the other simple semicyclicants and cocyclicants given in my last paper (Vol. XIX., pp. 392—398), most are projective and indeed principiant. This is not the case with $3x_{30}x_{11}-2x_{20}x_{21}$, which ω_1 does not annihilate. It is easily verified, however, that the semicyclicant of Vol. XIX., p. 395 (59), viz., $G(3x_{30}x_{11}-2x_{20}x_{21})$, or

$$Q = \begin{vmatrix} 2x_{20}, & x_{11} & \\ 3x_{30}, & x_{21}, & x_{20} \\ 4x_{40}, & x_{31}, & 2x_{30} \end{vmatrix} \quad\text{.......................(107),}$$

is annihilated by both ω_1 and ω_2. This, then, is a principiant semicyclicant, and the corresponding covariant in z of the cyclicogenitive forms, viz.,

$$\left(Q,\ \frac{1}{6}\Omega_2 Q,\ \frac{1}{6.5}\Omega_2^2 Q,\ \dots\ \frac{1}{6!}\Omega_2^6 Q\right)(-z_{01}, z_{10})^6 \quad\text{......(108),}$$

is a principiant cocyclicant.

G, G^2, G^3, ..., operating on (107), produce other projective (not necessarily principiant) semicyclicants of higher degree and first partial weight. Again, V_2, V_2^2, V_2^3, ... operating upon it produce a limited number of other principiant semicyclicants.

The family of surfaces whose criterion is Q, or the left-hand member of (108), has been alluded to in Vol. XIX., pp. 395, 396. It is now seen that the family includes more than has been there stated, viz., in fact, surfaces which cut planes parallel to $z = 0$ in curves in perspective, and not merely in homothetic curves.

25. The semicyclicant $\begin{vmatrix} x_{30}, & x_{20} & \\ x_{21}, & x_{11}, & x_{20} \\ x_{12}, & x_{02}, & x_{11} \end{vmatrix}$ ....(109),

of Vol. XIX., p. 397, § 18, or p. 13 (10), is annihilated by ω_1 and ω_2 as well as Ω_1, V_1, and V_2. It is then a principiant semicyclicant having the extra property of being annihilated by V_2, and its corresponding cocyclicant in z,

$$\begin{vmatrix} z_{30}, & z_{20}, & & z_{10}^3 \\ z_{21}, & z_{11}, & z_{20}, & 3z_{10}^2 z_{01} \\ z_{12}, & z_{02}, & z_{11}, & 3z_{10} z_{01}^2 \\ z_{03}, & & z_{02}, & z_{01}^3 \end{vmatrix} \quad \text{.....................(110),}$$

is a principiant cocyclicant whose leading coefficient is annihilated by V_2.

Operation on (109) with G, G^2, G^3, ... produces other projective, not shown to be principiant, semicyclicants. The other generators ω_2, V_2 are here annihilators.

26. Of the semicyclicants and cocyclicants discussed in Vol. XIX., pp. 398—405, which are of second partial weight zero, and one of which is obtained from every Sylvesterian pure reciprocant, all are projective, being annihilated by ω_1, and not merely those which are obtained from projective or principiant reciprocants. These last have the property of being also annihilated by ω_2. It is this fact which has led me to distinguish between two classes of projective semicyclicants, and call the more restricted class which ω_2 annihilates by Professor Sylvester's second name *principiant*.

The present conclusion may be stated concisely by saying that homogeneous and isobaric functions of the cyclicogenitive forms E_2, E_3, E_4, ..., which have the annihilator

$$4\frac{E_2^2}{2}\frac{d}{dE_3} + 5E_2E_3\frac{d}{dE_4} + 6\left(E_2E_4 + \tfrac{1}{2}E_3^2\right)\frac{d}{dE_5} + 7\left(E_2E_5 + E_3E_4\right)\frac{d}{dE_6} + \dots$$

..............(107),

are *projective* cocyclicants; and that *principiant* cocyclicants which are functions of E_2, E_3, E_4, ... have also the annihilator

$$E_2\frac{d}{dE_3}+2E_3\frac{d}{dE_4}+3E_4\frac{d}{dE_5}+\dots \quad\dots\dots\dots\dots\dots(108).$$

27. *Formation of projective or principiant cyclicants.*

It will now be proved that—*Every invariant of a principiant cocyclicant whose semicyclicant source is also annihilated by V_2, considered as a quantic in $-z_{01}$, z_{10}, is a projective cyclicant.*

If S be the semicyclicant source of such a cocyclicant, it is annihilated by ω_1, ω_2, Ω_1, V_1, V_2. $\Omega_2 S$ has then the annihilators ω_1, ω_2, V_1, V_2, by (75), (74), (87), (85). $\Omega_2^2 S$ has the same annihilators by the same identities, and so on. Thus all the coefficients of the cocyclicant have the annihilators ω_1, ω_2, V_1, V_2, and consequently any function of them has the same. Now, an invariant of the cocyclicant, being an invariant of the cyclicogenitive forms of which that cocyclicant is a covariant, is annihilated by Ω_1 and Ω_2. The invariant has then all the annihilators ω_1, ω_2, V_1, V_2, Ω_1, Ω_2, and is consequently a projective cyclicant—a persistent differential invariant for the general homographic transformation.

It may happen that a seminvariant of the cocyclicant occurs which is annihilated by Ω_2, through breaking up into factors, one of which is a function of second partial weight zero, and the other an invariant of the cyclicogenitive forms. Such seminvariants, with the first factor rejected, also give projective cyclicants.

We have examples of cocyclicants which yield principiant cyclicants in this manner in the quadratic cyclicogenitive form and in (110). For their discussion, see Vol. XIX., p. 16.

Other principiant cyclicants with the property in question are to be expected to be produced from any homogeneous and doubly isobaric forms of which ω_1, ω_2, V_1, V_2 are annihilators, as in Vol. XIX., p. 14.

I am not sure that this method will be very productive.

28. The following two propositions afford other methods for the production of projective cyclicants.

(*a*) *If S be a projective semicyclicant of type i, w_1, w_2, which is annihilated by V_2 as well as by ω_1, Ω_1, V_1, then $\omega_2^{w_1-w_2}S$ is a projective cyclicant.*

(*b*) *If S' be a principiant semicyclicant, i.e., one annihilated by ω_2 as well as by ω_1, Ω_1, V_1, of type i, w_1, w_2, then $V_2^{w_1-w_2}S'$ is a projective cyclicant.*

By Art. 19, if ω_1, Ω_1, V_1, V_2 annihilate S, they also annihilate $\omega_2 S$, $\omega_2^2 S$, $\omega_2^3 S$, &c., &c. Now, since the operation ω_2 diminishes w_1, by unity and leaves w_2 unaltered, the two partial weights of $\omega_2^{w_1-w_2} S$ are equal. Consequently, by (96), Ω_2 annihilates $\omega_2^{w_1-w_2} S$; and hence also, by (75), so does ω_2. $\omega_2^{w_1-w_2} S$ is then a projective cyclicant.

Again, if ω_1, ω_2, Ω_1, V_1 annihilate S', they annihilate $V_2 S'$, by Art. 20. In like manner they annihilate $V_2^2 S'$, $V_2^3 S'$, &c., &c. Now, V_2 increases w_2 by unity, and does not alter w_1. Thus the two partial weights of $V_2^{w_1-w_2} S'$ are equal; so that, by (96), Ω_2 annihilates $V_2^{w_1-w_2} S'$, and consequently so does V_2, by (87). Thus $V_2^{w_1-w_2} S'$ is a projective cyclicant.

The only possibility which may interfere with the success of these methods is that $\omega_2^{w_1-w_2}$ and $V_2^{w_1-w_2}$ be annihilators under the circumstances considered. I cannot see, however, that the complete system of alternant identities (72)—(96) gives us any reason for fearing this to be generally the case.*

29. One more method for the systematic calculation of projective semicyclicants and cyclicants will be exhibited. It has been shown (Vol. XIX., pp. 22, 23) how to obtain all the linearly independent pure cyclicants of a given type $\left(i, \frac{w}{2}, \frac{w}{2}\right)$; and the method has been extended (Vol. XIX., p. 379) to the obtaining of all the linearly independent semicyclicants of a given type (i, w_1, w_2). It has been proved, also, that a superior limit to the, and it may be the exact, number of these pure cyclicants or semicyclicants is the excess of the number of seminvariants of type (i, w_1, w_2) of the cyclicogenitive forms over the number of type $(i+1, w_1+1, w_2)$.

Now, let S_1, S_2, S_3, ... be a complete system of semicyclicants, or pure cyclicants of the type (i, w_1, w_2) thus determined, and let S_1', S_2', S_3', ... be a complete system of type (i, w_1, w_2-1). By (73) and (84), the operation of ω_1 on a semicyclicant generates another semicyclicant. Thus we must have

$$\omega_1(a_1 S_1 + a_2 S_2 + a_3 S_3 + \dots) \dots\dots\dots\dots\dots\dots (109),$$

* Should it be the case, we are none the less helped in our search for projective cyclicants. Suppose, for instance, that $\omega_2^m S = 0$ or $V_2^m S' = 0$, m being not greater than $w_1 - w_2$. Then $\omega_2^{m-1} S$ or $V_2^{m-1} S'$, as the case may be, is annihilated by ω_1, ω_2, V_1, V_2, Ω_1, in virtue of Art. 18 or 19. Consequently $\omega_2^{m-1} S$, or $V_2^{m-1} S'$, as the case may be, is the leading coefficient of such a cocyclicant as has, by Art. 26, the property that any one of its invariants is a projective cyclicant.

equal to such an expression as

$$(la_1+ma_2+na_3+\ldots)\,S_1'+(l'a_1+m'a_2+n'a_3+\ldots)\,S_2'$$
$$+(l''a_1+m''a_2+n''a_3+\ldots)\,S_3'+\ldots.$$

For ω_1 to be an annihilator of the sum on which it operates in (109), we must have simultaneously

$$la_1+ma_2+na_3+\ldots=0,$$
$$l'a_1+m'a_2+n'a_3+\ldots=0,$$
$$l''a_1+m''a_2+n''a_3+\ldots=0, \text{ \&c. \&c.},$$

a number of equations for the determination of $a_1, a_2, a_3, \ldots$ less than the number of those coefficients by the excess of the number of $S_1, S_2, \ldots$ over the number of $S_1', S_2', \ldots$.

By this means are found all the functions of the type (i, w_1, w_2) which ω_1 as well as Ω_1 and V_1 annihilate, *i.e.*, all the projective semicyclicants of the type. If $w_1=w_2$, then, by (96), Ω_2 is also an annihilator, and the functions are projective cyclicants.

If $N(i, w_1, w_2)$ denote the number of linearly independent seminvariants of type (i, w_1, w_2) of the cyclicogenitive forms, it is thus seen that the number of linearly independent projective semicyclicants of the type, or cyclicants if $w_1=w_2$, is likely, though not certain, to prove to be

$$N(i, w_1, w_2)-N(i+1, w_1+1, w_2)-N(i, w_1, w_2-1)$$
$$+N(i+1, w_1+1, w_2-1) \quad\ldots\ldots\ldots\ldots(110).$$

30. The resemblance of the operators $\omega_1, \omega_2, \Omega_1, \Omega_2$ to Mr. Forsyth's $\Delta_1, \Delta_2, \Delta_4, \Delta_3$ which, annihilate functions of the derivatives that are invariantic for homographic transformations of the independent variables only, is striking, but must not mislead. The distinction between ω_1, ω_1 and Δ_4, Δ_3, though at first sight slight, is essential. It is a remarkable conclusion from one of Mr. Forsyth's theorems that none of his invariants can be pure cyclicants, for all of them involve first derivatives. The important memoir here referred to is published in the *Phil. Trans.*, Vol. CLXXX., pp. 71—118.