one simple form of mark a besides those of mark x , so that O is the ordinary quantic $(x_1 - ax_0)^g a_0$, all the terms O_r will be equal, each being $= a_r$, in which case the coefficient of $x_1^g x_0^{g-r}$ is, apart from sign,

$\frac{g!}{r! (g-r)!} a_r$. In every case the coefficient of x_1^g , i.e., the leading term or source of the covariant O , will be represented by a single term $C_0 \Pi$, all the univalent forms substituted for those of mark x being of mark e . Since we may omit all simple forms of mark e and their bonds (Sec. 62), without affecting the value of the coefficient of the compound form containing them, we see that the graphical representation of the source C_0 of any covariant O is obtained by omitting from the graphical representation of the covariant all the nuclei of mark x , and the bonds connecting them to the other nuclei. Thus the source of the cubic-covariant which is the coefficient of

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On the Vibrations of an Elastic Circular Ring. (Abstract.)

By Mr. A. E. H. LOVE. Communicated January 12th, 1893.

The ring is supposed to be of small circular section of radius c , and the elastic central-line a circle of radius a . There are four ways of displacing the ring. A point on the central-line may move along the radius of the circle which is its primitive form, or perpendicularly to the plane of this circle, or along the tangent to this circle; and the circular sections may be displaced by rotation about the central-line. The modes of vibration fall into four classes, of which two are physically important.

CLASS I. *Flexural Vibrations in Plane of Ring.*—These were investigated by Hoppe in 1871 (*Crelle*, Bd. LXXIII.). The motion of a point on the elastic central-line is compounded of a displacement along the radius and a displacement along the tangent to the circle, so proportioned that the central-line remains unstretched and the nodes of the former displacement are the antinodes of the latter. There must be at least two wave-lengths to the circumference, and the frequency ($p/2\pi$) of the mode in which there are n wave-lengths to the circumference is given by the equation

$$p^2 = \frac{1}{4} \frac{n^2(n^2-1)^2}{1+n^2} \frac{E}{\rho_0} \frac{c^2}{a^4},$$

in which E is the Young's modulus, and ρ_0 the density of the material. Except for the numerical coefficient this is precisely similar to the formula for the lateral vibrations of a straight bar of the same material and section and of length πa . The sequence of component tones, when n is very great, is ultimately identical with that of the tones of a free-free bar of length πa , but the sequence for the low tones is quite different to that for a bar.

CLASS II. *Flexural Vibrations Perpendicular to the Plane of the Ring.* It is found to be impossible to make the ring vibrate freely so that each particle of the elastic central-line moves perpendicularly to the plane of the ring, unless at the same time the sections turn about the central-line through a certain angle. In other words the flexure perpendicular to the plane of the ring is always accompanied by *torsion*. As in Class I., there must be at least two wave-lengths to the circumference, and the frequency of the mode in which there are n wave-lengths to the circumference is given by the equation

$$p^2 = \frac{1}{4} \frac{n^2(n^2-1)^2}{1+\sigma+n^2} \frac{E}{\rho_0} \frac{c^2}{a^4},$$

where σ is the *Poisson's ratio* for the material, and the other constants have the same meaning as before. (For most hard solids σ is about $\frac{1}{4}$.) Since n must be at least 2, the sequence of tones is very nearly the same as in the vibrations of Class I., but the pitch is slightly lower, the ratio of the frequencies for the gravest tones being $\sqrt{\frac{21}{20}}$, which is very little greater than a *comma*. For the higher tones, as we should expect, there is no sensible difference.

These two classes include all that have much physical importance. The remaining types can be classified as follows :—

CLASS III. *Extensional Vibrations.*—The motion may be purely radial or partly radial and partly tangential. In the second case there will be an integral number of wave-lengths, and, when this number is n , we have the formula for the frequency

$$p^2 = (1 + n^2) \frac{E}{\rho_0} \frac{1}{a^2}.$$

Putting $n = 0$, we find the frequency of the purely radial vibrations. The pitch of any mode of extensional vibration of the ring is of the same order of magnitude as the pitch of the corresponding longitudinal vibration of a bar of length equal to half the circumference, the formula for the latter being in fact derived by writing n^2 for $1 + n^2$.

CLASS IV. *Torsional Vibrations.*—The motion consists of an angular displacement of the sections about the elastic central-line, accompanied by a relatively very small displacement of the points on this line perpendicular to the plane of the ring. When there are n wave-lengths to the circumference, the frequency is given by the formula

$$p^2 = (1 + \sigma + n^2) \frac{\mu}{\rho_0} \frac{1}{a^2},$$

in which μ is the *rigidity* of the material. There is one symmetrical mode for which n is zero, and, since $2\mu(1 + \sigma) = E$, the frequency of this mode is $\frac{1}{2}\sqrt{2}$ of that of the radial vibrations. The pitch of the torsional vibrations is comparable with that for a straight rod of length equal to half the circumference, the formula for the latter being in fact derived by writing n^2 in place of $1 + \sigma + n^2$. Formulæ equivalent to those given in connexion with Classes II. and IV. have been obtained by Mr. Basset (*Proc.*, December, 1891), but he has not interpreted his results.