If p be any number of the form $4m+1$, then

(i.)

$$\begin{aligned} &E(p) \\ &-2E(p-4)-2E(p-8) \\ &+3E(p-12)+3E(p-16)+3E(p-20) \\ &-4E(p-24)- \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &= \\ &\chi(p) \\ &+2\chi(p-4)+2\chi(p-8) \\ &+3\chi(p-12)+3\chi(p-16)+3\chi(p-20) \\ &+4\chi(p-24)- \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

(ii.)

$$\begin{aligned} &E(p) \\ &-2E(p-8)-2E(p-16) \\ &+3E(p-24)+3E(p-32)+3E(p-40) \\ &-4E(p-48)- \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &= \\ &(-)^m\{\chi(p) \\ &+2\chi(p-8)+2\chi(p-16) \\ &+3\chi(p-24)+3\chi(p-32)+3\chi(p-40) \\ &+4\chi(p-28)+ \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots\}, \end{aligned}$$

the series being continued so long as the arguments remain positive.

The Relations of the Intersections of a Circle with a Triangle.

By Mr. H. M. Taylor.

[*Read Feb.* 14*th*, 1884.]

If, in a given triangle ABC, a triangle $\alpha\beta\gamma$ be inscribed, and the circumscribed circle of the triangle $\alpha\beta\gamma$ cut the sides BC, CA, AB again respectively in α', β', γ', then, as will be proved, if the triangle $\alpha\beta\gamma$ remains constant in shape (see Fig. 1)—

(1) The angles of the triangle $\alpha'\beta'\gamma'$ are determinate.

(2) The lines $\alpha\beta'$, $\beta\gamma'$, $\gamma\alpha'$, $\alpha'\beta$, $\beta'\gamma$, $\gamma'\alpha$ remain constant in direction.

(3) If the lines $\beta\gamma'$, $\gamma\alpha'$, $\alpha\beta'$ (see Fig. 2) form a triangle $A'B'C'$, and

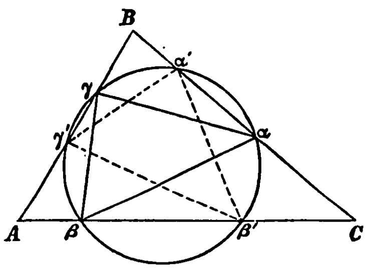

FIG. 1.

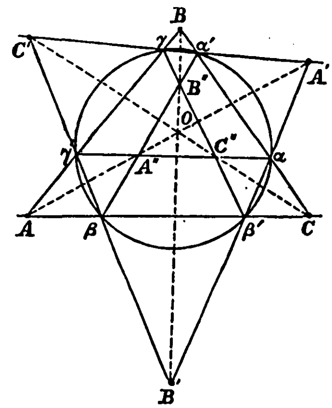

FIG. 2.

the lines $\gamma\beta'$, $\alpha\gamma'$, $\beta\alpha'$ form a triangle $A''B''C''$, the triangles ABC, $A'B'C'$, $A''B''C''$ are copolar, their common pole O being a fixed point.

(4) The ratio of the radii of the circles ABC, $\alpha\beta\gamma$ is proportional to the sine of the inclination of one of the sides of the triangle $\alpha\beta\gamma$ to a fixed straight line, *i.e.*, $\frac{R}{r} = L\sin(\psi+\theta)$.

(5) The locus of the centre of the circle $\alpha\beta\gamma$ is a straight line.

(6) The envelope of any side of the triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ is a parabola which touches two sides of the triangle ABC.

(7) The circle $\alpha\beta\gamma$ envelopes a conic whose centre is at the centre of the minimum $(\alpha\beta\gamma)$ circle, and one of whose axes is in the direction of the locus of the centre of the $(\alpha\beta\gamma)$ circle.

1. *The angles of the triangle $\alpha'\beta'\gamma'$ are determinate.*

In Fig. 1, all the six points lie in the sides BC, CA, AB of the triangle ABC.

In this case, which may be regarded as the typical case, the relations which exist between the angles of the triangles ABC, $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, may be written

$$\left.\begin{aligned} \alpha+\alpha' &= B+C \\ \beta+\beta' &= C+A \\ \gamma+\gamma' &= A+B \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This may be proved at once, as follows (see Fig. 1):

$$\beta = \alpha\beta\gamma = \pi - \alpha\alpha'\gamma = B\alpha'\gamma,$$

$$\beta' = \alpha'\beta'\gamma' = \pi - \alpha'\gamma\gamma' = B\gamma\alpha',$$

therefore $\beta+\beta' = B\alpha'\gamma + B\gamma\alpha' = \pi - B = A + C.$

Similarly it can be shewn that

$$\alpha+\alpha' = B+C,$$

and $$\gamma+\gamma' = A+B.$$

We may select the pairs of triangles in *four* different ways, and the proposition applies equally to all the combinations.

The point γ may be taken with the following pairs:

$$\alpha, \beta; \quad \alpha, \beta'; \quad \alpha', \beta; \quad \alpha', \beta',$$

and it is easily seen that

$$\beta\alpha\gamma+\beta'\alpha'\gamma' = \beta'\alpha\gamma+\beta\alpha'\gamma' = \beta\alpha'\gamma+\beta'\alpha\gamma' = \beta'\alpha'\gamma+\beta\alpha\gamma',$$

the sum of each pair of angles being equal to the sum

$$\beta\alpha\gamma+\beta'\alpha\gamma'.$$

In Fig. 3, which may be regarded as the exceptional case, the circle cuts one side BC twice, but the other two sides produced. In this case

$$\left.\begin{aligned} \alpha+\alpha' &= \pi+A \\ \beta+\beta' &= B \\ \gamma+\gamma' &= C \end{aligned}\right\} \quad \ldots\ldots(2).$$

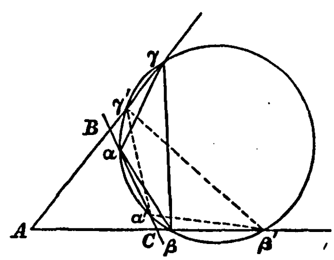

Fig. 3.

This may be proved thus (see Fig. 3):

$$\alpha = \beta\alpha\gamma = \pi-\beta\beta'\gamma,$$

and $$\alpha' = \beta'\alpha'\gamma' = \pi-\beta'\gamma\gamma',$$

therefore $$\alpha+\alpha' = \pi-A\beta'\gamma+\pi-A\gamma\beta' = \pi+A;$$

also $$\beta = \alpha\beta\gamma = \pi-\alpha\gamma'\gamma = B\gamma'\alpha,$$

and $$\beta' = \alpha'\beta'\gamma' = \pi-\alpha'\alpha\gamma' = B\alpha\gamma',$$

therefore $$\beta+\beta' = B\gamma'\alpha+B\alpha\gamma' = ABC = B;$$

and similarly $$\gamma+\gamma' = C.$$

The only special case of interest in the exceptional case (Fig. 3) is when the triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ become coincident, and the circle becomes an escribed circle of the triangle ABC. We shall therefore confine our attention for the future to the general case in which equations (1) hold.

Special Cases.

The most interesting special cases to consider are

Case I., where $$\left.\begin{aligned} \alpha' &= \alpha \\ \beta' &= \beta \\ \gamma' &= \gamma \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

whence
$$\left.\begin{aligned} \alpha &= \frac{\pi}{2}-\frac{A}{2} \\ \beta &= \frac{\pi}{2}-\frac{B}{2} \\ \gamma &= \frac{\pi}{2}-\frac{C}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4).$$

Case II., where
$$\left.\begin{aligned} \alpha &= A \\ \beta &= B \\ \gamma &= C \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(5),$$

whence
$$\left.\begin{aligned} \alpha' &= \pi-2A \\ \beta' &= \pi-2B \\ \gamma' &= \pi-2C \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(6).$$

Case III., where
$$\left.\begin{aligned} \alpha' &= \gamma \\ \beta' &= \alpha \\ \gamma' &= \beta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(7),$$

whence
$$\left.\begin{aligned} \alpha &= C = \beta' \\ \beta &= A = \gamma' \\ \gamma &= B = \alpha' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(8).$$

If we apply the general case of equations (1) to the triangle $A'B'C'$ (Fig. 2), we obtain at once the equations

$$\left.\begin{aligned} \beta+\gamma' &= B'+C' \\ \gamma+\alpha' &= C'+A' \\ \alpha+\beta' &= A'+B' \end{aligned}\right\},$$

whence
$$\left.\begin{aligned} A' &= C+\gamma-\beta \\ B' &= A+\alpha-\gamma \\ C' &= B+\beta-\alpha \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(9);$$

and, if we apply the equations (1) to the triangle $A''B''C''$, we obtain

$$\left.\begin{aligned} \gamma+\beta' &= B''+C'' \\ \alpha+\gamma' &= C''+A'' \\ \beta+\alpha' &= A''+B'' \end{aligned}\right\},$$

whence
$$\left.\begin{aligned} A'' &= B+\beta-\gamma \\ B'' &= C+\gamma-\alpha \\ C'' &= A+\alpha-\beta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(10).$$

2. *The lines $\alpha\beta'$, &c. remain constant in direction.*

It is seen at once, from Fig. 4, that the inclinations of $\alpha\beta$ to the sides CA, CB are γ, γ' respectively.

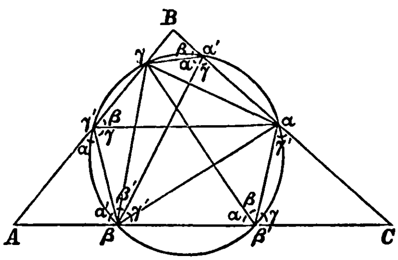

FIG. 4.

3. *The triangles ABC, $A'B'C'$, $A''B''C''$ are copolar, their common pole O being a fixed point.*

Now, applying Pascal's theorem to the points taken in the order $\alpha'\alpha\beta'\beta\gamma'\gamma$, we see that the pairs of lines $\alpha'\alpha$, $\beta\gamma'$; $\beta'\beta$, $\gamma\alpha'$; $\gamma'\gamma$, $\alpha\beta'$ intersect on a straight line; or (Fig. 2) that the pairs of sides BC, $B'C'$; CA, $C'A'$; AB, $A'B'$ of the triangles ABC, $A'B'C'$ intersect on a straight line. In other words, the triangles ABC, $A'B'C'$ are coaxal. They are therefore copolar, *i.e.*, AA', BB', CC' meet in a point, O suppose.

Again, applying Pascal's theorem to the points taken in the order $\alpha\alpha'\beta\beta'\gamma\gamma'$, we see that the pairs of lines

$$\alpha\alpha',\ \beta'\gamma;\quad \beta\beta',\ \gamma'\alpha;\quad \gamma\gamma',\ \alpha'\beta$$

intersect on a straight line; or (Fig. 2) that the pairs of sides BC, $B''C''$; CA, $C''A''$; AB, $A''B''$ of the triangles ABC, $A''B''C''$ intersect on a straight line.

In other words, the triangles ABC, $A''B''C''$ are coaxal. They are therefore copolar, *i.e.*, AA'', BB'', CC'' meet in a point.

Again, applying Pascal's theorem to the points taken in the order $\beta'\beta\alpha'\gamma\gamma'\alpha$, we see that the pairs of lines

$$\beta'\beta,\ \gamma\gamma';\quad \beta\alpha',\ \gamma'\alpha;\quad \alpha'\gamma,\ \alpha\beta'$$

intersect on a straight line; *i.e.*, (Fig. 2) $AA''A'$ is a straight line.

Similarly $BB''B'$, $CC''C'$ are straight lines.

Therefore we have three straight lines, viz., $AA''A'$, $BB''B'$, $CC''C'$, passing through the point O.

It may be remarked that, so far, this proposition is true for any conic section.

It appears, from Fig. 4, that as long as $\alpha\beta\gamma$ is of constant shape, the lines $\beta\gamma'$, $\gamma\alpha'$, $\alpha\beta'$ remain parallel to themselves; and the same remark applies to the lines $\beta'\gamma$, $\gamma'\alpha$, $\alpha'\beta$.

We see therefore (Fig. 2) that, since $\gamma'\beta$, $\gamma'A''$, $\beta A''$ are all fixed in direction, $AA''A'$ is a fixed straight line.

Similarly, $BB''B'$, $CC''C'$ are fixed straight lines, and therefore O is a fixed point.

4. $\dfrac{R}{r} = L \sin(\psi+\theta)$.

As the whole figure is determinate, when we fix three of the six points α, β, γ, α', β', γ', it is clear that we can express any angle and any line in the figure in terms of the sides and angles of the triangle ABC, of the angles α, β, γ, and of the inclination of one of the sides of the triangle $\alpha\beta\gamma$ to one of the sides of ABC.

If we call the angle $B\alpha\gamma$, θ, the inclinations of the other sides of $\alpha\beta\gamma$ to the corresponding sides of ABC are given in Fig. 5.

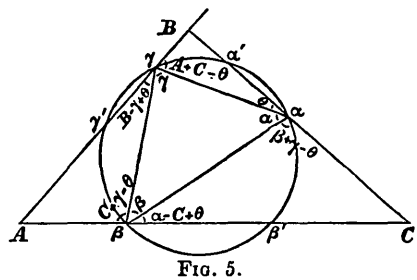

FIG. 5.

We will now establish the relation which exists between the angle θ, and the radius of the circumscribing circle of the triangle $\alpha\beta\gamma$, assuming the angles α, β, γ known.

We have (Fig. 5)

$$B\alpha = \gamma\alpha \frac{\sin B\gamma\alpha}{\sin \gamma B\alpha} = \gamma\alpha \frac{\sin (A+C-\theta)}{\sin B},$$

$$C\alpha = \beta\alpha \frac{\sin C\beta\alpha}{\sin \beta C\alpha} = \beta\alpha \frac{\sin (\alpha-C+\theta)}{\sin C},$$

and $$B\alpha + C\alpha = BC = a.$$

Therefore, substituting for $B\alpha$, $C\alpha$, and using the relations

$$\gamma\alpha = 2r \sin \beta,$$

$$\beta\alpha = 2r \sin \gamma,$$

where r is the radius of the circle $\alpha\beta\gamma$, we obtain

$$2r \sin \beta \frac{\sin (A+C-\theta)}{\sin B} + 2r \sin \gamma \frac{\sin (\alpha-C+\theta)}{\sin C} = a.$$

Now, if R be the radius of the circumscribing circle of the triangle ABC, we have $a = 2R \sin A$. Therefore we may write

$$\frac{R}{r} = \frac{\sin \beta \sin (B+\theta)}{\sin A \sin B} + \frac{\sin \gamma \sin (\alpha-C+\theta)}{\sin A \sin C} \quad\text{...........(11)}.$$

This equation may be written

$$\frac{R}{r} = L \sin (\psi+\theta) \quad\text{..........................(12)},$$

where
$$\left.\begin{aligned} L \sin \psi &= \frac{\sin \beta}{\sin A} + \frac{\sin \gamma \sin (\alpha-C)}{\sin A \sin C} \\ L \cos \psi &= \frac{\sin \beta}{\sin A} \cot B + \frac{\sin \gamma \cos (\alpha-C)}{\sin A \sin C} \end{aligned}\right\}.$$

It is clear that L must be a symmetrical function of A and α, B and β, C and γ. We will proceed to find $L^2 \sin^2 A \sin^2 B \sin^2 C$.

From the last equations we have

$$L^2 \sin^2 A = \left(\sin\beta + \frac{\sin\gamma \sin(\alpha - C)}{\sin C}\right)^2$$
$$+ \left(\sin\beta \cot B + \frac{\sin\gamma \cos(\alpha - C)}{\sin C}\right)^2$$
$$= \frac{\sin^2\beta}{\sin^2 B} + \frac{\sin^2\gamma}{\sin^2 C} + \frac{2\sin\beta\sin\gamma}{\sin B \sin C}\cos(B + C - \alpha),$$

therefore
$$L^2 \sin^2 A \sin^2 B \sin^2 C$$
$$= \sin^2\beta \sin^2 C + \sin^2\gamma \sin^2 B - 2\sin\beta\sin\gamma\sin B\sin C\cos(A+\alpha)$$
$$= \sin^2\beta\sin^2 C + \sin^2\gamma\sin^2 B - 2\cos\alpha\sin\beta\sin\gamma\cos A\sin B\sin C$$
$$+ 2\sin\alpha\sin\beta\sin\gamma\sin A\sin B\sin C$$
$$= \sin^2\beta\sin C(\sin A\cos B + \cos A\sin B)$$
$$+ \sin^2\gamma\sin B(\sin A\cos C + \cos A\sin C)$$
$$- 2\cos\alpha\sin\beta\sin\gamma\cos A\sin B\sin C$$
$$+ 2\sin\alpha\sin\beta\sin\gamma\sin A\sin B\sin C$$
$$= \sin^2\beta\sin A\cos B\sin C + \sin^2\gamma\sin A\sin B\cos C$$
$$+ \cos A\sin B\sin C(\sin^2\beta + \sin^2\gamma - 2\cos\alpha\sin\beta\sin\gamma)$$
$$+ 2\sin\alpha\sin\beta\sin\gamma\sin A\sin B\sin C$$
$$= \sin^2\alpha\cos A\sin B\sin C + \sin^2\beta\sin A\cos B\sin C$$
$$+ \sin^2\gamma\sin A\sin B\cos C$$
$$+ 2\sin\alpha\sin\beta\sin\gamma\sin A\sin B\sin C \quad \text{.................. (13)}.$$

5. *The locus of the centre of the circle* $\alpha\beta\gamma$ *is a straight line.*

It may be shewn in various ways that the locus of the centre of the circle $\alpha\beta\gamma$, when $\alpha\beta\gamma$ is of constant shape, is a straight line.

In Fig. 6 it is easily seen that, if x, y, z be the perpendiculars from

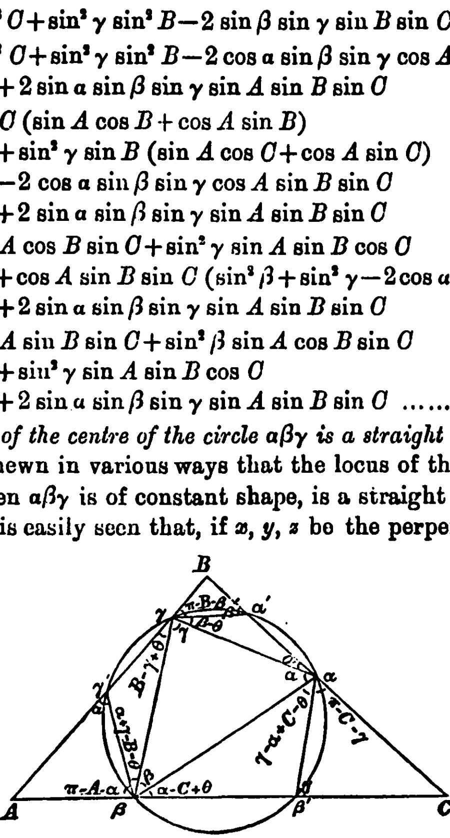

FIG. 6.

the centre of the circle $\alpha\beta\gamma$ on the sides of the triangle ABC, or, in other words, if x, y, z be the trilinear coordinates of the centre, using the triangle ABC as the triangle of reference, then

$$\left.\begin{aligned} x &= r\cos(\beta - \theta) \\ y &= r\cos(\gamma - \alpha + C - \theta) \\ z &= r\cos(\alpha + \gamma - B - \theta) \end{aligned}\right\} \quad \text{.................... (14)}.$$

The coordinates x_1, y_1, z_1 of the minimum $(\alpha\beta\gamma)$ circle are found as follows :—

From equations (12) and (14),

$$x_1 = \frac{R}{L}\cos\left(\beta+\psi-\frac{\pi}{2}\right) = \frac{R}{L}\sin(\beta+\psi)$$

$$= \frac{R}{L^2}(L\sin\psi\cos\beta + L\cos\psi\sin\beta)$$

$$= \frac{R}{L^2\sin A}\left\{\left(\sin\beta+\frac{\sin\gamma\sin(\alpha-C)}{\sin C}\right)\cos\beta + \left(\sin\beta\cot B+\frac{\sin\gamma\cos(\alpha-C)}{\sin C}\right)\sin\beta\right\}$$

$$\left.\begin{aligned} &= \frac{R}{L^2\sin A}\left(\frac{\sin\beta\sin\beta'}{\sin B}+\frac{\sin\gamma\sin\gamma'}{\sin C}\right) \\ \text{similarly}\quad y_1 &= \frac{R}{L^2\sin B}\left(\frac{\sin\gamma\sin\gamma'}{\sin C}+\frac{\sin\alpha\sin\alpha'}{\sin A}\right) \\ z_1 &= \frac{R}{L^2\sin C}\left(\frac{\sin\alpha\sin\alpha'}{\sin A}+\frac{\sin\beta\sin\beta'}{\sin B}\right)\end{aligned}\right\} \quad\ldots\ldots(14a).$$

From these equations, by eliminating r and θ, we obtain the equation of the straight line, which is the locus of the centre, *i.e.*,

$$x\sin(A+2\alpha)+y\sin(B+2\beta)+z\sin(C+2\gamma) = 0 \quad\ldots\ldots(15),$$

or $$x\sin(\alpha-\alpha')+y\sin(\beta-\beta')+z\sin(\gamma-\gamma') = 0.$$

We must observe, that Case I., *i.e.*, when $\alpha = A$, $\beta = B$, $\gamma = C$, is an exceptional one; for equations (14) then become

$$\left.\begin{aligned} x &= r\cos(B-\theta) \\ y &= r\cos(B-\theta) \\ z &= r\cos(B-\theta)\end{aligned}\right\}.$$

Hence we see that the centre is the point, whose coordinates satisfy the equations $x = y = z$, *i.e.*, the point is the centre of the inscribed circle of the triangle ABC.

In Case II., the equation (15) becomes

$$x\sin 3A+y\sin 3B+z\sin 3C = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

which is satisfied by the coordinates of the centre of the nine-point circle of the triangle ABC, which is a particular case of Case II.

In Case III., the equation (15) becomes

$$x\sin(A+2C)+y\sin(B+2A)+z\sin(C+2B) = 0,$$

which may be written

$$x\sin(B-C)+y\sin(C-A)+z\sin(A-B) = 0 \quad\ldots\ldots\ldots(17),$$

which is satisfied by the coordinates of the centre of the circumscribing circle of the triangle ABC, which is a particular case of Case III.

If a, a' (see Fig. 2) coincide, the point A' will coincide with them. We will call this point a''.

It will be seen (see Fig. 5) that, when a, a' coincide, $\theta=\beta$, and that

$$Ba'' = a\gamma \frac{\sin(A+C-\beta)}{\sin B} = \frac{2r\sin\beta\sin\beta'}{\sin B},$$

$$Ca'' = a\beta' \frac{\sin\gamma}{\sin C} = \frac{2r\sin\gamma\sin\gamma'}{\sin C},$$

wherefore a'' divides the side BC in the ratio

$$\frac{\sin\beta\sin\beta'}{\sin B} : \frac{\sin\gamma\sin\gamma'}{\sin C}.$$

If β'', γ'' be the points where BO, CO cut the opposite sides, we shall have

$$\left.\begin{aligned} A\gamma'' : \gamma''B &= \frac{\sin a\sin a'}{\sin A} : \frac{\sin\beta\sin\beta'}{\sin B} \\ Ba'' : a''C &= \frac{\sin\beta\sin\beta'}{\sin B} : \frac{\sin\gamma\sin\gamma'}{\sin C} \\ C\beta'' : \beta''A &= \frac{\sin\gamma\sin\gamma'}{\sin C} : \frac{\sin a\sin a'}{\sin A} \end{aligned}\right\} \quad \text{.............(18).}$$

It will be observed that the point O is the centre of mass of three masses proportional to

$$\frac{\sin A}{\sin a\sin a'},\quad \frac{\sin B}{\sin\beta\sin\beta'},\quad \frac{\sin C}{\sin\gamma\sin\gamma'},$$

placed at A, B, C respectively.

In Case I.,

$$Ba'' : a''C = \frac{\sin^2\beta}{\sin B} : \frac{\sin^2\gamma}{\sin C} = \frac{\cos^2\frac{B}{2}}{\sin B} : \frac{\cos^2\frac{C}{2}}{\sin C}$$

$$= \cot\frac{B}{2} : \cot\frac{C}{2} = s-b : s-c \quad \text{................(19),}$$

where $2s = a+b+c;$

therefore $Ba'' = s-b,\quad Ca'' = s-c;$

whence we see that a'' is the point of contact of the inscribed circle of ABC.

In Case III.,

$$Ba'' : a''C = \sin\beta' : \sin\gamma' = \sin 2B : \sin 2C$$

$$= \sin B\cos B : \sin C\cos C \quad \text{..................(20),}$$

therefore $$Ba'' = a\,\frac{\sin 2B}{\sin 2B+\sin 2C} = \frac{a\sin B\cos B}{\sin A\cos(B-C)}.$$

In Case III.,

$$Ba'' : a''C = \frac{\sin A \sin C}{\sin B} : \frac{\sin B \sin A}{\sin C} = \sin^2 C : \sin^2 B \ldots\ldots(21),$$

therefore
$$Ba'' = a\,\frac{\sin^2 C}{\sin^2 C + \sin^2 B} = \frac{ac^2}{b^2+c^2},$$

$$Ca'' = \frac{ab^2}{b^2+c^2}.$$

6. *The envelope of $\beta\gamma$ is a parabola which touches AB, AC.*

We can shew that each side of the triangle $\alpha\beta\gamma$, if it remain of constant shape, touches a parabola, which touches two sides of the triangle ABC.

Take the side $\beta\gamma$, and let us consider AC, AB as axes of oblique coordinates x and y.

Then the equation of $\beta\gamma$ may be written

$$\frac{x}{A\beta} + \frac{y}{A\gamma} = \frac{\beta\gamma}{\beta\gamma},$$

or (see Fig. 5),

$$\frac{x}{\sin(B-\gamma+\theta)} + \frac{y}{\sin(C+\gamma+\theta)} = \frac{2r\sin\alpha}{\sin A} = \frac{2R\sin\alpha}{L\sin A\sin(\psi-\theta)},$$

or
$$\frac{x}{\sin(\lambda+\theta)} + \frac{y}{\sin(\mu-\theta)} = \frac{M}{\sin(\psi-\theta)} \text{ suppose } \ldots\ldots\ldots(22),$$

$$2\sin(\psi-\theta)\{x\sin(\mu-\theta)+y\sin(\lambda+\theta)\} = 2M\sin(\lambda+\theta)\sin(\mu-\theta),$$

$$x\{\cos(\psi-\mu)-\cos(\psi+\mu-2\theta)\}+y\{\cos(\psi-\lambda-2\theta)-\cos(\psi+\lambda)\}$$
$$= M\{\cos(\lambda-\mu+2\theta)-\cos(\lambda+\mu)\},$$

which may be written

$$x\cos(\psi-\mu)-y\cos(\psi+\lambda)+M\cos(\lambda+\mu)$$
$$+\cos 2\theta\{-x\cos(\psi+\mu)+y\cos(\psi-\lambda)-M\cos(\lambda-\mu)\}$$
$$+\sin 2\theta\{-x\sin(\psi+\mu)+y\sin(\psi-\lambda)+M\sin(\lambda-\mu)\} = 0.$$

This we will write for the time

$$A + B\cos 2\theta + C\sin 2\theta = 0.$$

Therefore the envelope is

$$A^2 = B^2 + C^2.$$

Therefore the equation of the envelope required is

$$\{\ \ x\cos(\psi-\mu)-y\cos(\psi+\lambda)+M\cos(\lambda+\mu)\}^2$$
$$= \{-x\cos(\psi+\mu)+y\cos(\psi-\lambda)-M\cos(\lambda-\mu)\}^2$$
$$+\{-x\sin(\psi+\mu)+y\sin(\psi-\lambda)+M\sin(\lambda-\mu)\}^2,$$

which, being rearranged, gives

$$x^2 \sin^2(\psi-\mu)+y^2\sin^2(\psi+\lambda)-2xy\sin(\psi-\mu)\sin(\psi+\lambda)$$
$$-2xM\sin(\psi-\mu)\sin(\lambda+\mu)$$
$$-2yM\sin(\psi+\lambda)\sin(\lambda+\mu)+M^2\sin^2(\lambda+\mu)=0 \quad \text{.........(23).}$$

This is the equation of a parabola which touches the axes of coordinates at $x_1 = M.\dfrac{\sin(\lambda+\mu)}{\sin(\psi-\mu)}$,

and at $y_1 = M.\dfrac{\sin(\lambda+\mu)}{\sin(\psi+\lambda)}$.

Now $\lambda = B-\gamma, \quad \mu = C+\gamma$.

Therefore the coordinates of the points of contact are

$$x_1 = M.\frac{\sin A}{\sin(\psi-C-\gamma)}, \quad y_1 = M.\frac{\sin A}{\sin(\psi+B-\gamma)} \quad \text{.......(24).}$$

It will be seen that, since the inclination of $\beta\gamma$, $\beta'\gamma'$ to the sides AB, AC are merely interchanged, and their lengths are in the constant ratio $\dfrac{\sin a}{\sin a'}$, the corresponding enveloping parabolas will have the same relation.

In Case I., $\psi = -\dfrac{B}{2}, \quad \gamma = \dfrac{\pi}{2}-\dfrac{C}{2}$,

therefore
$$\psi - C-\gamma = -\frac{B}{2}-C-\frac{\pi}{2}+\frac{C}{2} = \frac{A}{2}-\pi,$$
$$\psi+B-\gamma = -\frac{B}{2}+B-\frac{\pi}{2}+\frac{C}{2} = -\frac{A}{2}.$$

Therefore the distances x_1, y_1 of the points of contact with the sides are equal. Now

$$M\sin A = \frac{2R\sin a}{L} = \frac{2R\cos\frac{A}{2}}{-\frac{R}{r_1}} = -2r_1\cos\frac{A}{2},$$

where r_1 is the radius of the inscribed circle of the triangle ABC.

Therefore
$$x_1 = y_1 = \frac{-2r_1\cos\frac{A}{2}}{-\sin\frac{A}{2}} = 2r_1\cot\frac{A}{2} = 2(s-a),$$

where $2s = a+b+c$.

The parabola therefore touches the sides AB, AC at double the distance from A of the points of contact of the inscribed circle of ABC.

In Case II., $\psi = C+\dfrac{\pi}{2}, \quad \gamma = C$,

therefore
$$\psi - C - \gamma = -C + \frac{\pi}{2}$$
$$\psi + B - \gamma = B + \frac{\pi}{2},$$

and
$$M = \frac{2R\sin\alpha}{L\sin A} = R;$$

therefore
$$\left.\begin{aligned} x_1 &= R\frac{\sin A}{\cos C} = \frac{a}{2\cos C} \\ y_1 &= R\frac{\sin A}{\cos B} = \frac{a}{2\cos B} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(25).$$

In Case III., $\psi = -\phi$, $L = -\frac{1}{\sin\phi}$, $\gamma = B$;

therefore
$$\psi - C - \gamma = -\phi - C - B = A - \phi - \pi,$$
$$\psi + B - \gamma = -\phi,$$
$$M = \frac{2R\sin\alpha}{L\sin A} = -\frac{2R\sin C\sin\phi}{\sin A} = -\frac{c\sin\phi}{\sin A},$$

therefore
$$\left.\begin{aligned} x_1 &= +\frac{c\sin\phi}{\sin(A-\phi)} = \frac{c\sin B\sin C}{\sin^2 A} \\ y_1 &= +\frac{c\sin\phi}{\sin\phi} = c \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(26).$$

Therefore the parabola touches AB at B and AC at a distance $b\frac{c^2}{a^2}$ from A.

7. *The envelope of the circle $\alpha\beta\gamma$ is a conic.*

Let x, y, z, x', y', z' be the coordinates of the centres of two equal circles of radius r; then, from equations (14),

$$x + x' = r\{\cos(\beta-\theta) + \cos(\beta-\theta')\} = 2r\cos\{\beta - \tfrac{1}{2}(\theta+\theta')\}\cos(\theta-\theta');$$

but, from equation (12),
$$\frac{R}{r} = L\sin(\psi+\theta) = L\sin(\psi+\theta'),$$

therefore
$$\theta + \psi = \pi - (\psi + \theta'),$$

or
$$\theta + \theta' = \pi - 2\psi = 2\theta_1 \text{ suppose},$$

if θ_1 be the value of θ which makes the radius a minimum.

Therefore, if $\bar{x}$ be the coordinate of the middle point of the line joining the centres of the pair of equal circles (r),

$$\bar{x} = \tfrac{1}{2}(x+x') = r\cos(\beta-\theta_1)\cos(\theta-\theta'),$$

similarly
$$\bar{y} = \tfrac{1}{2}(y+y') = r\cos(\gamma-\alpha+C-\theta_1)\cos(\theta-\theta'),$$
$$\bar{z} = \tfrac{1}{2}(z+z') = r\cos(\alpha+\gamma-B-\theta_1)\cos(\theta-\theta').$$

Therefore, if x_1, y_1, z_1 be the coordinates of the centre of the minimum circle
$$\frac{\bar{x}}{x_1} = \frac{\bar{y}}{y_1} = \frac{\bar{z}}{z_1},$$
or the point $(\bar{x}, \bar{y}, \bar{z})$ coincides with the point (x_1, y_1, z_1).

If r_1 be the minimum radius,
$$\frac{R}{r_1} = L,$$
$$\frac{R}{r} = L\sin(\psi+\theta) = L\sin(\tfrac{1}{2}\pi-\theta_1+\theta) = L\cos(\theta_1-\theta),$$
therefore
$$r\cos(\theta-\theta_1) = r_1.$$

Now
$$\begin{aligned} x-x_1 &= r\cos(\beta-\theta)-r_1\cos(\beta-\theta_1)\\ &= r\{\cos(\beta-\theta)-\cos(\theta-\theta_1)\cos(\beta-\theta_1)\}\\ &= r\{\cos(\beta-\theta_1+\theta_1-\theta)-\cos(\theta-\theta_1)\cos(\beta-\theta_1)\}\\ &= r\,.\sin(\theta-\theta_1)\sin(\beta-\theta_1) = \lambda\,.\,\delta, \end{aligned}$$
where δ is the distance between the centres of the circles r, r_1.

Therefore, eliminating θ from the equations
$$\left.\begin{aligned} r\sin(\theta-\theta_1)\sin(\beta-\theta_1) &= \lambda\delta\\ r\cos(\theta-\theta_1) &= r_1 \end{aligned}\right\},$$
we obtain
$$r^2\sin^2(\beta-\theta_1) = \lambda^2\delta^2+r_1^2\sin^2(\beta-\theta_1).$$

Now, if we take the centre of the circle r for the origin, and the locus of the centres of the $(\alpha\beta\gamma)$ circle as the axis of x, we may write the equation of the circle thus,
$$(x-\delta)^2+y^2 = \lambda^2\delta^2\operatorname{cosec}^2(\beta-\theta_1)+r_1^2,$$
or
$$x^2+y^2-r_1^2-2\delta x = \delta^2\{\lambda^2\operatorname{cosec}^2(\beta-\theta_1)-1\}.$$

The envelope therefore is
$$x^2\frac{\lambda^2}{\lambda^2-\sin^2(\beta-\theta_1)}+y^2 = r_1^2,$$
which is a conic section, whose axes are
$$r_1 \text{ and } r_1\sqrt{\left\{1-\frac{\sin^2(\beta-\theta_1)}{\sin^2\phi}\right\}},$$
where ϕ is the angle which the straight line (15), the locus of the centre, makes with BC.

We will now examine the three special cases marked I., II., III. separately.

Case I.,
$$\left.\begin{aligned} \alpha &= \alpha'\\ \beta &= \beta'\\ \gamma &= \gamma' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

This is the same as saying

$$\text{arc}\,\beta\gamma = \text{arc}\,\beta'\gamma' \text{ (see Fig. 7)},$$
&c.

therefore $\text{arc}\,\beta\beta' = \text{arc}\,\gamma\gamma'$
$= \text{arc}\,\alpha\alpha'$.

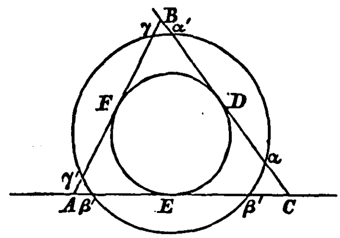

FIG. 7.

The chords $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are therefore equal to one another, and consequently equidistant from the centre of the circle.

It appears therefore that in this case the circle is concentric with the inscribed circle of the triangle ABC.

It can easily be seen that

$$\left.\begin{aligned} A\beta &= A\gamma' \\ B\gamma &= B\alpha' \\ C\alpha &= C\beta' \end{aligned}\right\}$$

and that $\beta\gamma'$ is parallel to $\beta'\gamma$,

$\gamma\alpha'$,, ,, $\gamma'\alpha$,

$\alpha\beta'$,, ,, $\alpha'\beta$.

It will be seen that the point O in this case is the intersection of AD, BE, CF, where D, E, F are the points of contact of the inscribed circle of the triangle ABC.

In Case I.,

$$\left.\begin{aligned} \alpha = \alpha' &= \frac{\pi}{2} - \frac{A}{2} \\ \beta = \beta' &= \frac{\pi}{2} - \frac{B}{2} \\ \gamma = \gamma' &= \frac{\pi}{2} - \frac{C}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Therefore, by equation (11),

$$\frac{R}{r} = \frac{\cos\frac{B}{2}\sin(B+\theta)}{\sin A \sin B} + \frac{\cos\frac{C}{2}\cos\left(\frac{A}{2}+C-\theta\right)}{\sin A \sin C},$$

or, rearranging,

$$\frac{R}{r} = \frac{1}{4}\,\frac{\sin(\theta+\tfrac{1}{2}B)}{\sin\tfrac{1}{2}A\,\sin\tfrac{1}{2}B\,\sin\tfrac{1}{2}C} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(27).$$

Here, if $\theta = \frac{\pi}{2} - \frac{B}{2}$, we have the case of the inscribed circle, when, as is well known,

$$\frac{R}{r} = \frac{1}{4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}.$$

It is evident that the inscribed circle is the minimum circle of Case I.

Case II., $\left.\begin{aligned} \alpha &= A \\ \beta &= B \\ \gamma &= C \end{aligned}\right\}$ ....(5)

(see Fig. 8).

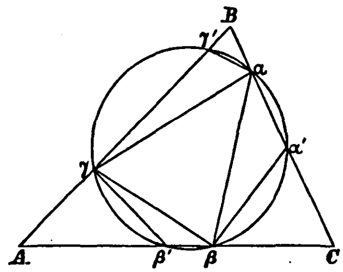

Fig. 8.

Here, if $A\alpha$, $B\beta$, $C\gamma$ meet in a point, the point is the centre of the triangle ABC, and α, β, γ are the middle points of BC, CA, AB.

In this case we have the nine-point circle of the triangle ABC.

$$\angle A\gamma'\beta = \angle \gamma\gamma'\beta = \angle \gamma\alpha\beta = \alpha = A.$$

Therefore $\beta\gamma' = \beta A$, similarly $\beta\alpha' = \beta C$;
similarly also $\gamma\alpha' = \gamma B$, $\alpha\beta' = \alpha C$; $\gamma\beta' = \gamma A$, $\alpha\gamma' = \alpha B$ ....(28).

It appears therefore that the perimeter of the hexagon $\alpha\beta'\gamma\alpha'\beta\gamma'$ is equal to the perimeter of the triangle ABC.

Again, from equation (11),

$$\frac{R}{r} = \frac{\sin(A+C-\theta)+\sin(A-C+\theta)}{\sin A}$$

$$= \frac{2\sin A\cos(C-\theta)}{\sin A} = 2\cos(C-\theta) = 2\sin\left(\frac{\pi}{2}+C-\theta\right) \text{...(29).}$$

If $\theta = C$, we have the case of the nine-point circle, and $\frac{R}{r} = 2$.

It is evident that the nine-point circle is the minimum circle of Case II.

Case III., $\left.\begin{aligned} \alpha' &= \gamma = B \\ \beta' &= \alpha = C \\ \gamma' &= \beta = A \end{aligned}\right\}$ ...(8).

Here $\beta'\gamma' = \alpha\beta$ (see Fig. 9),

therefore $\beta\gamma' = \alpha\beta'$,

therefore $\gamma'\alpha$ is parallel to $\beta\beta'$, *i.e.* to AC. Similarly $\alpha'\beta$ is parallel to AB and $\beta'\gamma$ to BC.

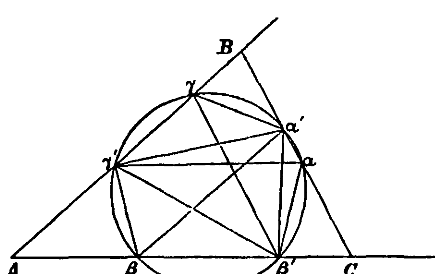

Fig. 9.

Again, in this case, because $Ca \,.\, Ca' = C\beta \,.\, C\beta'$,

$$\frac{Ca}{C\beta'} = \frac{C\beta}{Ca'} = \frac{CA}{CB} \quad\ldots\ldots\ldots\ldots(30).$$

The perpendicular distances of $\beta'\gamma$ from BC, and of $\gamma'a$ from CA, are $C\beta' \sin C$, and $Ca \sin C$ respectively; hence it follows that in this case the perpendicular distances of $a'\beta$, $\beta'\gamma$, $\gamma'a$ from the sides of the triangle ABC, AB, BC, CA respectively, are proportional to AB, BC, CA.

In the particular case when these three lines $a'\beta$, $\beta'\gamma$, $\gamma'a$ pass through a point, the circle becomes what has been called the "Triplicate Ratio" circle. (See *Quarterly Journal of Mathematics*, Vol. xix., p. 342.)

The point through which the three lines are drawn is the centre of similitude of the two triangles ABC, $A''B''C''$. (Fig. 2.)

In Case III. (Fig. 9), we see at once that the angles $Ba\gamma$, $C\beta a$, $A\gamma\beta$ are equal, since we have

$$a = C \quad \text{and} \quad \gamma = B.$$

In this case, therefore, the sides $a\gamma$, βa, $\gamma\beta$ of the triangle $a\beta\gamma$ make the same angle θ with the sides BC, CA, AB respectively of the triangle ABC.

It will be observed that the angle θ is the inclination of the sides of the triangle $a'\beta'\gamma'$ to those of ABC in the opposite direction.

Again, from equation (11),

$$\frac{R}{r} = \frac{\sin(A+C-\theta)}{\sin B} + \frac{\sin B \sin\theta}{\sin A \sin C} = \cos\theta + \sin\theta \left\{ \cot B + \frac{\sin B}{\sin A \sin C} \right\}$$

$$= \frac{\sin(\theta+\phi)}{\sin\phi} \quad\ldots\ldots\ldots\ldots(31),$$

if $$\cot\phi \equiv \cot B + \frac{\sin B}{\sin A \sin C} = \frac{1+\cos A \cos B \cos C}{\sin A \sin B \sin C}$$

$$= \cot A + \cot B + \cot C \quad\ldots\ldots\ldots\ldots(32).$$

If $\theta = \frac{\pi}{2}$, we have the case of the triangles $a\beta\gamma$, $a'\beta'\gamma'$, having their sides at right angles to those of ABC, when

$$\frac{R}{r} = \frac{1+\cos A \cos B \cos C}{\sin A \sin B \sin C} \quad\ldots\ldots\ldots\ldots(33).$$

(See *Trinity College Scholarship Examination Papers*, December, 1883.)

If $$\tan\theta = -\tan A \tan B \tan C,$$

then $$\frac{R}{r} = \frac{\sin\theta}{\sin A \sin B \sin C}$$

$$= \surd\{\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C\} \quad \text{.........(34)}.$$

This is the case where the points α, β, γ, α', β', γ' are the feet of the perpendiculars on the sides of ABC from D, E, F, the feet of the perpendiculars from A, B, C on the opposite sides. (See *Math. Messenger*, Vol. xi., p. 177.)

If $$\cot\theta = \cot A + \cot B + \cot C = \frac{1+\cos A \cos B \cos C}{\sin A \sin B \sin C},$$

then $$\frac{R}{r} = 2\cos\theta = \frac{2(1+\cos A \cos B \cos C)}{\surd\{(1+\cos A \cos B \cos C)^2 + \sin^2 A \sin^2 B \sin^2 C\}}.$$

This is the case of the "Triplicate Ratio" circle.

The numerator of the last fraction

$$= \sin^2 A + \sin^2 B + \sin^2 C,$$

and the denominator

$$= \surd\{\sin^2 A \sin^2 B + \sin^2 B \sin^2 C + \sin^2 C \sin^2 A\},$$

therefore $$\frac{R}{r} = \frac{a^2+b^2+c^2}{\surd(b^2c^2+c^2a^2+a^2b^2)} \quad \text{.....................(35)},$$

which is the form given by Mr. Tucker in the *Quarterly Journal*, *loc. cit.*

Now, since $$\frac{R}{r} = \frac{\sin(\theta+\phi)}{\sin\phi},$$

r is a minimum when $\theta+\phi = \frac{\pi}{2}$, in which case

$$\frac{R}{r} = \frac{1}{\sin\phi} = \surd\{1+(\cot A + \cot B + \cot C)^2\} \quad \text{.........(36)}.$$

If $\beta\alpha'$ be parallel to AB,

$$\angle\beta'\alpha C = \angle\alpha'\beta C = \angle BAC = A,$$

and $$\angle\alpha\beta' C = \angle\beta\alpha' C = B.$$

Hence we see that the triangle $\alpha\beta' C$ (and by symmetry each of the triangles $A\beta\gamma'$, $\alpha' B\gamma$) is similar to the triangle ABC, and

$$\gamma\alpha' = 2r\sin\theta = \alpha\beta' = \beta\gamma' = k \text{ suppose.}$$

Therefore $$\left.\begin{aligned} \frac{B\gamma}{a} &= \frac{B\alpha'}{c} = \frac{\gamma\alpha'}{b} \\ \frac{C\alpha}{b} &= \frac{C\beta}{a} = \frac{\alpha\beta'}{c} \\ \frac{A\beta}{c} &= \frac{A\gamma'}{b} = \frac{\beta\gamma'}{a} \end{aligned}\right\} \quad \text{.......................(37)},$$

whence

$$\left.\begin{array}{lll} B\gamma = \frac{a}{b}k, & C\alpha = \frac{b}{c}k, & A\beta = \frac{c}{a}k \\ B\alpha' = \frac{c}{b}k, & C\beta' = \frac{a}{c}k, & A\gamma' = \frac{b}{a}k \end{array}\right\} \quad \text{...........(38).}$$

Therefore $A\beta \, . \, B\gamma \, . \, C\alpha = k^3 = A\gamma' \, . \, B\alpha' \, . \, C\beta'$ ....(39).

It is evident that, in Case III., the centre of the circle $\alpha\beta\gamma$ is the centre of the inscribed circle of the triangle formed by producing the three chords $\beta\gamma'$, $\gamma\alpha'$, $\alpha\beta'$, *i.e.* (see Fig. 2) of the triangle $A'B'C'$.

If we wish to find the point O for Case III., we may find the position where α, α' coincide.

This will happen when

$$B\alpha' + C\alpha = BC,$$

or

$$\frac{c}{b}k + \frac{b}{c}k = a;$$

therefore

$$k = \frac{abc}{b^2+c^2} \quad \text{............................(40).}$$

If this point be called α'', we may write

$$B\alpha'' = \frac{ac^2}{b^2+c^2}, \qquad C\alpha'' = \frac{ab^2}{b^2+c^2};$$

therefore

$$\frac{B\alpha''}{C\alpha''} = \frac{c^2}{b^2},$$

or α'' divides the side BC in the ratio $c^2 : b^2$, as was proved before.

On the Induction of Electric Currents in Cylindrical and Spherical Conductors. By Horace Lamb, M.A.

[*Communicated Jan. 10th,* 1884.]

The following calculations relate chiefly to some cases of induction of electric currents in a long circular cylinder by variation of the magnetic field in which it is placed, this field being supposed uniform, with the lines of force parallel to the axis of the cylinder. For example, we may suppose the cylinder placed in the interior of a long hollow coil, round which a variable electric current is made to circulate. In one instance I have added the solution of the corresponding problem for the sphere. The theory of the *free* transverse currents in a circular cylinder has been recently given by Lord Rayleigh.* The

* *British Association Reports* for 1882, p. 446.