and the displacements are

$$\left.\begin{aligned} w &= -\frac{ga^2\rho}{2n}\cos\theta - \frac{ga^2\rho}{2n}\left[\cos\theta\log(1+\cos\theta)-1+\tfrac{1}{2}\cos\theta\right] \\ &\qquad\qquad -\frac{ga^2\rho}{4n}\,\frac{m+n}{3m-n}\cos\theta \\ u &= \frac{ga^2\rho}{2n}\sin\theta - \frac{ga^2\rho}{2n}\left[\tan\tfrac{1}{2}\theta-\sin\theta\log(1+\cos\theta)\right] \end{aligned}\right\}\ldots(38).$$

Hence, if $2h$ be the thickness, the deflection at the vertex is

$$\left[\tfrac{1}{2}+\log 2+\tfrac{1}{2}(m+n)/(3m-n)\right] W/8\pi nh \ldots\ldots\ldots\ldots\ldots(39),$$

where W is the weight of the bowl.

A Method of Transformation with the aid of Congruences of a Particular Type. *By* J. Brill, M.A.

[*Read Dec.* 13*th*, 1888.]

1. Suppose that we have a family (A) of surfaces. The orthogonal trajectories of this family will form a congruence (a) of curves. On one of the surfaces belonging to the family (A) draw a family of lines. The curves of the congruence (a) that meet each of these lines will form a surface; and the curves of the congruence (a) that meet all of these lines will form a family (B) of surfaces, which is such that the members of it intersect orthogonally the members of the family (A). The curves of intersection of the members of the family (A) with those of the family (B) will form a congruence (c). This congruence will possess the property that it is possible to draw within it* two families of surfaces, viz. the families (A) and (B), such that the members of the one intersect the members of the other orthogonally. Further, since the family of lines drawn on the selected surface of the family (A) are altogether arbitrary, it is evident that they may be chosen so that at least one other selected property may belong to the congruence. It is, however, conceivable that cases may

* By this expression it is intended that each surface is the locus of some singly infinite series of the curves of the congruence in question.

arise in which one other property may be selected, the choice of which property may not be altogether arbitrary.

We will take α and β, the parameters of the families (A) and (B), as the coordinates of the curves of the congruence (c). Then we have the three equations

$$\left.\begin{aligned}\left(\frac{\partial\alpha}{\partial x}\right)^2+\left(\frac{\partial\alpha}{\partial y}\right)^2+\left(\frac{\partial\alpha}{\partial z}\right)^2&=k_1^2\\ \left(\frac{\partial\beta}{\partial x}\right)^2+\left(\frac{\partial\beta}{\partial y}\right)^2+\left(\frac{\partial\beta}{\partial z}\right)^2&=k_2^2\\ \frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial x}+\frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial y}+\frac{\partial\alpha}{\partial z}\frac{\partial\beta}{\partial z}&=0\end{aligned}\right\}\text{.....................(I.).}$$

We proceed to discover what conditions must be satisfied by the families (A) and (B) in order that, if any family (C) of surfaces be drawn within the congruence, it may be always possible to draw another family (D) within the congruence, such that the members of it intersect orthogonally those of the family (C).

Suppose that ξ and η are the parameters of the families (C) and (D). We have the equations

$$\left.\begin{aligned}\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2+\left(\frac{\partial\xi}{\partial z}\right)^2&=h_1^2\\ \left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2+\left(\frac{\partial\eta}{\partial z}\right)^2&=h_2^2\\ \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial z}\frac{\partial\eta}{\partial z}&=0\end{aligned}\right\}\text{..................(II.).}$$

If we substitute for $\partial\xi/\partial x$, &c., their values in terms of $\partial\xi/\partial\alpha$ and $\partial\xi/\partial\beta$, and make use of equations (I.), we easily deduce

$$\left.\begin{aligned}k_1^2\left(\frac{\partial\xi}{\partial\alpha}\right)^2+k_2^2\left(\frac{\partial\xi}{\partial\beta}\right)^2&=h_1^2\\ k_1^2\left(\frac{\partial\eta}{\partial\alpha}\right)^2+k_2^2\left(\frac{\partial\eta}{\partial\beta}\right)^2&=h_2^2\\ k_1^2\frac{\partial\xi}{\partial\alpha}\frac{\partial\eta}{\partial\alpha}+k_2^2\frac{\partial\xi}{\partial\beta}\frac{\partial\eta}{\partial\beta}&=0\end{aligned}\right\}\text{.....................(III.).}$$

If we are given that $\xi=f(\alpha,\beta)$, and if we substitute the values of

$\partial\xi/\partial\alpha$ and $\partial\xi/\partial\beta$ given by this in the third of equations (III.), we see that this equation will enable us to determine η as a function of α and β, provided that k_2/k_1 is expressible as a function of α and β.

The geometrical interpretation of this condition is easy. Draw two consecutive surfaces belonging to each of the families (A) and (B). These will enclose a space or filament whose normal sections at all points of its length are rectangles having their sides in a constant ratio. It follows, when this condition is satisfied, that any two surfaces drawn within the congruence cut at a constant angle all along their line of intersection.

The above reasoning, of course, breaks down for points on the focal surface of the congruence.

3. The simplest case of a congruence belonging to the type discussed in the preceding article, is that of one consisting of parallel straight lines.

A series of cases may be produced in the following manner. In a given plane draw a family of parallel curves, and with these curves for bases draw a family of cylinders having their generators at right angles to the given plane. The intersections of these cylinders with a family of planes parallel to the given one, will constitute a congruence of the type we are considering. The simplest case of this series is that of the parallel straight lines mentioned above. The next in order of simplicity is that of a congruence of circles having their centres disposed along an axis and their planes at right angles to that axis.

A second series of cases may be obtained by drawing a family of parallel curves on a sphere, and through them drawing a family of cones having the centre of the sphere for vertex. If we cut these cones by a family of spheres concentric with the given one, the intersections will constitute a congruence of the required type.

A third series of cases may be obtained by drawing a family of plane curves possessing the property that, if we draw a normal at any point of one of them, the length of the portion of this normal intercepted between the curve and its consecutive is proportional to the distance of the point from a fixed straight line. If we revolve this figure about the fixed straight line, the congruence formed by the curves in their consecutive positions will be of the type in question.

We are not concerned with the question as to whether it is always possible to obtain a congruence of the type in question containing a given family of surfaces, nor with the problem of discovering the said congruences in cases where it is possible. It is sufficient for our present purpose to know that examples of this type of congruence do exist.

4. Since k_2/k_1 is expressible as a function of α and β, it is evident from equations (III.) that h_2/h_1 may be also so expressed. From the same equations we easily deduce

$$\frac{k_1 \dfrac{\partial \xi}{\partial \alpha}}{k_2 \dfrac{\partial \eta}{\partial \beta}} = -\frac{k_2 \dfrac{\partial \xi}{\partial \beta}}{k_1 \dfrac{\partial \eta}{\partial \alpha}} = \pm \frac{\left\{ k_1^2 \left(\dfrac{\partial \xi}{\partial \alpha}\right)^2 + k_2^2 \left(\dfrac{\partial \xi}{\partial \beta}\right)^2 \right\}^{\frac{1}{2}}}{\left\{ k_1^2 \left(\dfrac{\partial \eta}{\partial \alpha}\right)^2 + k_2^2 \left(\dfrac{\partial \eta}{\partial \beta}\right)^2 \right\}^{\frac{1}{2}}} = \pm \frac{h_1}{h_2}.$$

Thus we should have either

$$h_2 k_1 \frac{\partial \xi}{\partial \alpha} = h_1 k_2 \frac{\partial \eta}{\partial \beta} \quad \text{and} \quad h_2 k_2 \frac{\partial \xi}{\partial \beta} = -h_1 k_1 \frac{\partial \eta}{\partial \alpha},$$

or

$$h_2 k_1 \frac{\partial \xi}{\partial \alpha} = -h_1 k_2 \frac{\partial \eta}{\partial \beta} \quad \text{and} \quad h_2 k_2 \frac{\partial \xi}{\partial \beta} = h_1 k_1 \frac{\partial \eta}{\partial \alpha}.$$

These two forms are virtually identical. In developing our theory we will make use of the first form. However, in discussing any particular case, it would be necessary to ascertain carefully which parameter should be taken for ξ and which for η.

Now h_2/h_1 and k_2/k_1 are both functions of α and β. We will express them by the symbols u and v respectively. Then our equations take the form

$$u \frac{\partial \xi}{\partial \alpha} = v \frac{\partial \eta}{\partial \beta} \quad \text{and} \quad uv \frac{\partial \xi}{\partial \beta} = -\frac{\partial \eta}{\partial \alpha}.$$

Now $$u\,d\xi + i\,d\eta = u \left\{ \frac{\partial \xi}{\partial \alpha} d\alpha + \frac{\partial \xi}{\partial \beta} d\beta \right\} + i \left\{ \frac{\partial \eta}{\partial \alpha} d\alpha + \frac{\partial \eta}{\partial \beta} d\beta \right\}$$

$$= u \frac{\partial \xi}{\partial \alpha} \left(d\alpha + i \frac{d\beta}{v} \right) + i \frac{\partial \eta}{\partial \alpha} \left(d\alpha + i \frac{d\beta}{v} \right)$$

$$= \left\{ u \frac{\partial \xi}{\partial \alpha} + i \frac{\partial \eta}{\partial \alpha} \right\} \left(d\alpha + i \frac{d\beta}{v} \right).$$

Therefore $$\frac{u\,d\xi + i\,d\eta}{v\,d\alpha + i\,d\beta} = \frac{1}{v} \left\{ u \frac{\partial \xi}{\partial \alpha} + i \frac{\partial \eta}{\partial \alpha} \right\} = \frac{\partial \eta}{\partial \beta} - iu \frac{\partial \xi}{\partial \beta}.$$

Thus the value of the expression

$$\frac{u\,d\xi + i\,d\eta}{v\,da + i\,d\beta}$$

is independent of the value of the ratio $da : d\beta$. The same will be true of the expression

$$\frac{(m+in)(u\,d\xi + i\,d\eta)}{(p+iq)(v\,da + i\,d\beta)}.$$

Writing $f = m+in$ and $g = p+iq$, we will seek to determine f and g so that the expressions $f\,(u\,d\xi + i\,d\eta)$ and $g\,(v\,da + i\,d\beta)$ may be perfect differentials. The necessary conditions for this will be

$$\frac{\partial}{\partial \eta}(fu) = i\,\frac{\partial f}{\partial \xi} \quad \text{and} \quad \frac{\partial}{\partial \beta}(gv) = i\,\frac{\partial g}{\partial a}.$$

These equations will enable us to obtain suitable forms for f and g, and the solution of each will involve an arbitrary function. If we now write

$$dw = f\,(u\,d\xi + i\,d\eta) \quad \text{and} \quad d\zeta = g\,(v\,da + i\,d\beta),$$

$dw/d\zeta$ will possess a single definite value; and, if we further write $w = \lambda + i\mu$ and $\zeta = \gamma + i\delta$, we see that the value of the expression

$$\frac{d\lambda + i\,d\mu}{d\gamma + i\,d\delta}$$

is independent of the value of the ratio $d\gamma : d\delta$. This necessitates the relations

$$\frac{\partial \lambda}{\partial \gamma} = \frac{\partial \mu}{\partial \delta} \quad \text{and} \quad \frac{\partial \lambda}{\partial \delta} = -\frac{\partial \mu}{\partial \gamma}.$$

But these are the conditions that $\lambda + i\mu$ may be expressible as a function of $\gamma + i\delta$. Thus we see that we can draw within the congruence an indefinite number of systems containing two orthogonal families of surfaces, which possess properties somewhat analogous to those possessed by plane systems consisting of two orthogonal families of equipotential curves. All the systems of this character may be obtained from the expression

$$\int (p+iq)(v\,da + i\,d\beta)$$

with the aid of the general integral of the equation

$$\frac{\partial}{\partial\beta}(gv) = i\,\frac{\partial g}{\partial a}.*$$

It is to be noted, however, that the families of surfaces of which these systems consist are not necessarily equipotential.

5. We are now in a position, with the aid of congruences of the type we have been discussing, to develop a method of transformation in space of three dimensions analogous to the method of transformation by means of conjugate functions in a plane. In this method the curves of the congruence take the place of the points in the plane, and the surfaces of the congruence take the place of curves in the plane.

If we take γ and δ as the parameters of the two families of surfaces constituting a system of the type discussed at the end of the preceding article, and if we take λ and μ as the parameters of the two families constituting another such system, we have $\lambda + i\mu = F(\gamma + i\delta)$. Now take a congruence exactly like the one in which these systems are drawn; and make those families of surfaces within it, the ones exactly like which in the first congruence belong to the parameters λ and μ, correspond to the parameters γ and δ. Then any surface drawn within the first congruence will be transformed into a different surface within the second congruence.

It is easy to prove that any pair of transformed surfaces will cut at the same angle as the original pair of surfaces. Thus, from the equation
$$(p+iq)(vda+id\beta) = d\gamma + id\delta,$$
we deduce $\quad pvda - qd\beta = d\gamma$ and $qvda + pd\beta = d\delta$;

from which it follows that

$$\frac{\partial\gamma}{\partial a} = pv, \quad \frac{\partial\gamma}{\partial\beta} = -q, \quad \frac{\partial\delta}{\partial a} = qv, \quad \frac{\partial\delta}{\partial\beta} = p.$$

Therefore

$$k_1^2\left(\frac{\partial\gamma}{\partial a}\right)^2 + k_2^2\left(\frac{\partial\gamma}{\partial\beta}\right)^2 = k_1^2\left(\frac{\partial\delta}{\partial a}\right)^2 + k_2^2\left(\frac{\partial\delta}{\partial\beta}\right)^2 = (p^2+q^2)\,k_2^2 = k^2 \text{ say}.$$

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* The referee points out that all the congruences of the type considered can be obtained from the equations

$$\left(\frac{\partial\gamma}{\partial x}\right)^2 + \left(\frac{\partial\gamma}{\partial y}\right)^2 + \left(\frac{\partial\gamma}{\partial z}\right)^2 = \left(\frac{\partial\delta}{\partial x}\right)^2 + \left(\frac{\partial\delta}{\partial y}\right)^2 + \left(\frac{\partial\delta}{\partial z}\right)^2$$

and
$$\frac{\partial\gamma}{\partial x}\frac{\partial\delta}{\partial x} + \frac{\partial\gamma}{\partial y}\frac{\partial\delta}{\partial y} + \frac{\partial\gamma}{\partial z}\frac{\partial\delta}{\partial z} = 0.$$

Thus, if ρ and σ be the parameters of the two surfaces in question,

then
$$\left(\frac{\partial\rho}{\partial x}\right)^2+\left(\frac{\partial\rho}{\partial y}\right)^2+\left(\frac{\partial\rho}{\partial z}\right)^2=k^2\left\{\left(\frac{\partial\rho}{\partial\gamma}\right)^2+\left(\frac{\partial\rho}{\partial\delta}\right)^2\right\},$$
$$\left(\frac{\partial\sigma}{\partial x}\right)^2+\left(\frac{\partial\sigma}{\partial y}\right)^2+\left(\frac{\partial\sigma}{\partial z}\right)^2=k^2\left\{\left(\frac{\partial\sigma}{\partial\gamma}\right)^2+\left(\frac{\partial\sigma}{\partial\delta}\right)^2\right\},$$
$$\frac{\partial\rho}{\partial x}\frac{\partial\sigma}{\partial x}+\frac{\partial\rho}{\partial y}\frac{\partial\sigma}{\partial y}+\frac{\partial\rho}{\partial z}\frac{\partial\sigma}{\partial z}=k^2\left\{\frac{\partial\rho}{\partial\gamma}\frac{\partial\sigma}{\partial\gamma}+\frac{\partial\rho}{\partial\delta}\frac{\partial\sigma}{\partial\delta}\right\}.$$

Hence, if θ be the angle at which the surfaces cut

$$\cos\theta=\frac{\frac{\partial\rho}{\partial\gamma}\frac{\partial\sigma}{\partial\gamma}+\frac{\partial\rho}{\partial\delta}\frac{\partial\sigma}{\partial\delta}}{\left\{\left(\frac{\partial\rho}{\partial\gamma}\right)^2+\left(\frac{\partial\rho}{\partial\delta}\right)^2\right\}^{\frac{1}{2}}\left\{\left(\frac{\partial\sigma}{\partial\gamma}\right)^2+\left(\frac{\partial\sigma}{\partial\delta}\right)^2\right\}^{\frac{1}{2}}}$$

But we have

$$\left(\frac{\partial\lambda}{\partial\gamma}\right)^2+\left(\frac{\partial\lambda}{\partial\delta}\right)^2=\left(\frac{\partial\mu}{\partial\gamma}\right)^2+\left(\frac{\partial\mu}{\partial\delta}\right)^2=s^2 \text{ say,}$$

and
$$\frac{\partial\lambda}{\partial\gamma}\frac{\partial\mu}{\partial\gamma}+\frac{\partial\lambda}{\partial\delta}\frac{\partial\mu}{\partial\delta}=0.$$

Therefore
$$\left(\frac{\partial\rho}{\partial\gamma}\right)^2+\left(\frac{\partial\rho}{\partial\delta}\right)^2=s^2\left\{\left(\frac{\partial\rho}{\partial\lambda}\right)^2+\left(\frac{\partial\rho}{\partial\mu}\right)^2\right\},$$
$$\left(\frac{\partial\sigma}{\partial\gamma}\right)^2+\left(\frac{\partial\sigma}{\partial\delta}\right)^2=s^2\left\{\left(\frac{\partial\sigma}{\partial\lambda}\right)^2+\left(\frac{\partial\sigma}{\partial\mu}\right)^2\right\},$$

and
$$\frac{\partial\rho}{\partial\gamma}\frac{\partial\sigma}{\partial\gamma}+\frac{\partial\rho}{\partial\delta}\frac{\partial\sigma}{\partial\delta}=s^2\left\{\frac{\partial\rho}{\partial\lambda}\frac{\partial\sigma}{\partial\lambda}+\frac{\partial\rho}{\partial\mu}\frac{\partial\sigma}{\partial\mu}\right\}.$$

Therefore
$$\cos\theta=\frac{\frac{\partial\rho}{\partial\lambda}\frac{\partial\sigma}{\partial\lambda}+\frac{\partial\rho}{\partial\mu}\frac{\partial\sigma}{\partial\mu}}{\left\{\left(\frac{\partial\rho}{\partial\lambda}\right)^2+\left(\frac{\partial\rho}{\partial\mu}\right)^2\right\}^{\frac{1}{2}}\left\{\left(\frac{\partial\sigma}{\partial\lambda}\right)^2+\left(\frac{\partial\sigma}{\partial\mu}\right)^2\right\}^{\frac{1}{2}}};$$

and the second side of this equation denotes the cosine of the angle at which the transformed surfaces cut.

Further, it is easily seen that we may utilise our results to obtain a correspondence between two different congruences of the type in question, which is exactly similar in character to the geographical correspondence of two surfaces.

6. If we apply our method to the simplest congruence of the specified type, viz., the one consisting of parallel straight lines, it reduces to the ordinary method of transformation by means of functions of a complex variable as applied to the plane. Also, if we apply our method to the congruence of circles having their centres disposed along an axis, and their planes at right angles to this axis, we shall have a method applicable to surfaces of revolution having a common axis.

In conclusion, I may state that I think I possess a clue to the establishment of a correspondence between any two congruences whatsoever, and that I hope to make a communication to the Society on this subject at some future time.

Thursday, January 10th, 1889.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

Mr. G. H. Bryan, M.A., St. Peter's College, Cambridge, and Mr. W. W. Taylor, M.A., late Scholar of Queen's College, Oxford, were elected members, and Miss Meyer was admitted into the Society.*

The auditor (Mr. Heppel) made his report, which on the motion of Sir J. Cockle, seconded by Rev. T. C. Simmons, was adopted. Upon the motion of Mr. Basset, seconded by Dr. Glaisher, the Treasurer's report was adopted.

Subsequently a vote of thanks was passed to the Auditor on the motion of Major Macmahon, R.A., seconded by Dr. Glaisher.

Mr. Basset made a few remarks on the Steady Motion and Stability of Dynamical Systems.

Dr. Glaisher gave several forms of expression of Bernoulli's Numbers derived from the consideration of Lemniscate properties.

The President read a paper on "Results of Ternary Quadric Operators on Products of Forms of any Orders" (Sir J. Cockle in the Chair).

Mr. Jenkins read a note by Mr. Christie "On a Theorem in Combinations."

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* By an oversight it was omitted to be stated that Mr. R. W. Hogg was admitted into the Society at the November meeting.