Further Applications of Matrix Notation to Integration Problems.

By H. F. BAKER. Received and communicated April 10th, 1902.

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2. Gives the finite equations of the adjoint group with canonical parameters, showing that the exponential matrix Δ of a former note (*Proc. Lond. Math. Soc.*, Vol. xxxiv., 1902, p. 91) enters therein; and so gives a simple proof of the so-called exponential theorem.
3. Remarks on the connection with Lie's formulæ.
4. Shows that any transformation of the adjoint group can be resolved into a succession of two transformations respectively of the first and second parameter groups.
5. Shows that this leads to a result including as a particular case the theorem that the characteristic determinantal equation allows the adjoint group.
6. Gives another proof of this theorem of invariance.
7. Develops the result further, establishing in particular the equation

$$\psi(e^t, \Delta_{e^t} x) = \Delta_{e^t(1-t)} \psi(x, e^t).$$
8. Remarks on the translation from the first to the second parameter group.
9. Obtains a proof of the existence of and a formula for the general integral of any set of simultaneous linear differential equations with variable coefficients, in a form valid for the whole of the Mittag-Leffler star region over which the integrals exist. This appears a remarkable result.
10. Remarks that the theorem and method of § 9 is particularly applicable to establish the existence of and to calculate the integrals for a system of differential equations with *real* independent variable, the coefficients not being necessarily continuous, so long as they are integrable.
- 11, 12. Remark on the generalization of the results of §§ 9, 10.

1. The finite equations of a continuous group of n variables $x_1, \dots, x_n$ and r parameters $a_1, \dots, a_r$, being

$$x_i = f_i(x^0, a),$$

leading to equations

$$f_i[j(x^0, a), b] = f_i[x^0, \phi(a, b)],$$

so that

$$y_e = \phi_e(y^0, a) \quad \text{and} \quad z_e = \phi_e(a, z^0),$$

for $\sigma = 1, \dots, r$, are the finite equations respectively of the first and second parameter groups, we have the three sets of differential equations

$$\frac{\partial x}{\partial a} \alpha(a) = \xi(x), \quad \frac{\partial y}{\partial a} \alpha(a) = \alpha(y), \quad \frac{\partial z}{\partial a} \beta(a) = \beta(z),$$

where $\alpha(a)$ is a matrix of r rows and columns, and $\xi(x)$ a matrix of n rows and r columns. If, now, in the general infinitesimal transformation of the group

$$\sum_{\sigma=1}^r e_{\sigma} P_{\sigma} = \sum_{\sigma=1}^r e_{\sigma} \left(\xi_{1\sigma} \frac{\partial}{\partial x_1} + \dots + \xi_{n\sigma} \frac{\partial}{\partial x_n} \right),$$

we change the independent variables to $x_1^0, \dots, x_n^0$ by means of the equations of the group

$$x_i = f_i(x^0, a),$$

the infinitesimal transformation becomes $\sum_{\sigma=1}^r e_{\sigma}^0 J_{\sigma}^0$, where P_{σ}^0 contains $x_1^0, \dots, x_n^0$, just as P_{σ} contains $x_1, \dots, x_n$ and $e_1, \dots, e_r$ are linearly deduced from $e_1^0, \dots, e_r^0$ by equations, depending on $a_1, \dots, a_r$ only, which for varying $a_1, \dots, a_r$ are the finite equations of the adjoint group.

These equations are, in fact,

$$e = a^{-1} \beta . e^0,$$

namely,*

$$e_{\sigma} = \sum_{\tau=1}^r \sum_{\rho=1}^r (a^{-1})_{\sigma\rho} \beta_{\rho\tau} e_{\tau}^0.$$

2. The infinitesimal equations of the adjoint group are, however,

$$- \left(E_{1\sigma} \frac{\partial}{\partial e_1} + \dots + E_{r\sigma} \frac{\partial}{\partial e_r} \right),$$

where

$$E_{\sigma\sigma} = \sum_{\tau=1}^r e_{\sigma\tau} e_{\tau},$$

and the finite equations of the adjoint group, with *canonical parameters* $e'_1, \dots, e'_r$, are therefore to be obtained by solving the r ordinary differential equations

$$\frac{de}{dt} = -Ee' = E'e,$$

* The notation in the *Encyk. Math. Wiss.*, Vol. II., A. 6, p. 406, seems an unfortunate change from Lie's for the matrix α in *Transformationsgruppen*, Vol. I., p. 34. If the quantity $\alpha_{\sigma\sigma}$ of the *Encyk.* be called $\gamma_{\sigma\sigma}$, then the matrix γ is the negative of the matrix $\beta(a)$ belonging to the second parameter group.

making $e = e^0$, when $t = 0$, and afterwards putting $t = 1$. These equations are, however, obviously satisfied* by

$$e = \left(1 + tE' + \frac{t^2}{2!} E'^2 + \frac{t^3}{3!} E'^3 + \dots\right) e^0.$$

Thus, if
$$\Delta' = 1 + E' + \frac{E'^2}{2!} + \frac{E'^3}{3!} + \dots,$$

the finite equations of the adjoint group are $e = \Delta' e^0$, the parameters being canonical. Using these same parameters to write the finite equations of the original group in the form

$$x_i = F_i(x^0, e'),$$

we have the result that the equations

$$x_i = F_i(x^0, e'), \quad e = \Delta' e^0$$

lead to
$$\sum_{\sigma=1}^r e_{\sigma} P_{\sigma} = \sum_{\sigma=1}^r e_{\sigma}^0 P_{\sigma}^0.$$

Considering then a further set

$$y_i = F_i(x, e''), \quad f = \Delta'' e', \quad \sum f_{\sigma} Q_{\sigma} = \sum e_{\sigma} P_{\sigma},$$

we obtain the set

$$y_i = F_i(x^0, e'''), \quad f = \Delta'' \Delta' e^0, \quad \sum f_{\sigma} Q_{\sigma} = \sum e_{\sigma}^0 P_{\sigma}^0,$$

showing that the equations of the parameter group, with canonical variables and parameters

$$e_{\sigma}''' = \psi_{\sigma}(e', e''),$$

lead to
$$\Delta''' = \Delta'' \Delta'.$$

Thus we have a very simple proof of the so-called exponential theorem.

* A particular example of this integration is the proof of the formulæ for the nine cosines between two sets of rectangular axes by integrating the equations

$$\frac{dx}{dt} = ny - mz, \quad \frac{dy}{dt} = -nx + lz, \quad \frac{dz}{dt} = mx - ly.$$

The matrix

$$E = \begin{pmatrix} 0 & n & -m \\ -n & 0 & l \\ m & -l & 0 \end{pmatrix}$$

satisfies the equation

$$E^3 = -(l^2 + m^2 + n^2) E;$$

cf. *Proc. Lond. Math. Soc.*, Vol. xxxiv., p. 116.

3. The problem of finding canonical variables for a group of which some form of the finite equations is known is thus, for a group without special infinitesimal transformations, that of solving the r^2 equations

$$\alpha^{-1}(u) \beta(u) = \Delta',$$

the matrices ρ and σ of Lie [*Transformationsgruppen*, Vol. III., p. 613, equation (24)] being these two respectively. And by expressing that the two parameter groups are reciprocal, or otherwise [Lie, *Transformationsgruppen*, Vol. III., p. 616, equation (27)], it can be shown that

$$(e'_1 A_1 + \dots + e'_r A_r)(\alpha^{-1}\beta) = E'(\alpha^{-1}\beta),$$

where

$$A_\sigma = \sum_{\rho=1}^r a_{\rho\sigma} \frac{\partial}{\partial a_\rho}$$

is the general infinitesimal transformation of the first parameter group, while the exponential theorem in general variables has a form

$$\alpha_2^{-1}\beta_2 = \alpha_1^{-1}\beta_1 \cdot \alpha^{-1}\beta.$$

4. We prove now that any transformation $y = \Delta'x$ of the adjoint group can be resolved into a succession of two transformations

$$z_\sigma = \psi_\sigma(x, e'), \quad y_\sigma = \psi_\sigma(-e', z),$$

respectively, of the first and second parameter groups. Herein the variables x, y, z , as well as the parameters e' , are canonical, a fact we emphasise by using the functional sign ψ instead of the more usual ϕ .

Using infinitesimal transformations, we have

$$\begin{aligned} y_\sigma &= z_\sigma - [e'_1 \beta_{\sigma 1}(z) + \dots + e'_r \beta_{\sigma r}(z)] + \dots \\ &= x_\sigma + e'_1 [\alpha_{\sigma 1}(x) - \beta_{\sigma 1}(x)] + \dots + e'_r [\alpha_{\sigma r}(x) - \beta_{\sigma r}(x)] + \dots \end{aligned}$$

On the other hand (*Proc. Lond. Math. Soc.*, Vol. XXXIV., 1902, p. 97), by Schur's formulæ,

$$\alpha - \beta = \alpha - \alpha \Delta = -E.$$

Thus the infinitesimal transformations of the compound transformation are these of the adjoint group.

5. This remark is equivalent with

$$\Delta'x = \psi[-e', \psi(x, e')],$$

namely, as

$$\psi(e, e') = -\psi(-e', -e),$$

with

$$-\Delta'x = \psi[-\psi(x, e'), e'],$$

and gives $-\psi(x, e') = \psi(-\Delta'x, -e') = -\psi(e', \Delta'x);$

and hence $\psi_e(e', \Delta'x) = \psi_e(x, e'),$

a result* leading, by the exponential theorem, to

$$\Delta_y \Delta_{e'} = \Delta_{e'} \Delta_x,$$

as a consequence solely of the linear equations

$$y = \Delta'x.$$

As these last are unaffected by multiplying each of $x_1, \dots, x_r; y_1, \dots, y_r$ by the arbitrary numerical quantity λ , we can, by comparing coefficients of λ in the equation

$$\Delta_{\lambda y} \Delta_{e'} = \Delta_{e'} \Delta_{\lambda x},$$

infer that

$$E_y \Delta_{e'} = \Delta_{e'} E_x;$$

and hence, if θ be an arbitrary numerical quantity, as the determinant of $\Delta_{e'}$ is not zero, we see that

$$|E_x + \theta| = |E_y + \theta|,$$

namely, that the so-called characteristic equation expressed by the vanishing of the determinant $|E_x + \theta|$ allows the transformations of the adjoint group. The equation

$$E_y + \theta = \Delta_{e'} (E_x + \theta) \Delta_{e'}$$

shows that the greatest common divisor, in θ , of the first minors of $|E_x + \theta|$ agrees with that for $|E_y + \theta|$, and hence that, if

$$\chi(E) = 0$$

be the algebraic equation satisfied by the matrix E , the equation

$$\chi(\omega) = 0,$$

for arbitrary ω , allows the adjoint group. Further, the determinantal equation

$$|\Delta_x - e^{-\omega}| = 0,$$

which is satisfied by all the roots θ of the characteristic equation, also allows the adjoint group; this is equivalent with the fact that the determinantal equation

$$|\alpha(u) - \lambda \beta(u)| = 0,$$

for arbitrary numerical λ , allows the adjoint group; and *herein the variables $a_1, \dots, a_r$ need not be canonical.*

* Re-proving that, if T_e be the transformation of the original group with canonical parameters e , the canonical parameters of $T_f = T_e T_e^{-1}$ are $f = \Delta'e$; or conversely.

6. In the general case when the group has no special infinitesimal transformations the equation

$$\Delta_y \Delta_{e'} = \Delta_{e'} \Delta_x$$

necessarily involves the equations

$$\psi_*(e', y) = \psi_*(x, e'),$$

as was proved in my previous note (*Proc. Lond. Math. Soc.*, Vol. xxxiv., 1902, p. 112); so that we have the identities

$$\psi_*(e', \Delta'x) = \psi_*(x, e'),$$

which therefore hold also in the special case. We give now a direct proof of the equation

$$\Delta_y \Delta_{e'} = \Delta_{e'} \Delta_x,$$

where

$$y = \Delta'x$$

(cf. Engel, *Math. Annal.*, Vol. xxxi., 1882, p. 262, footnote).

If $e^{(1)} = E'e$, and e'' be a further independent set, we have (*Proc. Lond. Math. Soc.*, Vol. xxxiv., p. 93)

$$E^{(0)}e'' = -E''e^{(1)} = -E''E'e = EE''e' + E'Ee'' = E'Ee'' - EE'e'';$$

so that

$$E^{(1)} = E'E - EE'.$$

Put, similarly,

$$e^{(2)} = E'E^{(1)} = E'^2e, \quad e^{(3)} = E'e^{(2)} = E'^3e,$$

and so on; then

$$\begin{aligned} \Delta'e &= e + E'e + \frac{E'^2e}{2!} + \frac{E'^3e}{3!} + \dots \\ &= e + e^{(1)} + \frac{1}{2!}e^{(2)} + \frac{1}{3!}e^{(3)} + \dots; \end{aligned}$$

and therefore, if $f = \Delta'e$, as E_f is linear in f ,

$$\begin{aligned} E_f &= E + E^{(1)} + \frac{1}{2!}E^{(2)} + \frac{1}{3!}E^{(3)} + \dots \\ &= E + (E'E - EE') + \frac{1}{2!}(E'E^{(1)} - E^{(1)}E') + \frac{1}{3!}(E'E^{(2)} - E^{(2)}E') + \dots \\ &= E + (E'E - EE') + \frac{1}{2!}(E'^2E - 2E'E'E' + EE'^2) \\ &\quad + \frac{1}{3!}(E'^3E - 3E'^2EE' + 3E'E'E'^2 - EE'^3) + \dots \end{aligned}$$

$$\begin{aligned}
&= \left(1 + E' + \frac{E'^2}{2!} + \dots\right) E - \left(1 + E' + \frac{E'^2}{2!} + \dots\right) EE' \\
&\quad + \frac{1}{2!} \left(1 + E' + \frac{E'^2}{2!} + \dots\right) EE'^2 - \dots \\
&= \Delta' E (\Delta')^{-1},
\end{aligned}$$

giving

$$E_f \Delta' = \Delta' E$$

as a consequence of

$$f = \Delta' e.$$

From this

$$E_f^k \Delta' = \Delta' E^k \quad \text{and} \quad \Delta_f \Delta' = \Delta' \Delta_e.$$

7. Hence it follows that the two sets of differential equations

$$\frac{d\eta}{dt} = \beta(\eta) e', \quad \frac{d\xi}{dt} = \alpha(\xi) e'$$

are transformable into one another by the linear equations

$$\eta = \Delta_{e'(1-t)} \xi.$$

And this leads to $\psi(e't, \Delta_{e'x}) = \Delta_{e'(1-t)} \psi(x, e't)$,

including the former result.

$$\text{For, from } \eta = \Delta_{e'(1-t)} \xi = \Delta_{e't} \Delta_{e'} \xi, \quad \frac{d\xi}{dt} = \alpha(\xi) e',$$

we have

$$\begin{aligned}
\frac{d\eta}{dt} &= -E' \Delta_{e't} \Delta_{e'} \xi + \Delta_{e't} \Delta_{e'} \frac{d\xi}{dt} \\
&= -E' \eta + \Delta_{e'(1-t)} \alpha(\xi) e',
\end{aligned}$$

while, by Schur's formulæ, from

$$E_\eta \Delta_{e'(1-t)} = \Delta_{e'(1-t)} E_\xi,$$

we have (*Proc. Lond. Math. Soc.*, Vol. XXXIV., p. 96)

$$\alpha(\eta) \Delta_{e'(1-t)} = \Delta_{e'(1-t)} \alpha(\xi),$$

and hence

$$\begin{aligned}
\frac{d\eta}{dt} &= -E' \eta + \alpha(\eta) \Delta_{e'(1-t)} e' \\
&= E_\eta e' + \alpha(\eta) e' \\
&= [E_\eta + \alpha(\eta)] e' \\
&= \beta(\eta) e',
\end{aligned}$$

since

$$\alpha - \beta = -E$$

(*ib.*, p. 97). This proves the transformation in question.

8. It should perhaps be remarked that, if in

$$x = \psi(x^0, e')$$

we put

$$x^0 = \psi(e^0, -y^0),$$

we obtain

$$x = \psi[e^0, \psi(-y^0, e')] = \psi[e^0, -\psi(-e', y^0)] = \psi(e^0, -y),$$

provided

$$y = \psi(-e', y^0).$$

These equations give the translation of the first parameter group into the second.

9. The method used above for obtaining the finite equations of the adjoint group can be generalized as follows to obtain an expression for the integrals of any set of simultaneous linear equations *with variable coefficients*.

Let the equations be

$$\frac{dx_i}{dt} = u_{i1}x_1 + \dots + u_{in}x_n \quad (i = 1, \dots, n),$$

and let $t = t^0$, which, by a change of independent variable, we shall suppose to be $t = 0$, be a value of t about which every one of the n^2 coefficients u_{ij} is developable, and therefore single-valued, finite, and continuous. By barriers passing to infinity from the infinities or branch points of these n^2 coefficients, we define a simply-connected region of the plane over the whole of the finite portion of which this character remains; and we suppose t limited to this finite region.

Now, writing the equations in the form

$$\frac{dx}{dt} = ux,$$

form in succession the matrices

$$u^{(1)} = \int_0^t u dt, \quad u^{(2)} = \int_0^t uu^{(1)} dt, \quad \dots, \quad u^{(n+1)} = \int_0^t uu^{(n)} dt, \quad \dots,$$

where, by the integral of a matrix is meant simply the matrix whose elements are the integrals of the elements of the original; if the operation of integrating a matrix from 0 to t be denoted by the prefix Q , these successive matrices may be succinctly denoted by

$$u^{(1)} = Qu, \quad u^{(2)} = QuQu, \quad u^{(3)} = QuQuQu, \quad \dots,$$

it being understood that each symbol Q operates on all that follows it.

It is now to be proved that the elements of the matrix

$$\nabla = 1 + Qu + Qu Qu + Qu Qu Qu + \dots \text{ to infinity}$$

are all uniformly and absolutely converging over the whole of the star region before described, and that the values

$$x = \nabla x^0$$

represent the solutions of the original differential equations which reduce to the arbitrary values $x_1^0, \dots, x_n^0$ for $t = 0$.

Let M_{ij} be a real positive quantity not exceeded by the modulus of u_{ij} for any value of t within the region under consideration; every one of the functions

$$\begin{aligned} u_{ij}^{(1)}(t) &= \int_0^t u_{ij}(t) dt, \\ u_{ij}^{(2)}(t) &= \int_0^t [u_{i1}(t) u_{1j}^{(1)}(t) + \dots + u_{in}(t) u_{nj}^{(1)}(t)] dt, \\ &\dots \dots \dots \dots \dots \dots \dots \dots \end{aligned}$$

is developable, single-valued, finite, and continuous about every point throughout the star region under consideration; let t_1 be a particular value of t on the path of integration from 0 to t , and s_1 the length of the path of integration to this point, the whole arc from 0 to t being s ; then

$$|u_{ij}^{(1)}(t)| \leq M_{ij} \int_0^s ds_1 \leq s M_{ij},$$

and, in the same way,

$$|u_{ij}^{(1)}(t_1)| \leq s_1 M_{ij};$$

$$\text{also } |u_{ij}^{(2)}(t)| \leq \int_0^s s_1 ds_1 (M_{i1} M_{1j} + M_{i2} M_{2j} + \dots + M_{in} M_{nj})$$

$$\leq (M^2)_{ij} \int_0^s s_1 ds_1$$

$$\leq \frac{1}{2} s^2 (M^2)_{ij},$$

where M denotes the matrix of the quantities M_{ij} , and M^2 the square of this matrix; in the same way,

$$|u_{ij}^{(2)}(t_1)| \leq \frac{1}{2} s_1^2 (M^2)_{ij};$$

$$\text{also } |u_{ij}^{(3)}(t)| \leq \int_0^s \frac{1}{2} s_1^2 ds_1 [M_{i1} (M^2)_{1j} + \dots + M_{in} (M^2)_{nj}]$$

$$\leq \frac{s^3}{3!} (M^3)_{ij};$$

this process can be continued indefinitely. It follows that each of the n^2 infinite series constituting the elements of the matrix

$$\nabla = 1 + u^{(1)} + u^{(2)} + u^{(3)} + \dots$$

has terms whose moduli are respectively equal to or less than the real positive terms of the corresponding infinite series constituting the elements of the matrix

$$1 + sM + \frac{s^2}{2!} M^2 + \frac{s^3}{3!} M^3 + \dots$$

This last is, however, certainly convergent for all finite values of s ; its sum is, in fact, given in the case in which the algebraic equation satisfied by the matrix M has unequal roots by the formula given, *Proc. Lond. Math. Soc.*, Vol. xxxiv., p. 114, which can be easily modified to suit the case of repeated roots. Each of the elements of the matrix ∇ is thus an absolutely and uniformly converging series for the whole of the star region considered, and represents an analytic function of t , single-valued, finite, continuous, and developable throughout; and differentiation and integration term by term of the series are permissible.

Hence, if we take

$$x = (1 + Qu + QuQu + \dots) x^0,$$

$$\text{we obtain} \quad \frac{dx}{dt} = u(1 + Qu + QuQu + \dots) x^0 = ux,$$

showing that $x = \nabla x^0$ satisfies the differential equations and reduces to x_0 when $t = 0$.*

* If $\omega_1, \dots, \omega_r$ be functions of t , and $A_1, \dots, A_r$ the infinitesimal transformations of the first parameter group (cf. §§ 1, 2), we have the important application that solutions of the partial equation

$$\frac{\partial F}{\partial t} + \sum_{\sigma=1}^r \omega_{\sigma} A_{\sigma} F = 0$$

are given by $\beta^{-1} a e$, where $e_1, \dots, e_r$ satisfy the linear equations

$$\frac{de}{dt} = -E\omega.$$

The formula of the text leads, for the non-homogeneous equation

$$\frac{dx}{dt} = ux + c,$$

to the solutions

$$x = \nabla x_0 + \nabla Q \nabla^{-1} c;$$

thus the solutions of the equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^n (c_i + u_{i1} x_1 + \dots + u_{in} x_n) \frac{\partial f}{\partial x_i} = 0,$$

reducing to x when $t = 0$, are $\nabla^{-1} x - Q \nabla^{-1} c$.

Example (i.)—For an ordinary differential equation with one independent and one dependent variable

$$\frac{dx}{dt} = ux,$$

we easily verify that

$$QuQu = \frac{1}{2!}(Qu)^2, \quad QuQuQu = \frac{1}{3!}(Qu)^3, \text{ \&c.,}$$

and so get the ordinary solution

$$x = (\exp. Qu) x^0.$$

Example (ii.)—The ordinary linear differential equation of the second order

$$\frac{d^2x}{dt^2} = wx,$$

is equivalent to the pair $dx/dt = x'$, $dx'/dt = wx$,

namely, to

$$\left(\frac{dx}{dt}, \frac{dx'}{dt} \right) = \begin{pmatrix} 0 & 1 \\ w & 0 \end{pmatrix} (x, x').$$

Putting then

$$u = \begin{pmatrix} 0 & 1 \\ w & 0 \end{pmatrix},$$

we find

$$x = \Delta_1 x_0 + \Delta_2 x'_0,$$

where Δ_1, Δ_2 are the series

$$\Delta_1 = 1 + Q^2w + Q^2wQ^2w + Q^2wQ^2wQ^2w + \dots,$$

$$\Delta_2 = t + Q^2wt + Q^2wQ^2wt + Q^2wQ^2wQ^2wt + \dots,$$

as the solution for which x and dx/dt have the values x_0 and x'_0 for $t = 0$, it being understood that Q^2w denotes $Q(Qw)$, and so on, and Q^2wt denotes $Q[Q(wt)]$, and so on. It is manifest that the terms $Q^2w, Q^2wt, Q^2wQ^2w, Q^2wQ^2wt, \dots$ vanish respectively to the orders 2, 3, 4, 5, ..., when $t = 0$, and, if w be a linear aggregate $\lambda_1 w_1 + \lambda_2 w_2 + \dots$ of other functions $w_1, w_2, \dots$ with constant coefficients, $\lambda_1, \lambda_2, \dots$, they are homogeneous polynomials in $\lambda_1, \lambda_2, \dots$, respectively, of dimensions 1, 1, 2, 2, ...

For instance, if in Bessel's equation

$$\xi^2 d^2x/d\xi^2 + \xi dx/d\xi + (\xi^2 - n^2)x = 0$$

we put, with arbitrary a ,

$$\xi = ae^{t'}, \quad m = \frac{1}{4}n^2, \quad c = \frac{1}{4}a,$$

we obtain $d^2x/dt^2 = (m - ce')x$ or $w = m - ce'$.

Hence we find

$$\begin{aligned} \Delta_1 = 1 + & \left\{ m \frac{t^2}{2!} - c(e^t - 1 - t) \right\} \\ & + \left\{ m^2 \frac{t^4}{4!} + mc \left[4 + 2t + \frac{t^2}{2!} + \frac{t^3}{3!} - e^t (4 - 2t + \frac{1}{2}t^2) \right] \right. \\ & \left. + c^2 \left[-\frac{5}{24} - \frac{1}{2}t + e^t (1 - t) + \frac{1}{4}e^{2t} \right] \right\} + \dots, \\ \Delta_2 = t + & \left\{ m \frac{t^3}{3!} - c[(t-2)e^t + t + 2] \right\} \\ & + \left\{ m^2 \frac{t^5}{5!} + mc \left[-8 - 4t - t^2 - \frac{t^3}{3!} + e^t (8 - 4t + t^2 - \frac{t^3}{3!}) \right] \right. \\ & \left. + c^2 \left[\frac{3}{24} + \frac{1}{4}t + te^t + \frac{1}{4}(t-3)e^{2t} \right] \right\} + \dots \end{aligned}$$

Since these are convergent for all finite values of t , they can be rearranged in powers of t , and will then agree with the power series found in the ordinary way.

If, on the other hand, we take Legendre's equation

$$(t^2 - 1) d^2y/dt^2 + 2t dy/dt = n(n+1)y,$$

and put $y\sqrt{t^2-1} = x$,

it reduces to $d^2y/dt^2 = wx$,

with $w = n(n+1)(t^2-1)^{-1} - (t^2-1)^{-2}$.

Setting aside the practical difficulty of expressing the successive quadratures of w in explicit form, the series obtained by the theorem is valid for the whole finite plane of t with the exception of the two barriers joining the points $t = -1$, $t = 1$ to infinity. But when rearranged in powers of t it ceases to converge outside the circle of radius unity whose centre is at $t = 0$. In other words, the theorem may be looked upon as giving a rearrangement of the ordinary power series solutions, so that they continue to converge over the whole star region described.

10. In the previous section we have for simplicity dealt with the case of analytical functions of complex variables. But it is clear that if in the n linear equations

$$\frac{dx}{dt} = ux$$

the elements u_{ij} be functions of the real variable t , which between $t = 0$ and $t = a$ are single valued, throughout finite, and integrable, the method establishes the existence of integrals of the equations within these limits, and gives a rule for calculating them, which, for instance, with a mechanical integrator may be of practical utility for purposes of computation.

11. If the roots $\theta_1, \theta_2, \dots$ of the algebraic equation satisfied by a matrix M , supposed unequal, be within the circle of convergence of the series

$$f(x) = 1 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots,$$

it is immediately proved that the series

$$1 + a_1 \frac{M}{1!} + a_2 \frac{M^2}{2!} + a_3 \frac{M^3}{3!} + \dots$$

is convergent, its sum being given by the formula of *Proc. Lond. Math. Soc.*, Vol. xxxiv., 1902, p. 114, by replacing therein $e^x, e^y, \dots$, respectively by $f(\theta_1), f(\theta_2), \dots$. The case of equal roots follows at once from this formula.

If, now, $u_1, u_2, \dots$ be matrices of the same order, such that for a certain range of values of the independent variable t their elements are less in absolute value than those of a matrix M which are real and positive, if $c_1, c_2, \dots$ be numerical quantities whose moduli are $a_1, a_2, \dots$, it is easily proved, as in § 9, that the convergence of the series

$$\nabla = 1 + c_1 Qu_1 + c_2 Qu_1 Qu_2 + c_3 Qu_1 Qu_2 Qu_3 + \dots$$

is involved in that of

$$1 + a_1 M + \frac{a_2 M^2}{2!} + \frac{a_3 M^3}{3!} + \dots$$

In particular, if $f(x)$ be an integral function, the series ∇ converges in a star region excluding only the singularities of the elements of the matrices $u_1, u_2, \dots$.

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12. The discussion of the particular case when $u_1, u_2, \dots$ are single finite integrable functions of t is scarcely cognate to the present note, but seems worth remark. Suppose that, for instance,

$$u_r = 1 + u_1^{(r)} t + u_2^{(r)} t^2 + \dots$$

are all polynomials in t , reducing to unity when $t=0$; then the series

$$\nabla = 1 + c_1 Qu_1 + c_2 Qu_1 Qu_2 + \dots$$

is a series of polynomials converging within a star figure excluding the singularities of $u_1, u_2, \dots$, and lying within the circle of convergence of the series

$$1 + a_1 Mt + \frac{a_2}{2!} M^2 t^2 + \dots,$$

where

$$a_r = |c_r| \quad \text{and} \quad |u_r| < M;$$

and, if

$$\nabla = F(t),$$

we have

$$c_r = \left[\frac{1}{u_r} \frac{d}{dt} \frac{1}{u_{r-1}} \frac{d}{dt} \dots \frac{1}{u_1} \frac{d}{dt} F(t) \right]_{t=0}.$$

When each of $u_1, u_2, u_3, \dots$ reduces to unity, we have Taylor's theorem; the same also, after a change of variable, when

$$u_1 = u_2 = \dots$$

In general, if it were possible to choose the polynomials $u_1, u_2, \dots$, for a given function $F(t)$, so that the function

$$1 + c_1 t + \frac{c_2 t^2}{2!} + \dots$$

was an integral function, we should have an expansion of $F(t)$ in polynomials valid over the whole of a finite star region excluding only the singularities of $F(t)$ and $u_1, u_2, \dots$.