III. (e). $d = a$, $e = h$: $x^2 = ax$, $xy = ay$, $yx = hx$, $y^2 = hy$; or, writing as we may do $a=1$, $h=1$, this is $x^2=x$, $xy=y$; $yx=x$; $y^2=y$. Introducing here the new symbol $z, = x-y$, we have $z^2=0$, $xz=z$, $zx=0$, $yz=z$, $zy=0$. Thus x, z form the system $x^2=x$, $xz=z$, $zx=0$, $z^2=0$ (or, what is the same thing, y, z form a system $y^2=y$, $yz=z$, $zy=0$, $z^2=0$); each of these is Peirce's system (b_3).

The conclusion is that every non-commutative associative system is either Peirce's system (b_3), or else the omitted system (d_3). Hence, disregarding the mixed systems, every associative system is either (a_3), (b_3), (c_3), or (d_3).

19. It may be proper to show that the systems (b_3), $x^2=x$, $xy=y$, $yx=0$, $y^2=0$, and (d_3), $x^2=x$, $xy=0$, $yx=y$, $y^2=0$, or say

	a	b	c	d	e	f	g	h
(b_3)	1	0	0	1	0	0	0	0
(d_3)	1	0	0	0	0	1	0	0

are really distinct from each other. Observe that they each belong to the case 10, $\Omega = 0$, an infinity of idems and 1 nil; viz., in each of them writing $z = x + \beta y$, β an arbitrary coefficient, we have $z^2 = x^2 + \beta(xy + yx) + \beta^2 y^2 = x + \beta y = z$, we have z an idem, and y is the only nil. And, this being so, we have in the first system $zy = y$, $yz = 0$, viz., the system is $z^2 = z$, $zy = y$, $yz = 0$, $y^2 = 0$, retaining, when we write z for x , its original form. And similarly, in the second system, $zy = 0$, $yz = y$; viz., the system is $z^2 = z$, $zy = 0$, $yz = y$, $y^2 = 0$, retaining, when we write therein z for x , its original form. The two are thus distinct systems, in no wise transformable the one into the other.

On Electrical Oscillations and the effects produced by the motion of an Electrified Sphere. By J. J. THOMSON, M.A., Fellow and Assistant Lecturer of Trinity College, Cambridge.

[Read April 3rd, 1884.]

In this paper two problems are discussed which, though physically different, are yet capable of solution by almost the same mathematical treatment.

The first problem treats of the vibrations which take place in the electrical distribution on the surface of a thin spherical shell when the

cause which produced an irregular distribution over the surface is suddenly withdrawn. The electrical oscillations, which occur when the oppositely charged plates of a condenser are connected, are due to the interchange of energy between the kinetic energy possessed by the currents in the wire joining the plates of the condenser and the potential energy of the electricity distributed over these plates. In the problem we are considering, the oscillations are due to the interchange of energy between the currents which start off to equalize the distribution of electricity and the potential energy of this distribution. It is necessary to distinguish between two cases. The first is when the electrical system which produced the irregular distribution over the shell is inside it. In this case, if the shell be made of a perfectly conducting material, there will be neither magnetic nor electro-motive forces outside it; thus all the energy will be kept inside the shell, so that, were it not for the resistance of the shell, the vibrations would last for ever. In the second case, the electrical system which produces the distribution is outside the shell. In this case there will be neither magnetic nor electro-motive forces inside the shell, but a magnetic field will be produced outside the shell, and energy will continually be propagated from the shell into the surrounding dielectric. It is proved in the following paper, that the dissipation of the energy of the electrical distribution due to this cause is so great that the whole of the energy will be dissipated in the course of a few oscillations.

he period of the oscillations in both cases is investigated, and it is proved that the time of oscillation is comparable with the time taken by light to cross over the diameter of the spherical shell. In particular, if a distribution of electricity over the shell represented by the first zonal harmonic be produced by an internal electrical system, the period of vibration equals the time taken by light to pass over a distance of 1.14 times the diameter of the shell. In this case the magnetic forces inside the shell might be made sufficiently large to be easily measurable, so that this result seems to afford a promising way of testing Maxwell's theory experimentally. The very rapid dissipation of the energy by dissipation into the surrounding medium forbids us to expect that a collection of small particles could be made luminous by electrical means; the vibrations would not last long enough for us to see them.

It would require a very sudden disturbance to produce these oscillations, as their period is very small. Supposing, for example, the disturbance to be produced by the motion of an electrified point, to give rise to these oscillations it would have to move with a velocity comparable with that of light. We might get a sudden disturbance, however, by increasing the charge on a conductor until the potential became so great that a spark passed; the time occupied in

the discharge would be small, and we might expect the electricity on the neighbouring conductors to vibrate.

We might get, by means of suitable electrical systems, vibrations of all periods, from those of Light to those of Sound.

The case of a solid sphere is not given in this paper, because the energy due to any distribution of electricity on its surface is very rapidly dissipated into the surrounding dielectric, just as in the case of a spherical shell excited by an external electrical system.

In the second part of this paper, the effects produced by the motion of an originally uniformly electrified sphere are discussed. I considered this problem in a paper published in the *Philosophical Magazine*, for April 1881, neglecting the self-induction of the displacement currents; in this paper, however, these are taken into account, and the sphere is supposed to be of finite size. The most interesting results of this investigation are, that, whether the velocity of the sphere be uniform or variable, the charged sphere produces the same magnetic force as unit length of a current whose intensity is ve and which flows in the direction of motion of the sphere,— v being the velocity of, and e the charge on, the sphere. If the sphere is vibrating in a magnetic field perpendicularly to the direction of the magnetic induction, the force on the sphere is perpendicular both to its direction of motion and the magnetic induction, and

$$= evb \left\{ \frac{2}{3} \frac{\mu}{\mu'} - 1 \right\},$$

where b is the magnetic induction, and μ and μ' the coefficients of magnetic permeability of the dielectric and the sphere respectively. It is also proved that the distribution of electricity on the surface of the sphere will no longer remain uniform, but that, if the velocity be changing very rapidly, the negative electricity will be heaped around the extremity of the radius of the sphere which is parallel to its acceleration, while, if the velocity of the sphere be small, it will be heaped around the extremity of the radius parallel to the direction of motion. These effects are, of course, only very small for all practicable velocities and accelerations. The original charge on the sphere is supposed to be positive.

In the problems we are about to consider, a spherical conductor charged with electricity is placed in the midst of an insulating medium which stretches to infinity in all directions. In the dielectric there will be what Maxwell calls electric displacements, and, if these vary for any reason, their variations produce the same effects as ordinary electric currents; thus they will give rise to magnetic forces, possess self-induction, and induce currents in the substance of the conductor. The investigation of these effects is the object of the following paper.

The following is the notation we shall employ :— f , g , h are the com-

ponents of electric displacement parallel to the axes of x, y, z respectively; u, v, w are the components of the electric current in the conductor.

F, G, H , are the components of the vector potential of magnetic induction.

K is the specific inductive capacity of the dielectric, σ is the specific resistance of the conductor, μ and μ' are the coefficients of magnetic permeability of the dielectric and the conductor.

ϕ is the potential due to the electricity in the field.

The equations satisfied by these quantities in the dielectric are

$$\frac{4\pi}{K} f = -\frac{dF}{dt} - \frac{d\phi}{dx} \dots\dots\dots (1),$$

$$\nabla^2 F = -4\pi\mu \frac{df}{dt} \dots\dots\dots (2),$$

with symmetrical equations for g, h, G, H ; also

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 \dots\dots\dots (3),$$

since there is no free electricity in the dielectric.

We have also
$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0 \dots\dots\dots (4).$$

In the conductor the equations are

$$\sigma u = -\frac{dF}{dt} - \frac{d\phi}{dx} \dots\dots\dots (5),$$

with symmetrical equations for v and w .

$$\nabla^2 F = -4\pi\mu'u \dots\dots\dots (6),$$

with symmetrical equations for G and H

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0,$$

and

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = -\frac{d\epsilon}{dt} \dots\dots\dots (7),$$

where ϵ is the volume density of the free electricity in the conductor.

Thus F may be looked upon as the potential due to a distribution of matter of density $\mu'u$ throughout the conductor and $\mu' \frac{df}{dt}$ throughout the dielectric. Thus F, G, H , and all their differential coefficients will be finite and continuous as long as $u, v, w, \frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt}$ are finite.

We notice also that, if there is no electricity except on the surface of the conductor,

$$\nabla^2 \phi = 0.$$

Differentiating equation (1) with respect to t , we get

$$\frac{4\pi}{K} \frac{df}{dt} = -\frac{d^2 F}{dt^2} - \frac{d^2 \phi}{dx dt} \dots\dots\dots(8),$$

or, substituting for $\frac{df}{dt}$ its value from equation (2), we have

$$-\frac{1}{\mu K} \nabla^2 F = -\frac{d^2 F}{dt^2} - \frac{d^2 \phi}{dt dx} \dots\dots\dots(9).$$

We shall suppose that all the functions concerned vary as e^{ipt} where p may be composite. With this supposition, equation (9) becomes

$$-\frac{1}{\mu K} \nabla^2 F = p^2 F - ip \frac{d\phi}{dx} \dots\dots\dots(10).$$

Since $\nabla^2 \phi = 0$, a particular solution of this differential equation is

$$F = \frac{i}{p} \frac{d\phi}{dx}.$$

The complementary function is the solution of the differential equation

$$(\nabla^2 + p^2 \mu K) F = 0,$$

or, writing $\lambda^2 = p^2 \mu K$,

$$(\nabla^2 + \lambda^2) F = 0 \dots\dots\dots(11).$$

Let F vary as y_n , the surface harmonic of the n^{th} order. Then, if $F = Y_n F'$, where F' is a function of r the distance of a point from the centre of the spherical conductor, we can easily prove that F' must satisfy the differential equation

$$\frac{d^2 F'}{dr^2} + \frac{2}{r} \frac{dF'}{dr} + \left(\lambda^2 - \frac{n \cdot n + 1}{r^2} \right) F' = 0 \dots\dots\dots(12).$$

This equation has been discussed by various writers. The following are references to some of the more important papers on the subject:—

Stokes, "On the Communication of Vibrations from a Vibrating Body to a surrounding Gas." *Phil. Trans.*, 1868.

Lord Rayleigh, "Theory of Sound." Chap. xvii., vol. II.

Professor C. Niven, "On the Conduction of Heat in Ellipsoids of Revolution." *Phil. Trans.* Part I. 1880.

Professor C. Niven, "On the Induction of Electric Currents in Infinite Plates and Spherical Shells." *Phil. Trans.* Part II. 1881.

Professor H. Lamb, "On the Vibrations of an Elastic Sphere," and "On the Oscillations of a Viscous Spheroid." *Proc. Lond. Math. Soc.*, 1882.

Professor H. Lamb, "On Electrical Motions in a Spherical Conductor." *Phil. Trans.* Part II. 1883.

V. Helmholtz, *Collected Papers*, s. 320, Vol. I.

Heine, *Kugelfunctionen*, s. 240, Vol. I.

The differential equation (12) is satisfied by any of the following functions

$$(\lambda r)^n \left(\frac{1}{\lambda r} \frac{d}{d \cdot \lambda r} \right)^n \frac{\sin \lambda r}{\lambda r}$$

which we shall denote by $S_n(\lambda r)$,

$$\left. \begin{aligned} &(\lambda r)^n \left(\frac{1}{\lambda r} \frac{d}{d \cdot \lambda r} \right)^n \frac{\cos \lambda r}{\lambda r} \\ &(i\lambda r)^n \left(\frac{1}{i\lambda r} \frac{d}{d \cdot i\lambda r} \right)^n \frac{e^{i\lambda r}}{i\lambda r} \\ &(i\lambda r)^n \left(\frac{1}{i\lambda r} \frac{d}{d \cdot i\lambda r} \right)^n \frac{e^{-i\lambda r}}{i\lambda r} \end{aligned} \right\} \dots\dots\dots (13).$$

The first of these is the only one which remains finite when r vanishes, so it is the solution which must be taken whenever r can vanish.

To determine which solution must be taken outside the sphere when r cannot vanish but can become infinite, we notice that, since the currents fade away, owing to the resistance of the conductor, p will equal $\alpha + i\beta$ where β is positive, therefore λ is of the form $\alpha' + i\beta'$ where β' is positive; now the third of the above solutions is the only one which does not become infinite when r is infinite if λ is of this form. So that, outside the sphere,

$$A (i\lambda r)^n \left(\frac{1}{i\lambda r} \frac{d}{d \cdot (i\lambda r)} \right)^n \frac{e^{i\lambda r}}{i\lambda r}$$

is the solution for our purpose of the differential equation (12), and this will still be the case when we neglect the resistance of the conductor, as we consider this as a particular case of the more general problem where the resistance is taken into account.

In the conductor we have, by substituting in (5) the value of u from (6),

$$\frac{-\sigma}{4\pi\mu} \nabla^2 F = -\frac{dF}{dt} - \frac{d\phi}{dx},$$

or, since F varies as e^{ipt} ,

$$\frac{-\sigma}{4\pi\mu} \nabla^2 F = -ipF - \frac{d\phi}{dx} \dots\dots\dots (14);$$

since $\nabla^2 \phi = 0$, a particular solution of this differential equation is

$$F = -\frac{1}{ip} \frac{d\phi}{dx}.$$

The complementary function is the solution of the equation

$$(\nabla^2 + \lambda'^2) F = 0 \dots\dots\dots (15),$$

if

$$\lambda'^2 = \frac{4\pi ip\mu'}{\sigma}.$$

Thus, if $F = Y_n F'$, where Y_n is a surface harmonic, $F' = A' S_n(\lambda' r)$, since F' is finite when $r = 0$.

Let us now consider a case where the mathematical conditions are as simple as possible. Let us suppose that we have a very thin spherical shell made of some material which is a good conductor, and that over this surface the distribution of electricity can be initially expressed by a zonal harmonic of the n^{th} degree Q_n , and that the cause which produced this distribution is suddenly withdrawn, we shall now investigate what will happen to the electricity on the surface of the shell.

We shall suppose that the electrical distribution on the shell can still be expressed by $f(t) Q_n$, where $f(t)$ is a function of the time. We shall find that we can satisfy all the conditions of the problem with this assumption.

Let us take the axis of the zonal harmonic Q_n as the axis of z . Let us suppose, as before, that all the quantities involved in this problem vary as $e^{i\mu t}$. Then, outside the sphere, ϕ can be written

$$A e^{i\mu t} \frac{d^n}{dz^n} \frac{1}{r} \dots \dots \dots (16).$$

Let $(i\lambda r)^n \left(\frac{d}{i\lambda r \cdot d i\lambda r} \right) \frac{e^{i\mu r}}{i\lambda r}$ be denoted by $E_n(\lambda r)$. Since the quantities F, G, H satisfy equations of the form

$$\nabla^2 F + \lambda^2 F = i p \mu K \frac{d\phi}{dz},$$

one solution of these equations is, if B be an arbitrary constant,

$$F = \frac{iA}{p} e^{i\mu t} \frac{d^{n+1}}{dz^n dx} \frac{1}{r} + B e^{i\mu t} E_{n+1}(\lambda r) r^{n+2} \frac{d^{n+1}}{dz^n dx} \frac{1}{r} \dots \dots \dots (17),$$

with symmetrical expressions for G and H ; these values however do not satisfy the condition

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0.$$

If, however, we add to F the term

$$(-1)^{n+1} B e^{i\mu t} \frac{n+1!}{n} \frac{E_{n-1}(\lambda r)}{r^{n-1}} \frac{d}{dx} (r^n Q_n),$$

and make similar additions to the expressions for G and H , we get

$$F = \frac{iA}{p} e^{i\mu t} \frac{d^{n+1}}{dz^n dx} \frac{1}{r} + B e^{i\mu t} E_{n+1}(\lambda r) r^{n+2} \frac{d^{n+1}}{dz^n dx} \frac{1}{r} \\ + (-1)^n B \frac{(n+1)!}{n} e^{i\mu t} \frac{E_{n-1}(\lambda r)}{r^{n-1}} \frac{d}{dx} (r^n Q_n) \dots (18),$$

$$G = \frac{iA}{p} \epsilon^{ipt} \frac{d^{n+1}}{dz^n dy} \frac{1}{r} + B \epsilon^{ipt} E_{n+1}(\lambda r) r^{n+2} \frac{d^{n+1}}{dz^n dy} \frac{1}{r} \\ + (-1)^n B \frac{(n+1)!}{n} \epsilon^{ipt} \frac{E_{n-1}(\lambda r)}{r^{n-1}} \frac{d}{dy} (r^n Q_n) \dots (19),$$

$$H = \frac{iA}{p} \epsilon^{ipt} \frac{d^{n+1}}{dz^{n+1}} \frac{1}{r} B \epsilon^{ipt} E_{n+1}(\lambda r) r^{n+2} \frac{d^{n+1}}{dz^{n+1}} \frac{1}{r} \\ + (-1)^n \frac{B(n+1)!}{n} \epsilon^{ipt} \frac{E_{n-1}(\lambda r)}{r^{n-1}} \frac{d}{dz} (r^n Q_n) \dots (20).$$

We can prove that these expressions for F , G , H satisfy the condition

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0,$$

for, since $\nabla^2 \frac{1}{r} = 0$, the first terms in the expression for F , G , H satisfy this condition; for the same reason the second terms give

$$B \epsilon^{ipt} \frac{d}{dr} [E_{n+1}(\lambda r) r^{n+2}] \frac{d}{dr} \frac{d^n}{dz^n} \frac{1}{r} \\ = (-1)^{n+1} B (n+1)! \frac{Q_n}{r^{n+2}} \epsilon^{ipt} \frac{d}{dr} [E_{n+1}(\lambda r) r^{n+2}] \dots (21),$$

and, since $\nabla^2 (r^n Q_n) = 0$, the third terms give

$$(-1)^{n+1} B (n+1)! r^{n-1} Q_n \frac{d}{dr} \left(\frac{E_{n-1}(\lambda r)}{r^{n-1}} \right) \dots \dots \dots (22).$$

Now

$$E_n(\lambda r) = (i\lambda r)^n \left(\frac{1}{i\lambda r} \frac{d}{d(i\lambda r)} \right)^n \frac{\epsilon^{i\lambda r}}{i\lambda r},$$

and therefore

$$= (i\lambda r)^{n-1} \frac{d}{d \cdot i\lambda r} \frac{E_{n-1}(\lambda r)}{(i\lambda r)^{n-1}} = \frac{r^{n-1}}{i\lambda} \frac{d}{dr} \frac{E_{n-1}(\lambda r)}{r^{n-1}}.$$

Thus

$$E_{n+1}(\lambda r) = (i\lambda r)^n \frac{d}{d \cdot i\lambda r} \frac{E_n(\lambda r)}{(i\lambda r)^n} \dots \dots \dots (23),$$

therefore

$$\frac{d}{d(i\lambda r)} E_{n+1}(\lambda r) + \frac{n+2}{i\lambda r} E_{n+1}(\lambda r) \\ = (i\lambda r)^n \left\{ \left(\frac{d}{d \cdot i\lambda r} \right)^2 \frac{E_n(\lambda r)}{(i\lambda r)^n} + \frac{2(n+1)}{i\lambda r} \frac{d}{d \cdot i\lambda r} \frac{E_n(\lambda r)}{(i\lambda r)^n} \right\} \\ = E_n(\lambda r) \dots \dots \dots (24),$$

since equation (12) can be written

$$\left(\frac{d}{d(i\lambda r)} \right)^2 \frac{E_n(\lambda r)}{(i\lambda r)^n} + 2(n+1) \frac{d}{d(i\lambda r)} \frac{E_n(\lambda r)}{(i\lambda r)^n} - \frac{E_n(\lambda r)}{(i\lambda r)^n} = 0.$$

Equation (24) can be written

$$\frac{d}{d(i\lambda r)} \{ (i\lambda r)^{n+2} E_{n+1}(\lambda r) \} = - (i\lambda r)^{n+2} E_n(\lambda r),$$

or
$$\frac{d}{dr} \{r^{n+2} E_{n+1}(\lambda r)\} = -i\lambda \cdot r^{n+2} E_n(\lambda r) \dots\dots\dots(25);$$

hence, by equation (23),

$$\frac{d}{dr} \{r^{n+2} E_{n+1}(\lambda r)\} = -r^{2n+1} \frac{d}{dr} \frac{E_{n-1}(\lambda r)}{r^{n-1}} \dots\dots\dots(26);$$

hence, from (21) and (22), we see that

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0.$$

In connection with these equations we may remark that

$$\frac{d}{dx} (r^n Q_n) = -x r^{n-2} \frac{dQ_{n-1}}{d\mu} \dots\dots\dots(27),$$

$$\frac{d}{dy} (r^n Q_n) = -y r^{n-2} \frac{dQ_{n-1}}{d\mu} \dots\dots\dots(28),$$

$$\frac{d}{dz} (r^n Q_n) = n r^{n-1} Q_{n-1} \dots\dots\dots(29).$$

Inside the spherical shell,

$$\phi = (-1)^n n! A e^{i\mu t} \frac{r^n}{a^{2n+1}} Q_n \dots\dots\dots(30),$$

where a is the radius of the shell.

In the dielectric inside the shell, the differential equations are the same as outside. Since, however, F , G , H remain finite when $r = 0$, we must use the functions $S_n(\lambda r)$ instead of $E_n(\lambda r)$. Thus, inside the shell, we shall have

$$F = (-1)^n \frac{iA}{p} \frac{n!}{a^{2n+1}} e^{i\mu t} \frac{d}{dx} (r^n Q_n) + C e^{i\mu t} S_{n+1}(\lambda r) r^{n+2} \frac{d^{n+1}}{dz^n dx} \frac{1}{r} \\ + (-1)^{n+1} O \frac{(n+1)!}{n} e^{i\mu t} \frac{S_{n-1}(\lambda r)}{r^{n-1}} \frac{d}{dx} (r^n Q_n) \dots\dots\dots(31),$$

$$G = (-1)^n \frac{iA}{p} \frac{n!}{a^{2n+1}} e^{i\mu t} \frac{d}{dy} (r^n Q_n) + C e^{i\mu t} S_{n+1}(\lambda r) r^{n+2} \frac{d^{n+1}}{dz^n dy} \frac{1}{r} \\ + (-1)^{n+1} O \frac{(n+1)!}{n} e^{i\mu t} \frac{S_{n-1}(\lambda r)}{r^{n-1}} \frac{d}{dy} (r^n Q_n) \dots\dots\dots(32),$$

$$H = (-1)^n \frac{iA}{p} \frac{n!}{a^{2n+1}} e^{i\mu t} \frac{d}{dz} (r^n Q_n) + C e^{i\mu t} S_{n+1}(\lambda r) r^{n+2} \frac{d^{n+1}}{dz^{n+1}} \frac{1}{r} \\ + (-1)^{n+1} O \frac{(n+1)!}{n} e^{i\mu t} \frac{S_{n-1}(\lambda r)}{r^{n-1}} \frac{d}{dz} (r^n Q_n) \dots\dots\dots(33).$$

Since the electrical distribution on the shell keeps changing, there must be surface currents to carry the electricity from one part of the sphere to another, let q be the intensity of the current at any time,

then $2\pi a \sin \theta q$ is the quantity of electricity which in unit time crosses the small circle of colatitude θ , so that, if σ be the surface density of the electricity at any point of the shell, we have

$$a \frac{d}{d\theta} (q \sin \theta) d\phi d\theta + a^2 \sin \theta d\theta d\phi \frac{d\sigma}{dt} = 0,$$

therefore
$$\frac{d}{d\theta} (q \sin \theta) + a \sin \theta \frac{d\sigma}{dt} = 0.$$

Now
$$\sigma = (-1)^n \frac{2n+1}{4\pi} \frac{n!}{a^{n+2}} A e^{i\mu t} Q_n \dots\dots\dots (34),$$

therefore
$$\frac{d}{d\theta} (q \sin \theta) + a \sin \theta (-1)^n \frac{2n+1}{4\pi} \frac{n!}{a^{n+2}} i\mu \cdot K A e^{i\mu t} Q_n = 0,$$

therefore
$$q \sin \theta = \frac{(-1)^n 2n+1 \cdot n!}{4\pi \cdot a^{n+1}} i\mu \cdot K A e^{i\mu t} \int_1^n Q_n d\mu,$$

therefore
$$q = \frac{(-1)^{n+1} (2n+1)(n-1)!}{4\pi \cdot n+1 \cdot a^{n+1}} i\mu \cdot K A e^{i\mu t} \sin \theta \frac{dQ_n}{d\mu} \dots\dots (35),$$

where $\mu = \cos \theta$.

To find the conditions which must be satisfied by F, G, H , we must remember that F, G, H are the potentials due to distribution of matter whose volume density is finite everywhere, except at the surface of the shell where the surface densities are $\mu q \cos \theta \cos \phi$, $\mu q \cos \theta \sin \phi$, $-\mu q \sin \theta$ respectively. Thus F, G, H are continuous at the surface,

and
$$\left(\frac{dF}{dr}\right) \text{ inside} - \left(\frac{dF}{dr}\right) \text{ outside} = 4\pi \mu q \cos \theta \cos \phi \dots\dots (36),$$

with symmetrical equations for G and H .

The continuity of F, G, H requires that

$$\frac{iA}{p} + B a^{n+2} E_{n+1}(\lambda a) = C a^{n+2} S_{n+1}(\lambda a) \dots\dots\dots (37),$$

and
$$-(n+1) B E_{n+1}(\lambda a) = -\frac{iA}{p} \frac{n}{a^{n+2}} + (n+1) C S_{n+1}(\lambda a) \dots\dots (38).$$

Equation (36) gives, since

$$\begin{aligned} (2n+1) \sin \theta \cos \theta \cos \phi \frac{dQ_n}{d\mu} \\ = (-1)^{n+1} \frac{1}{n-1!} a^{n+2} \frac{d^{n+1}}{dx^n dx} \frac{1}{r} - \frac{n+1}{a^{n-1}} \frac{d}{dx} (r^n Q_n) \dots\dots (39), \end{aligned}$$

$$\begin{aligned} n \cdot n-1 \cdot \frac{iA}{p} - (n+1) C a^{n+2} \frac{d}{da} S_{n+1}(\lambda a) - (n+1) B a^{n+2} \frac{d}{da} E_{n+1}(\lambda a) \\ = \mu i p K a^3 A \dots\dots\dots (40), \end{aligned}$$

$$\text{and} \quad (n+2) \frac{iA}{p} + Ca^{n+3} \frac{d}{da} S_{n+1}(\lambda a) + Ba^{n+3} \frac{d}{da} E_{n+1}(\lambda a) \\ = \frac{1}{n+1} ip\mu Ka^3 A \dots\dots\dots(41).$$

Equations (37), (38), (40), and (41) are not independent, but any two of them may be deduced from the other two by the help of the relations (23) and (25). The electromotive force along a tangent to the spherical shell in a meridian plane

$$= \cos \theta \cos \phi \left(-\frac{dF}{dt} - \frac{d\phi}{dx} \right) + \cos \theta \sin \phi \left(-\frac{dG}{dt} - \frac{d\phi}{dy} \right) \\ - \sin \theta \left(-\frac{dH}{dt} - \frac{d\phi}{dz} \right) \dots\dots\dots(42),$$

and this equals σq if σ be the resistance of the shell. If σ be very small compared with pa , σq will be very small, and may be taken equal to zero, and the electromotive force along the tangent in this case must vanish. Substituting for F , G , and H their values from equations (31), we see that the electromotive force is

$$(-1)^{n+1} n! ip C \left(S_{n+1}(\lambda a) - \frac{n+1}{n} S_{n-1}(\lambda a) \right) \frac{dQ_n}{d\theta} \dots\dots\dots(43),$$

and for this to vanish, supposing C not to be zero,

$$S_{n+1}(\lambda a) - \frac{n+1}{n} S_{n-1}(\lambda a) = 0 \dots\dots\dots(44).$$

This equation determines λ and therefore p , the frequency of the vibration executed by the distribution of electricity. Since the values of λa or $\sqrt{\mu K} pa$ given by this equation are finite, and since $\sqrt{\mu K}$ is approximately equal to the time taken by light to travel the unit of length, we see that the time of vibration is comparable with the time taken by light to pass over a diameter of the spherical shell. This result justifies the neglect of σ/pa in comparison with the expression (42). The electrical vibrations which take place when the opposite plates of a condenser are connected by a coil whose coefficient of self-induction is considerable, are very much slower than the vibrations we have just been considering, and it would be theoretically possible to arrange a series of electrical systems whose times of vibration would range from those of light to those of sound. In order to get a numerical result, let us suppose $n=1$, so that the density of the electricity on the shell varies as $\cos \theta$; equation (44) gives

$$S_2(\lambda a) = 2 S_0(\lambda a) \dots\dots\dots(45),$$

$$\text{or} \quad -\frac{\sin(\lambda a)}{\lambda a} - \frac{3 \cos \lambda a}{(\lambda a)^2} + \frac{3 \sin \lambda a}{(\lambda a)^3} = \frac{2 \sin \lambda a}{\lambda a},$$

$$\text{or} \quad \dots \tan \lambda a = \frac{\lambda a}{1 - \lambda^2 a^2} \dots (46).$$

I find, by the method of trial and error, that the least root of this equation other than zero [which does not satisfy (45)] is 2.745. There are of course an infinite number of other roots greater than this; the relative importance of the vibrations corresponding to the various roots will depend upon the initial disposition of the field.

The time of vibration corresponding to the root 2.745 is the same as the time of vibration of a ray of light whose wave-length is 1.14 times the diameter of the shell. Since the time of vibration is so small, any disturbance which can give rise to the vibrations must be very sudden.

Still considering the case $n = 1$, we find, from equations (37) and (38), that when C is not zero B must vanish; thus, in this case, there are neither magnetic nor electromotive forces outside the shell. This corresponds to the case when the electrified system which initially produced the irregular distribution of electricity over the shell was inside the shell; when this system is suddenly discharged, the electricity which was on the shell vibrates, but no magnetic or electromotive forces are produced outside the shell; the efforts of the displacement currents in the dielectric inside the shell to produce such forces are just counterbalanced by the action of the current in the spherical shell. Since there is no magnetic force outside the shell, the system does not lose any of its energy by propagation into the surrounding dielectric, so that, if it were not for the resistance of the shell, the vibrations would last for ever.

Since $B = 0$, we find, from equation (37), that

$$C = \frac{iA}{pS_1(\lambda a) a^3} \dots (47),$$

or, as $S_1(\lambda a) = 2 S_0(\lambda a)$,

$$= \frac{iA}{2p a^3 S_0(\lambda a)} \dots (48),$$

when $\lambda a = 2.745$ and $S_0(\lambda a) = \frac{\sin \lambda a}{\lambda a}$.

Substituting this value of λa , we find

$$C = -\frac{7.1 \cdot iA}{2pa^3} = \frac{2\pi i \times 7.1 \sigma}{3Kp} \dots (50),$$

if σ be the maximum surface density of the electricity on the spherical shell.

Now, inside the shell,

$$\left. \begin{aligned} F &= C e^{i p t} S_2(\lambda r) r^3 \frac{d^2}{dx dz} \frac{1}{r} \\ G &= C e^{i p t} S_2(\lambda r) r^3 \frac{d^2}{dy dz} \frac{1}{r} \\ H &= C e^{i p t} S_2(\lambda r) r^3 \frac{d^2}{dz^2} \frac{1}{r} + 2C e^{i p t} S_0(\lambda r) - \frac{iA}{pa^3} e^{i p t} \end{aligned} \right\} \dots\dots (51).$$

Therefore, if α, β, γ be the components of the magnetic force, substituting these values for H and G in the equation

$$\mu \alpha = \frac{dH}{dy} - \frac{dG}{dz},$$

we get

$$\mu \alpha = 3C e^{i p t} \frac{y}{r} \frac{d}{dr} S_0(\lambda r).$$

Substituting for C and taking the real part, we get

$$\begin{aligned} \alpha &= -\frac{14.2\pi\sigma}{p\mu K} \sin pt \frac{y}{r} \frac{dS_0(\lambda r)}{dr} \\ &= -\frac{14.2\pi\sigma}{\sqrt{\mu K}} \sin pt \frac{y}{r} \frac{dS_0(\lambda r)}{d\lambda r} \\ &= -\frac{14.2\pi\sigma}{\sqrt{\mu K}} \sin pt \frac{y}{r} S_1(\lambda r) \dots\dots\dots (52). \end{aligned}$$

$$\begin{aligned} \text{Similarly} \quad \beta &= \frac{14.2\pi\sigma}{\sqrt{\mu K}} \sin pt \frac{x}{r} S_1(\lambda r) \dots\dots\dots (53), \\ \gamma &= 0. \end{aligned}$$

So that the resultant magnetic force is perpendicular to the meridian plane and

$$= \frac{14.2\pi\sigma}{\sqrt{\mu K}} \sin pt S_1(\lambda r) \dots\dots\dots (54).$$

This corresponds to a stationary vibration, and the magnetic force vanishes when $S_1(\lambda r) = 0$. This force could easily be made large enough to measure if its value were stationary; it is the very rapid changes in its value which would make the observations difficult. It seems, however, to afford a promising way of testing Maxwell's theory.

Equation (43) can also be satisfied by making C vanish. This corresponds to the case when the electrified system which produced the irregular distribution of electricity was outside the shell. Since $C = 0$, there are neither electromotive nor magnetic forces inside the shell, so that the currents in the shell produce inside it equal and opposite effects to the displacement currents outside.

In this case, energy is propagated from the shell through the medium

so that the electrical distribution on the shell gradually dies away. Hence we see that the value of λ will be no longer wholly real.

We see, from equations (36) and (37), that the values of λ are given by the equation

$$E_2(\lambda a) + 2E_0(\lambda a) = 0 \dots\dots\dots(55),$$

or

$$\epsilon^{i\lambda a} \left\{ \frac{i}{(\lambda a)^3} - \frac{i}{\lambda a} - \frac{1}{\lambda^2 a^2} \right\} = 0;$$

therefore

$$\lambda a = \frac{i}{2} \pm \sqrt{\frac{3}{4}} \dots\dots\dots(56),$$

therefore

$$p = \frac{iv}{2a} \pm \sqrt{\frac{3}{4}} \frac{v}{a}, \text{ when } v = 1/\sqrt{\mu K},$$

therefore

$$\epsilon^{ip t} = \epsilon^{-\frac{v}{2a} t} e^{\pm i \sqrt{\frac{3}{4}} \frac{v}{a} t}$$

Since the real negative power to which ϵ is raised is so large, the energy will all be dissipated into space in the course of a few vibrations, so that we could not expect any measurable result in this case.

We shall now proceed to investigate the case of a metal sphere charged with electricity and vibrating in a dielectric, and we shall suppose that the motion is confined to the conductor, and that the sphere moves through the dielectric without putting it in motion. This supposition is, of course, not quite accurate, but it does not lead to any error of importance, and it simplifies the equations very much. Let the centre of the sphere vibrate along the axis of z , and let the period of vibration be $2\pi/p$, so that the velocity of the centre of the sphere is $\omega e^{ip t}$. Then, if we take rectangular axes passing through the centre of the sphere and moving with it, using the same notation as before, the following equations are true inside the sphere :

$$\sigma u = -b\omega e^{ip t} - \frac{dF}{dt} - \frac{d\psi}{dx} \dots\dots\dots(57),$$

$$\sigma v = a\omega e^{ip t} - \frac{dG}{dt} - \frac{d\psi}{dy} \dots\dots\dots(58),$$

$$\sigma w = -\frac{dH}{dt} - \frac{d\psi}{dz} \dots\dots\dots(59);$$

therefore

$$-\frac{\sigma}{4\pi\mu'} \nabla^2 F = -b\omega e^{ip t} - \frac{dF}{dt} - \frac{d\psi}{dx} \dots\dots\dots(60),$$

$$-\frac{\sigma}{4\pi\mu'} \nabla^2 G = a\omega e^{ip t} - \frac{dG}{dt} - \frac{d\psi}{dy} \dots\dots\dots(61),$$

$$-\frac{\sigma}{4\pi\mu'} \nabla^2 H = -\frac{dH}{dt} - \frac{d\psi}{dz} \dots\dots\dots(62),$$

where a, b, c are the components of magnetic induction.

In the dielectric outside the sphere, the equations are, if $\frac{\delta}{\delta t}$ denote partial differentiation with respect to t ,

$$\frac{4\pi}{K}f = -\frac{dF}{dt} - \frac{d\psi}{dx} = -\left(\frac{\delta}{\delta t} - \omega\epsilon^{ip t} \frac{d}{dz}\right)F - \frac{d\psi}{dz} \dots\dots\dots (63),$$

$$\frac{4\pi}{K}g = -\frac{dG}{dt} - \frac{d\psi}{dy} = -\left(\frac{\delta}{\delta t} - \omega\epsilon^{ip t} \frac{d}{dz}\right)G - \frac{d\psi}{dy} \dots\dots\dots (64),$$

$$\frac{4\pi}{K}h = -\frac{dH}{dt} - \frac{d\psi}{dz} = -\left(\frac{\delta}{\delta t} - \omega\epsilon^{ip t} \frac{d}{dz}\right)H - \frac{d\psi}{dz} \dots\dots\dots (65),$$

therefore $-\frac{1}{\mu K} \nabla^2 F = -\left(\frac{\delta}{\delta t} - \omega\epsilon^{ip t} \frac{d}{dz}\right)^2 F - \left(\frac{\delta}{\delta t} - \omega\epsilon^{ip t}\right) \frac{d\psi}{dz} \dots\dots (66),$

with similar equations for G and H .

From the form of these equations we see that, if there is no magnetic field except that produced by the moving sphere, the solution will take the form

$$\begin{aligned} \psi &= \psi_0 + \psi_1 \epsilon^{ip t} + \psi'_1 \epsilon^{-ip t} + \psi_2 \epsilon^{2ip t} + \psi'_2 \epsilon^{-2ip t} + \&c., \\ F &= F_1 \epsilon^{ip t} + F'_1 \epsilon^{-ip t} + F_2 \epsilon^{2ip t} + F'_2 \epsilon^{-2ip t} + \dots \end{aligned}$$

If we substitute these values in the equations, we see that we may put $\psi'_1, \psi'_2, \psi'_3, \dots F'_1, F'_2, \&c.$ all equal to zero.

If e be the quantity of electricity on the sphere,

$$\psi_0 = \frac{e}{Kr} \dots\dots\dots (67).$$

Equating the coefficient of $\epsilon^{ip t}$ to zero in equations (60) and (66), we

get $-\frac{\sigma}{4\pi\mu} \nabla^2 F_1 = -\frac{dF_1}{dt} - \frac{d\psi_1}{dx}$, inside the sphere $\dots\dots\dots (68),$

with similar equations for G_1 and H_1 , and outside the sphere we have

$$-\frac{1}{\mu K} \nabla^2 F_1 = -\frac{\delta^2 F_1}{\delta t^2} - \frac{d^2 \psi_1}{dx dt} + \frac{\omega e}{K} \frac{d^2}{dx dz} \frac{1}{r} \dots\dots\dots (69),$$

with similar equations for G_1 and H_1 .

The form of equation (69) suggests that we should put

$$\psi_1 = B \frac{d}{dz} \frac{1}{r} \dots\dots\dots (70).$$

The particular integral of equation (69) is

$$F_1 = \frac{iB}{p} \frac{d^2}{dx dz} \frac{1}{r} - \frac{\omega e}{Kp^3} \frac{d^2}{dx dz} \frac{1}{r}.$$

The complementary function is that solution of the differential equation $\nabla^2 F_1 + \mu K p^2 F_1 = 0 \dots\dots\dots (71)$

which, when considered as a function of the angular coordinates of

the position of a point, varies as $r^3 \frac{d^2}{dx dz} \frac{1}{r}$; it must also correspond to a wave propagated outwards from the sphere; hence we see that, since the time enters through the factor $e^{i\mu t}$, the solution is of the form

$$O(i\lambda r)^3 r^3 \frac{d^2}{dx dz} \frac{1}{r} \left\{ \frac{1}{i\lambda r} \frac{d}{d(i\lambda r)} \right\}^2 \frac{e^{-i\lambda r}}{i\lambda r} \dots\dots\dots (72),$$

where

$$\lambda^2 = \mu K p^2.$$

We shall denote this solution by

$$O\mathfrak{G}_2(i\lambda r) r^3 \frac{d^2}{dx dz} \frac{1}{r}.$$

Thus

$$F_1 = Oe^{i\mu t} \mathfrak{G}_2(i\lambda r) r^3 \frac{d^2}{dx dz} \frac{1}{r} + \frac{iBe^{i\mu t}}{p} \frac{d^2}{dx dz} \frac{1}{r} - \frac{\omega e e^{i\mu t}}{Kp^2} \frac{d^2}{dx dz} \frac{1}{r} \dots (73),$$

$$G_1 = Ce^{i\mu t} \mathfrak{G}_2(i\lambda r) r^3 \frac{d^2}{dy dz} \frac{1}{r} + \frac{iBe^{i\mu t}}{p} \frac{d^2}{dy dz} \frac{1}{r} - \frac{\omega e e^{i\mu t}}{Kp^2} \frac{d^2}{dy dz} \frac{1}{r} \dots (74),$$

$$H_1 = Ce^{i\mu t} \mathfrak{G}_2(i\lambda r) r^3 \frac{d^2}{dz^2} \frac{1}{r} - 2O\mathfrak{G}_0(i\lambda r) + \frac{iBe^{i\mu t}}{p} \frac{d^2}{dz^2} \frac{1}{r} - \frac{\omega e e^{i\mu t}}{Kp^2} \frac{d^2}{dz^2} \frac{1}{r} \dots (75),$$

where $\mathfrak{G}_0(i\lambda r) = \frac{e^{-i\lambda r}}{i\lambda r}$, and is introduced to make the expressions for F_1 , G_1 , H_1 satisfy the condition

$$\frac{dF_1}{dx} + \frac{dG_1}{dy} + \frac{dH_1}{dz} = 0.$$

Inside the sphere, the differential equations for F_1 and G_1 are of the form

$$-\frac{\sigma}{4\pi\mu} \nabla^2 F_1 = -ipF_1 \dots\dots\dots (76),$$

if $\lambda_1^2 = -4\pi ip\mu/\sigma$. The solution of this equation is

$$F_1 = De^{i\mu t} S_2(\lambda_1 r) r^3 \frac{d^2}{dx dz} \frac{1}{r} \dots\dots\dots (77),$$

where

$$S_2(\lambda_1 r) = (\lambda_1 r)^2 \left(\frac{1}{\lambda_1 r} \frac{d}{d(\lambda_1 r)} \right)^2 \frac{\sin \lambda_1 r}{\lambda_1 r},$$

as this function satisfies the differential equation, and does not become infinite when $r = 0$. Similarly,

$$G_1 = De^{i\mu t} S_2(\lambda_1 r) r^3 \frac{d^2}{dy dz} \frac{1}{r} \dots\dots\dots (78),$$

the differential equation for H_1 is

$$-\frac{\sigma}{4\mu} \nabla^2 H_1 = -ipH_1 + \frac{B}{a^2} \dots\dots\dots (79),$$

therefore $H_1 = D e^{i p t} S_2(\lambda_1 r) r^3 \frac{d^2}{dz^2} \frac{1}{r} + 2 D e^{i p t} S_0(\lambda_1 r) + \frac{B}{i p a^3} e^{i p t} \dots (80),$

where $S_0(\lambda_1 r) = \frac{\sin \lambda_1 r}{\lambda_1 r}$, and is introduced for the same reason as $\mathfrak{G}_0(i \lambda r)$ in the value of H_1 outside the sphere.

Since the quantities F_1, G_1, H_1 are continuous, we may equate the expressions for their values outside and inside when $r = a$; therefore

$$C \mathfrak{G}_2(i \lambda a) - D S_2(\lambda_1 a) + \frac{i B}{p a^3} = \frac{\omega e}{p^3 a^3 K} \dots \dots \dots (81),$$

$$-2 C \mathfrak{G}_0(i \lambda a) - 2 D S_0(\lambda_1 a) + \frac{i B}{p a^3} = 0 \dots \dots \dots (82).$$

Since there are no surface currents in this case, the differential coefficients of F_1, G_1, H_1 with regard to r are continuous; hence we get

$$C \frac{d \mathfrak{G}_2(i \lambda a)}{da} - D \frac{d S_2(\lambda_1 a)}{da} - \frac{3 i B}{p a^4} = - \frac{3 \omega}{p^3 a^4} \frac{e}{K} \dots \dots \dots (83),$$

$$-2 C \frac{d \mathfrak{G}_0(i \lambda a)}{da} - 2 D \frac{d S_0(\lambda_1 a)}{da} = 0 \dots \dots \dots (84).$$

We can show, as before, that equations (81), (82), (83), and (84) are not independent. Taking the first, second, and fourth of these, we find

$$C \Delta = - \frac{\omega e}{p^3 a^3 K} \frac{d S_0(\lambda a)}{da} \dots \dots \dots (85),$$

$$+ D \Delta = \frac{\omega}{p^3 a^3} \frac{e}{K} \frac{d \mathfrak{G}_0(i \lambda a)}{da} \dots \dots \dots (86),$$

$$\frac{i B}{p a^3} \Delta = - 2 \frac{\omega}{p^3 a^3} \frac{e}{K} \left\{ \mathfrak{G}_0(i \lambda a) \frac{d S_0(\lambda_1 a)}{da} - S_0(\lambda_1 a) \frac{d \mathfrak{G}_0(i \lambda a)}{da} \right\} \dots (87),$$

where Δ denotes the determinant

$$\begin{vmatrix} \mathfrak{G}_2(i \lambda a), & S_2(\lambda_1 a), & 1 \\ -2 \mathfrak{G}_0(i \lambda a), & 2 S_0(\lambda_1 a), & +1 \\ -\frac{d \mathfrak{G}_0(i \lambda a)}{da}, & \frac{d S_0(\lambda_1 a)}{da}, & 0 \end{vmatrix}.$$

Since $\lambda^3 = \mu K p^3$ and $\mu K = 1/v^2$, where v is approximately equal to the velocity of light, we see that, unless the sphere be vibrating so rapidly that its period is comparable with the time light takes to travel over a diameter of the sphere, λa will be very small.

$$\text{Now,} \quad \mathfrak{G}_2(i \lambda a) = \frac{3 e^{-i \lambda a}}{(i \lambda a)^3} + \frac{3 e^{-i \lambda a}}{(i \lambda a)^2} + \frac{e^{-i \lambda a}}{i \lambda a},$$

$$\mathfrak{G}_0(i \lambda a) = \frac{e^{-i \lambda a}}{i \lambda a},$$

and, as $i \lambda a$ is very small, the first term of $\mathfrak{G}_2(i \lambda a)$ is very large com-

pared with the others, and also with $\mathfrak{C}_0(i\lambda a)$ and $\frac{d\mathfrak{C}_0(i\lambda a)}{da}$. Thus

$$\text{approximately} \quad \Delta = - \frac{3e^{-i\lambda a}}{(i\lambda a)^3} \frac{dS_0(\lambda_1 a)}{da} \dots\dots\dots(88).$$

$$\text{Therefore} \quad C = \frac{1}{3} (i\lambda a)^3 e^{i\lambda} \frac{\omega e}{p^3 a^3 K} \dots\dots\dots(89).$$

$$\text{The term} \quad 2C\mathfrak{C}_0(i\lambda r) e^{ipt} = \frac{2}{3} \frac{\omega e (i\lambda a)^3}{p^3 a^3 K} \frac{e^{i[pt-\lambda(r-a)]}}{i\lambda r} \dots\dots\dots(90).$$

$$\begin{aligned} \text{The term } O\mathfrak{C}_2(i\lambda r) e^{ipt} \\ = \frac{1}{3} \frac{\omega e (i\lambda a)^3}{p^3 a^3 K} e^{i[pt-\lambda(r-a)]} \left\{ \frac{3}{(i\lambda r)^3} + \frac{3}{(i\lambda r)^2} + \frac{1}{i\lambda r} \right\} \dots\dots(91). \end{aligned}$$

Thus, outside the sphere,

$$\begin{aligned} F_1 = \frac{1}{3} \frac{\omega e (i\lambda a)^3}{p^3 a^3 K} e^{i[pt-\lambda(r-a)]} \left\{ \frac{3}{(i\lambda r)^3} + \frac{3}{(i\lambda r)^2} + \frac{1}{(i\lambda r)} \right\} r^3 \frac{d^3}{dx dz} \frac{1}{r} \\ + \frac{iB}{p} \frac{d^3}{dx dz} \frac{1}{r} - \frac{\omega e}{p^3 a^3 K} \frac{d^3}{dx dz} \frac{1}{r} \dots\dots\dots(92), \end{aligned}$$

with a similar expression for G_1 ;

$$\begin{aligned} H_1 = \frac{1}{3} \frac{\omega e (i\lambda a)^3}{p^3 a^3 K} e^{i[pt-\lambda(r-a)]} \left\{ \frac{3}{(i\lambda r)^3} + \frac{3}{(i\lambda r)^2} + \frac{1}{i\lambda r} \right\} r^3 \frac{d^3}{dz^3} \frac{1}{r} \\ - \frac{2}{3} \frac{\omega e (i\lambda a)^3}{p^3 a^3 K} \frac{e^{i[pt-\lambda(r-a)]}}{i\lambda r} + \frac{iB}{p} \frac{d^3}{dz^3} \frac{1}{r} - \frac{\omega e}{p^3 a^3 K} \frac{d^3}{dz^3} \frac{1}{r} \dots\dots(93). \end{aligned}$$

If a_1, b_1, c_1 be those components of magnetic induction outside the sphere which vary as e^{ipt} ,

$$a_1 = \frac{dH_1}{dy} - \frac{dG_1}{dz}.$$

Differentiating, we find

$$a_1 = \frac{\omega e}{p^3 a^3 K} (i\lambda a)^3 e^{i[pt-\lambda(r-a)]} \left\{ \frac{1}{i\lambda} + r \right\} \frac{y}{r^3} \dots\dots\dots(94).$$

Taking the real part, since $\lambda^2 = p^2 \mu K$,

$$\begin{aligned} a_1 &= -\mu \omega e \{ \cos [pt-\lambda(r-a)] - \lambda r \sin [pt-\lambda(r-a)] \} \frac{y}{r^3} \\ &= \mu \omega e \frac{d}{dy} \frac{\cos [pt-\lambda(r-a)]}{r} \dots\dots\dots(95). \end{aligned}$$

Similarly,

$$\begin{aligned} b_1 &= \mu \omega e \{ \cos [pt-\lambda(r-a)] - \lambda r \sin [pt-\lambda(r-a)] \} \frac{z}{r^3} \\ &= -\mu \omega e \frac{d}{dz} \frac{\cos [pt-\lambda(r-a)]}{r} \dots\dots\dots(96); \\ c_1 &= 0. \end{aligned}$$

If $p = 0$, or if the velocity be constant, then $\lambda = 0$, and

$$a_1 = -\mu \frac{\omega e y}{r^3},$$

$$b_1 = \mu \frac{\omega e x}{r^3}$$

$$c_1 = 0,$$

or the moving charged sphere produces the same magnetic forces as a unit length of a current of strength ωe .

The expressions

$$a_1 = m \frac{d}{dy} \frac{\cos(pt - \lambda r)}{r},$$

$$b_1 = -m \frac{d}{dx} \frac{\cos(pt - \lambda r)}{r},$$

$$c_1 = 0,$$

satisfy equations of the form

$$\nabla^2 a_1 = \mu K \frac{d^2 a}{dt^2},$$

and they make the line integral

$$\int (a_1 dx + b_1 dy + c_1 dz),$$

taken round a circuit for which r is indefinitely small,

$$= 2\pi m \mu \cos pt;$$

hence these expressions are the x, y, z components of the magnetic induction due to an element of current of unit length and intensity $m \cos pt$ flowing along the axis of z . Comparing these expressions with equations (95) and (96), we see that, if r be large compared with a , so that we may write λr instead of $\lambda(r - a)$, the magnetic force produced by a sphere whose radius is a , charged with a quantity e of electricity and moving along the axis of z , so that its velocity is expressed by $\omega \cos pt$, is the same as that produced by an element of current along the axis of z of unit length and intensity $\omega e \cos pt$, so that the equivalence of the moving sphere and the current still holds when the velocity of the sphere is variable.

We will now calculate the mechanical force on the sphere, when it is placed in a strong magnetic field where the magnetic induction is parallel to the axis of y and is equal to b .

The mechanical force parallel to the axis of x per unit of volume is

$$-wb + eP \dots \dots \dots (97),*$$

* See Prof. G. F. Fitzgerald, "On Electromagnetic Actions produced by the Motion of the Earth," *Trans. Roy. Soc.*, Dublin, 1883.

where P is the electromotive force parallel to the axis of x . Now

$$-wb = \frac{1}{\sigma} \left(\frac{dH}{dt} + \frac{d\psi}{dz} \right) b.$$

The whole force parallel to the axis of x is therefore

$$\frac{b}{\sigma} \iiint \left(\frac{dH}{dt} + \frac{d\psi}{dz} \right) dx dy dz + \iiint \rho P dx dy dz.$$

Substituting the value of H given in equation (80), and retaining only those terms which do not vanish on integration, we find that the first member of this expression equals

$$\begin{aligned} & 2ipD \frac{b}{\sigma} e^{ipt} \int_0^a S_0(\lambda_1 r) 4\pi r^2 dr \\ &= 8i\pi pD \frac{b}{\sigma} e^{ipt} \frac{(\sin \lambda_1 a - \lambda_1 a \cos \lambda_1 a)}{\lambda_1^3} \\ &= 8i\pi pD \frac{ba^3}{\sigma \lambda_1^2} e^{ipt} \frac{dS_0(\lambda_1 a)}{da} \dots\dots\dots (98); \end{aligned}$$

but, from equation (86),

$$\frac{3D}{(i\lambda a)^3} e^{-i\lambda a} \frac{dS_0(\lambda_1 a)}{da} = \frac{\omega e}{p^3 a^3 K} \frac{e^{-i\lambda a}}{i\lambda a^3} \dots\dots\dots (99),$$

approximately, if λa be small.

Substituting this value for D , (98) becomes

$$-\frac{8i\pi}{3\sigma\lambda_1^2} p\mu b\omega e^{ipt} \dots\dots\dots (100).$$

Now $\lambda_1^2 = -\frac{4\pi}{\sigma} ip\mu'$, therefore (100) becomes, taking only the real part,

$$\frac{2}{3} \frac{\mu}{\mu'} e\omega b \cos pt \dots\dots\dots (101).$$

Since

$$P = -b\omega e^{ipt} - \frac{dF}{dt} - \frac{d\psi}{dx},$$

$$\iiint \rho P dx dy dz = -eb\omega e^{ipt},$$

or, taking the real part, $= -eb\omega \cos pt \dots\dots\dots (102).$

Thus the force on the sphere

$$= eb\omega \cos pt \left\{ \frac{2}{3} \frac{\mu}{\mu'} - 1 \right\} \dots\dots\dots (103).$$

If $\mu = \mu'$, the force $= -\frac{1}{3}eb\omega \cos pt$, or, *i.e.*, one-third of the force on an element of unit length of current whose intensity equals $\omega \cos pt$.

We shall now discuss the value of the electrostatic potential. From

equation (87) we get, supposing λa to be small,

$$\frac{3iB}{pa^3} \frac{\epsilon^{-i\lambda a}}{(i\lambda a)^3} \frac{dS_0(\lambda_1 a)}{da} = \frac{2\omega e}{p^2 a^3 K} \frac{\epsilon^{-i\lambda a}}{i\lambda a} \left\{ \frac{dS_0(\lambda_1 a)}{da} + \frac{S_0(\lambda_1 a)}{a} \right\},$$

therefore
$$iB = -\frac{2}{3}\omega e \mu p a^3 \left\{ 1 + \frac{S_0(\lambda_1 a)}{\lambda_1 a S_1(\lambda_1 a)} \right\} \dots\dots\dots(104).$$

If $\lambda_1 a$ be large,
$$iB = -\frac{2}{3}\omega e \mu p a^2,$$

therefore the real part of
$$Be^{i\omega t} \frac{d}{dz} \frac{1}{r}$$

$$= -\frac{2}{3}\mu e \omega a^2 p \sin pt \frac{d}{dz} \frac{1}{r} \dots\dots\dots(105);$$

therefore the maximum surface density of the electricity represented by this distribution

$$= \frac{\mu K p e \omega \sin pt}{2\pi a},$$

$$= \frac{\mu K e}{2\pi a} \text{ (acceleration of the sphere).}$$

From (105) we see that, if an electrified sphere be moving in a dielectric, the distribution of electricity over the sphere will not remain uniform, but the positive electricity will be heaped up towards the extremity of the radius vector, which points in the direction opposite to that of the acceleration. The original charge on the sphere is supposed to be positive.

If $\lambda_1 a$ be small, we get, from equation (104),

$$iB = \frac{2\omega e \mu p}{\lambda_1^3},$$

or, since $\lambda_1^2 = -\frac{4\pi i p \mu}{\sigma}$,

$$B = \frac{1}{2\pi} \frac{\sigma \mu}{\mu_1} \omega e.$$

Therefore the real part of $Be^{i\omega t} \frac{d}{dz} \frac{1}{r}$ is

$$\frac{\omega e}{2\pi} \frac{\mu}{\mu_1} \sigma \cos pt \frac{d}{dz} \frac{1}{r};$$

the maximum surface density of the electricity represented by this distribution is

$$\frac{3\omega e K}{8\pi^2} \frac{\mu}{\mu_1} \frac{\sigma \cos pt}{a^3},$$

$$= \frac{3e\sigma}{8\pi^2} \frac{\mu K}{\mu'} \frac{\text{(velocity of the sphere)}}{a^3},$$

and the electricity is heaped towards the extremity of the radius which points in the direction opposite to the direction of motion of the sphere.

If we substitute the values of F_1 , G_1 , H_1 , which we have just found in equations (63), (64), &c., we shall get equations to find F_2 , G_2 , H_2 , which we may solve in the same way. We shall find, however, that these quantities are small compared with F_1 , G_1 , H_1 .

Thursday, May 8th, 1884.

Prof. HENRICI, F.R.S., President, in the Chair.

Mr. J. Brill, B.A., Scholar of St. John's College, Cambridge, Lecturer in Mathematics at the University College, Aberystwith, was elected a Member; and Prof. Cremona, of Rome, was admitted into the Society.

The following communications were made :—

"Motion of a Network of Particles with some Analogies to Conjugate Functions," Dr. E. J. Routh, F.R.S.

"On a Subsidiary Elliptic Function $pm(u, k)$," Mr. J. Griffiths.

"On the Homogeneous Equation of a Plane Section of a Geometrical Surface," J. J. Walker, F.R.S.

"On a Birational Transformation of Space of Three Dimensions, of the Sixth Degree, the inverse of which is of the fifth," Prof. Cremona.

"Some Properties of Two Lines in the Plane of a Triangle," Mr. R. Tucker.

The following presents were received :—

"Educational Times," May, 1884.

"Integral Calculus," by B. Williamson, 5th ed., 8vo; London, 1884.

"Johns Hopkins University Circulars," Vol. 3, No. 29.

"Mittheilungen der Mathematischen Gesellschaft in Hamburg," No. 4; Hamburg, 1884.

"Beiblätter zu den Annalen der Physik und Chemie," B. 8, St. 3.

"Théorie Élémentaire des Séries Récurentes," by M. Maurice D'Ocagne. (Extracted from "Nouvelles Annales de Mathématiques.")

"Sur un Algorithme Algébrique," par M. Maurice D'Ocagne. 8vo, pamphlet.

"Sur l'Enveloppe de certaines Droites Variables," by Maurice D'Ocagne, 8vo, pamphlet.

"Archives Néerlandaises des Sciences Exactes et Naturelles," T. xviii., Nos. 2, 3, 4, and 5, 1883; T. xix., No. 1, 1884; Harlem, 1884.

"Bulletin des Sciences Mathématiques et Astronomiques," t. viii., February and March, 1884.

"Atti della R. Accademia dei Lincei," Vol. 8, F. 7, 8, 9, 10; 1884.

"Bulletin Astronomique et Météorologique de Rio de Janeiro," Sept. 1881, and Nov., 1883.

"Acta Mathematica," Vol. iii., Pt. 4.

"Programm des Gymnasiums zu Recklinghausen," liv. Schuljahr 1883—4. ("Untersuchungen über ähnliche Punktreihen auf den Seiten eines Dreiecks und auf deren Mittelsenkrechten, sowie über Kongruente Strahlenbüschel aus den Ecken desselben;" ein Beitrag zur Geometrie des Brocardschen Kreises vom Oberlehrer A. Artzt.)

On a Subsidiary Elliptic Function $\text{pm}(u, k)$.*

By JOHN GRIFFITHS, M.A.

[Read May 8th, 1884.]

I think there is an advantage in using a subsidiary elliptic function, $\text{pm}(u, k)$, as I propose to write it, in conjunction with the ordinary $\text{am}(u, k)$.

The theory of transformation leads in a very remarkable way to addition-theorems as regards both these functions.

For example, in a recent communication to the Society, it was shown by me that

$$\begin{aligned} \text{am}(3u, k) = 3 \text{am}(u, k) - \text{am}(Mu, \lambda) + \text{am}(M_1u, \lambda_1) \\ + \text{am}(M_2u, \lambda_2) + \text{am}(M_3u, \lambda_3); \end{aligned}$$

where $\lambda, \lambda_1, \lambda_2, \lambda_3$ are the roots of the modular equation in the cubic transformation. This being so, it can be proved that there is the correlative addition-theorem

$$\text{pm}(3u, k) = 3 \text{pm}(u, k) - \text{pm}(M_1u, \lambda) + \dots^*$$

1. *Definition of* $\text{pm}(u, k)$.

The new function is derived from $\text{am}(u, k)$ by the equation

$$\sin \text{pm}(u, k) = k \sin \text{am}(u, k) = k \text{sn}(u, k),$$

or, say,

$$\text{sm}(u, k) = k \text{sn}(u, k).$$

Differentiating this with regard to u , we have

$$\cos \text{pm } u \frac{d}{dt} \text{pm } u = k \text{cn } u \text{dn } u,$$

or, since $\cos \text{pm } u = \text{dn } u$,

$$\frac{d}{dt} \text{pm } u = k \text{cn } u.$$

* The notation " $\text{pm}(u, k)$ " has been adopted at Prof. Cayley's suggestion. Since $\sin \text{pm}(u, k) = \sin \text{am}\left(ku, \frac{1}{k}\right)$, he thinks that $\text{pm}(u, k) = \text{am}\left(ku, \frac{1}{k}\right)$ is the best definition of $\text{pm}(u, k)$.