Prof. Sylvester then gave an explanation of some of the processes employed in his paper on "Reducible Cyclodes."

The following present was made to the Society :—

"Crelle, 70 Band, Zweites Heft."

The following paper, an account of which was given by the author on April 8th, could not be inserted in the account of the Proceedings of that day :—

On the Focal Properties of Homographic Figures. By HENRY J. STEPHEN SMITH, F.R.S., Savilian Professor of Geometry in the University of Oxford.

A.—FOCAL PROPERTIES OF TWO HOMOGRAPHIC PLANE FIGURES.

(1.) *Two Plane Figures in Perspective.*

WE consider two plane figures Ω and ω in perspective with one another; we denote the centre of the perspective by S, and the axis of the perspective (or the line of intersection of the two planes) by $\Omega_1\omega_1$; we exclude the cases in which the straight lines at an infinite distance in the two planes are corresponding lines; *i.e.*, we suppose that the centre of perspective is not at an infinite distance, and that the planes are not parallel. Let OY, $o'y$ be the vanishing lines of the planes Ω and ω, or the straight lines which in the planes Ω and ω correspond to the straight lines at an infinite distance in the planes ω and Ω; the plane Ω is divided by OY into two regions (Ω_1) and (Ω_2); similarly, $o'y$ divides ω into two corresponding regions (ω_1) and (ω_2). Let (Ω_1) be that region of Ω in which $\Omega_1\omega_1$ is situated; then $\Omega_1\omega_1$ is also situated in (ω_1); and it will be seen that if P, p are corresponding points in the regions (Ω_1), (ω_1), the radii vectores SP, Sp are of the same sign; but if P, p are corresponding points in (Ω_2), (ω_2), the radii vectores SP, Sp are of opposite signs; or, in the language of some writers on perspective, (Ω_1) and (ω_1) are projections of one another, but (Ω_2) and (ω_2) are transprojections of one another.

(2.) *The Correspondence of Directions.*

If the positive and negative directions on any straight line in either of the planes Ω and ω are regarded as determined, the corresponding directions on the corresponding line are also determined; viz., if a point move in the positive direction on a straight line in either plane, its image in the other plane moves in the positive direction on the corresponding straight line. Hence, if P,Q are two points in the same region of Ω, and p, q their images, which are of course in the corresponding region of ω, the direction from P to Q along the finite segment PQ is of the same sign as the direction from p to q along the finite segment pq; but if P, Q are in opposite regions of Ω, so that the finite segment PQ is divided internally by OY in the point A, the direction $p \infty q$ will correspond to the direction PAQ, and the directions of the finite segments PQ, pq will be of opposite signs. We may add that if A is any point whatever on OY, to the directions PA, QA there will correspond similar or dissimilar directions on the parallel straight lines which are the images of the lines PA, QA, according as P and Q are in the same region, or in different regions of Ω. And in particular, if in the plane Ω there be drawn any parallel to the vanishing lines, the corresponding line in the plane ω will also be parallel to the vanishing lines, but the corresponding directions on the two parallels will be similar or dissimilar, according as they lie in the regions (Ω_1), (ω_1), or in the regions (Ω_2), (ω_2).

Again, if in the plane Ω we consider one of the two directions of rotation round any point as positive, (say, for example, that direction of rotation which viewed from S appears right-handed,) the signs of the directions of rotation will thereby be fixed for each point of the plane ω; but for all points in the region (ω_1) that direction of rotation, which viewed from S is right-handed, will be positive, whereas for all points of the region (ω_2) the same direction of rotation must be considered negative; it being inconsistent with the perspective relation to regard one and the same direction of rotation as being positive for all points of the one plane, and also to regard one and the same direction of rotation as being positive for all points of the other plane.

(3.) *The Equiangular Points, or Foci.*

Through S draw two lines perpendicular to the planes which bisect the dihedral angle formed by the intersecting planes Ω and ω. Let these perpendiculars meet the plane Ω in F_1, F_2, and the plane ω in f_1, f_2; let also SF_2f_2 be perpendicular to that bisecting plane which lies in the same angle with S; then SF_1, Sf_1 are of the same sign, and F_1, f_1 lie in the regions (Ω_1), (ω_1) respectively; but SF_2, Sf_2 are of opposite signs, and F_2, f_2 lie respectively in the regions (Ω_2), (ω_2).

Since each of the lines F_1f_1, F_2f_2, is at right angles to the line of intersection of Ω and ω, and is besides equally inclined to those two planes, any dihedral angle of which the axis is either F_1f_1, or F_2f_2, is intersected in two equal rectilineal angles by the planes Ω and ω. We thus obtain the theorem:

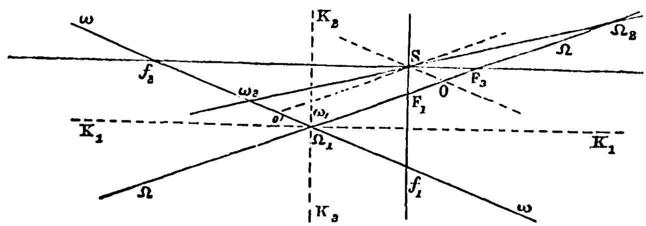
"Angles in the plane Ω at the points F_1, F_2, are projected into equal angles in the plane ω at the points f_1, f_2."

Or, more precisely,

"The angle contained by given directions on two straight lines lying in the plane Ω, and intersecting at F_1 or F_2, is equal to the angle formed in the plane ω by the corresponding directions on the corresponding straight lines, which intersect at f_1 or f_2."

For brevity, we shall call F_1, F_2, f_1, f_2, the *foci* of the perspective in the planes Ω and ω respectively. And since the directions of rotation, as viewed from S, are the same round F_1 and f_1, but are opposite round F_2 and f_2, we shall call F_1, f_1 the similar foci, and F_2, f_2 the dissimilar foci.

In the annexed diagram the plane of the paper is the plane of sym-



metry; *i.e.*, it is the plane passing through S, and cutting Ω, ω at right angles; $O\Omega_1$, $o'\omega_1$ are the traces of the given planes Ω and ω, K_1, K_2 are the traces of the bisecting planes; F_1F_2, f_1f_2 are the foci, which lie of course in the plane of symmetry; the axis of the perspective $\Omega_1\omega_1$ is perpendicular to the plane of the paper at Ω_1 or ω_1, and the vanishing lines OY, $o'y$ are perpendicular to the same plane at the points O, o', which we shall term the *centres* of the two planes. If Y, y are the points at infinity on the vanishing lines, Y and y are corresponding points; but it will be observed that the vanishing lines are not corresponding lines, nor the centres corresponding points. We may term the lines F_1OF_2, $f_1o'f_2$, which are corresponding lines at right angles to the vanishing lines, the focal axes of the two planes; so that the centre of each plane corresponds to the point at an infinite distance on the focal axis of the other. Since SO, So' are parallel to the traces of the planes Ω, ω, and SF_1f_1, SF_2f_2 to the traces of the bisecting planes, the figure $SO\Omega_1o'$ is a parallelogram, the triangles SOF_1, SOF_2, $So'f_1$, $So'f_2$, are isosceles, F_1F_2 and f_1f_2 are bisected at the centres O and o', SO is equal to OF_1 or OF_2, So' to $o'f_1$ or $o'f_2$; whence we find

$$\Omega_1F_1 = \omega_1f_1 = \tfrac{1}{2}(f_2f_1 + F_2F_1),$$
$$\Omega_1F_2 = -\omega_1f_2 = \tfrac{1}{2}(f_2f_1 - F_2F_1).$$

These equations, as all others in this paper, are to be interpreted, with regard to sign as well as magnitude; we shall avoid the use of the sign $=$, when we have to speak of equality irrespective of sign.

(4.) *The Equi-Segmental Axes, or Cyclic Lines.*

Any point of the line $\Omega_1\omega_1$, in which the planes Ω and ω intersect, considered as a point in either plane, has itself for its corresponding point in the other plane. If we do not attend to the coincidence of the corresponding points, we may express this by saying that $\Omega_1\omega_1$, is an *equi-segmental* line in either plane; *i.e.*, that to any segment of $\Omega_1\omega_1$, considered as a line in either plane, an equal segment corresponds in the other plane. But besides this coincident pair of equi-segmental lines, there is another pair of corresponding equi-segmental lines which are not coincident. Through S extend a plane parallel to the plane containing the vanishing lines, and let it meet the focal axes of the two planes in Ω_2 and ω_2. The lines O_2Y, $\omega_2 y$, parallel to the vanishing lines, are equi-segmental lines. For if P, p are corresponding points on those lines, the radii vectores SP, Sp, $S\Omega_2$, $S\omega_2$, are respectively equal and of opposite signs, and the angles $\Omega_1 SP$, $\omega_1 Sp$, are equal; so that P, p lie at equal distances from the plane of symmetry, but on opposite sides of that plane; *i.e.*, the axes $\Omega_2 Y$, $\omega_2 y$ are equi-segmental. We may term the coincident axes $\Omega_1 y_1$, $\omega_1 y$, the similar axes, and the axes $\Omega_2 Y$, $\omega_2 y$ the dissimilar axes. Since $\Omega_1\Omega_2$ is double of $\Omega_1 O$, *i. e.* of $o'S$, or $o'f_1$, and similarly $\omega_1\omega_2$ is double of $\omega_1 o'$, *i.e.* of OS or OF_1, we see that in either plane the equi-segmental axes are situated symmetrically with respect to the centre of that plane, and that the semi-distance between the foci is equal to the semi-distance between the equi-segmental axes in the other plane. The semi-distance between the foci in either plane may conveniently be called the *parameter* of that plane. Designating the parameters of the planes Ω and ω by C and c, we shall find, if $C=c$, that the foci of each plane lie on its equi-segmental lines, and that the centre of perspective lies on one of the two bisecting planes. But if C and c are unequal, if for example $C<c$, the foci lie between the equi-segmental lines in Ω, and outside them in ω, and the distances $c+C$ and $c-C$, between a focus and the nearer and further equi-segmental line, are the same for both planes.

If we cause one of the two planes, for example the plane ω, to revolve round the axis of intersection of the two planes, the two figures, as is well known, will continue in perspective; and the locus of the centre of perspective will be a circle, lying in the plane of symmetry, and described on F_1F_2 as diameter. At one of the coincidences of the two planes which take place during a complete revolution, the similar foci come to coincide with one another and with the centre of perspective; and, in like manner, at the other coincidence, the dissimilar foci coincide with one another and with the centre of perspective; the similar foci continuing similar, and the dissimilar foci continuing dissimilar during the whole revolution. If, however, we bring together corresponding points in the dissimilar axes (which we may conceive done by causing either plane to rotate through an angle of 180° round an axis perpendicular to the plane $\Omega_2\omega_2$ at S), the two figures will again be in perspective, but the foci, which were before similar, will become dissimilar, and *vice versâ*. Thus the two foci, and the two equi-segmental axes, in either plane, are not absolutely distinguished as similar to, or dissimilar from, their corresponding foci or axes; these denominations being, in fact, relative to one or other of the two ways in which the planes can be placed in perspective with one another, and changing when we pass from one of those ways to the other. In every case, if a focus or axis be regarded as similar, the nearer axis or focus is also to be regarded as similar.

The equi-segmental lines may, perhaps, be called the cyclic lines. This denomination is suggested by an analogy which will come before us presently.

(5.) *Any two Homographic Plane Figures.*

Since any two homographic plane figures, such that the lines at an infinite distance in the two figures are not corresponding lines, (this limitation is to be always understood in what follows when we speak of two homographic plane figures,) can be placed in perspective with one another, it appears that in any two such homographic systems there exist two pair of corresponding foci, and two pair of equi-segmental axes. This we shall now show independently of all considerations of perspective. Let P_1, P_2, q_1, q_2 be the imaginary circular points at an infinite distance in the planes Ω and ω respectively; and to P_1P_2, q_1q_2, let p_1p_2, Q_1Q_2 correspond in the planes ω and Ω respectively. The lines P_1P_2, p_1p_2; Q_1Q_2, q_1q_2, will be pairs of corresponding lines; P_1P_2, q_1q_2 being the lines at an infinite distance in the two planes, and p_1p_2, Q_1Q_2 the vanishing lines. Further, the three diagonal points of the quadrangle $P_1P_2Q_1Q_2$ (which are all real) will correspond to the three diagonal points of the quadrangle $p_1p_2q_1q_2$; of these three pairs of corresponding points, one pair are the points Y, y at an infinite distance on the vanishing lines; the two other pairs are the two pairs of foci. For if F, f are corresponding diagonal points (other than Y, y) of the two imaginary quadrangles, the homographic pencils at F, f are equiangular, because the imaginary circular asymptotes FP_1, FP_2 correspond in the pencil at F to the imaginary lines fp_1, fp_2, or fq_1, fq_2, *i.e.* to the imaginary circular asymptotes in the pencil at f. To determine the two pair of foci in two given homographic planes Ω and ω, we consider a pair of rectangular points at an infinite distance in each plane; let A_1A_2, b_1b_2 be these pairs of points; a_1a_2, B_1B_2 the pairs of points corresponding to them. The lines B_1B_2, a_1a_2 are the vanishing lines of the two planes; the centre of either plane is the point corresponding to the point at an infinite distance in the direction perpendicular to that of the vanishing line in the other plane; the focal axes are the lines perpendicular to the vanishing lines of the two planes at their respective centres; lastly, the foci are the points of intersection of the focal axes by the circles described on B_1B_2, a_1a_2 as diameters, and are situated in each plane symmetrically with regard to its vanishing line. If, assuming that we view each plane from a determinate region in space, we regard the rotations round F_1, f_1 as similar, it is evident that the rotations round F_2, f_2 must be dissimilar, and *vice versâ*; otherwise the two homographic figures would be similar, and the lines at an infinite distance would be corresponding lines, contrary to the hypothesis. We might prove the same thing, by imagining the planes of the two figures to coincide. The circular asymptotes at F_1, f_1, and again at F_2, f_2 will then be corresponding lines. But the correspondence in one case will be direct, and in the other inverse (*i. e.*, in the one case, those asymptotes which run to the same imaginary circular point at an infinite distance will be corresponding lines, in the other case asymptotes running to opposite circular points will correspond). And, since the locus of the intersections of corresponding rays in two equiangular pencils is a circle, or an equilateral hyperbola, according as the rotations of the two pencils are in the same or in opposite directions, we infer that two equiangular pencils in the same plane have the same direction of rotation, or opposite directions, according as the circular asymptotes of the two pencils correspond directly or inversely.

As we have obtained the foci in each plane by a quadratic construction (which seems inevitable) we have still to determine their correspondence, and the corresponding directions of rotation round each. To do this, we have only to observe that each plane is divided by its focal and vanishing axes into four regions, which correspond to one another in a manner which is readily ascertained, because when we pass in either plane from one region A into another region B across one of their common boundaries, we must simultaneously pass in the other plane from the region corresponding to A into the region corresponding to B, and must traverse the corresponding boundary; (the line at an infinite distance is a common boundary, it will be observed, of two diametrically opposite regions). Thus we have only to ascertain in either plane the region which corresponds to a given region in the other; the correspondence of the remaining regions is then known, and with it the correspondence of the foci. Lastly, if O′, o be the points at an infinite distance on the focal axes of the two planes, the direction of rotation from FO′ to FY corresponds to the direction of rotation from fo to fy, the rotating radii vectores being supposed to move in corresponding regions in the two planes.

If A, a are corresponding points in the regions (Ω_2), (ω_2) respectively, the angles F_2F_1A, f_2f_1a, are both acute; they are, therefore, equal to one another, since, by virtue of the equiangularity of the pencils at F_1, f_1, they must be either supplementary or equal. Observing that the directions F_1F_2, F_1A are opposite in sign to the directions f_1f_2, f_1a, we see that angles between corresponding directions on corresponding radii vectores are equal, in which form we have already stated the equiangular property of the foci.

The determination of the foci requires (as we have seen) the construction of three points in each plane corresponding to three points at an infinite distance in the other plane. And thus the actual determination of the foci, though very elementary in theory, is in actual practice somewhat troublesome. But when the foci have once been determined, the homographic representation of either plane upon the other can be carried out very rapidly; since, if A be any given point of Ω, F_1 the further, and F_2 the nearer focus, we have only to make the angle $f_2 f_1 a$ equal to the angle F_2F_1A, and the angle $f_1 f_2 a$ supplementary to the angle F_1F_2A; the directions in which these angles are to be measured being at once indicated by the correspondence of the regions in which A and a are situated.

The anharmonic equation $(O, X, F_1, \infty) = (\infty, x, f_1, o')$, or $OX \, . \, o'x = -C \, . \, c$, in which X, x denote corresponding points on the focal axes, suffices to prove that the parallels to the vanishing lines at Ω_1, ω_1; Ω_2, ω_2, are corresponding lines. And that these lines are equi-segmental, will then follow from the equiangular property of the foci, since $F_1\Omega_1 = f_1\omega_1$, $F_1\Omega_2 = -f_1\omega_2$. The image in either plane of any given indefinite line in the other is most easily found by making the intercepts on the equi-segmental lines in the first plane equal in sign and in magnitude to the corresponding intercepts in the second plane; so that, if the two intercepts are drawn in the same direction from the focal axis in one plane, they are drawn in opposite directions in the other plane.

(6.) *Homographic Plane Figures placed Homologically.*

It will be observed that any two homographic plane figures can be made homological, or put in plane perspective with one another, in four different ways. For we can take either pair of equi-segmental axes for the axis of homology, and either pair of foci for the centre of homology. And it is sometimes of importance, in the theory of homological figures, to consider the non-coincident foci and equi-segmental axes, as well as the two foci which are united in the centre of homology, and the two equi-segmental lines which are united in the axis of homology.

For example, if we regard a conic section as homological with itself, any point in the plane of the conic being the centre of homology, and its polar the axis of homology, the foot of the perpendicular let fall from the pole upon the polar will represent the second pair of foci (which in this case are coincident because the parameters are equal); and in like manner the second pair of equi-segmental lines will be represented by the parallel to the polar through the pole. Thus we have the elementary properties of a conic section, " angles subtended at the foot of the perpendicular by chords passing through the pole are bisected by the polar;" "the pole is the point of bisection of intercepts on the parallel to the polar made by tangents at the extremities of chords passing through the pole," &c.

Again, if we regard two conics as homological, a point of intersection of their common tangents being the centre of homology, and the axis of homology being either of the two common chords which pass through the intersection of the polars of the centre of homology with regard to the two conics; it will be found that there is a second pair of foci, situated on the perpendicular let fall from the centre of homology upon the axis of homology, and that corresponding points of the two conics subtend equiangular pencils (with opposite rotations) at these two points. And, in like manner, corresponding lines in the two figures determine equal intercepts (but measured in opposite directions) upon two axes situated at the same distances from the second pair of foci, that the axis of homology is from the centre of homology.

(7.) *Homographic Plane Figures placed Symmetrically.*

The equality of the parameters of two plane homographic figures is the necessary and sufficient condition that they should be capable of being so placed in the same plane that each point shall have but one corresponding point. For if the two figures can be so placed as to have this symmetrical relation to one another, the imaginary points corresponding to the imaginary circular points at an infinite distance must coincide; and hence the imaginary chords Q_1Q_2, p_1p_2, and with them the real parameters, must be equal. Conversely, we can always render the two figures capable of a symmetric position by altering the linear dimensions of either of them in the ratio of its parameter to the parameter of the other; since after this alteration the imaginary chords Q_1Q_2, p_1p_2, will be equal, and can be made to coincide. There are then two positions of symmetry—viz., the two positions of homology in which the vanishing lines coincide. It is sometimes convenient to imagine the scale of one of the figures altered in the parametric ratio; we shall express this by saying that the figures are reduced to the same scale.

(8.) *Metrical Properties of the Focal Radii Vectores.*

The following elementary properties of the foci of two homographic plane figures are frequently useful:—

(*a*.) The focal radii vectores of two corresponding points P and p satisfy the equation

$$\frac{F_1P}{F_2P} + \frac{f_1p}{f_2p} = 0.$$

The truth of this equation, so far as absolute magnitude is concerned, appears immediately from a comparison of the triangles F_1PF_2, f_1pf_2; the two ratios are of opposite signs, because the radii vectores drawn from the nearer foci are of the same sign, and the radii vectores drawn from the further foci are of opposite signs.

(β.) The rectangle contained by the sum of the radii vectores of P, and the difference of the radii vectores of p, is equal to the rectangle contained by the difference of the radii vectores of P and the sum of the radii vectores of p; and either of these rectangles is equal to four times the rectangle of the parameters. This relation is expressed by the single equation

$$(F_1P+F_2P)\,(f_1p+f_2p) = F_1F_2\,.\,f_1f_2,$$

if we observe that in it the signs of F_1P and F_2P are arbitrary, and that the signs of f_1p and f_2p depend on the signs of F_1P and F_2P respectively. The truth of the equation may be inferred immediately from the elementary theorem, that if a straight line bisecting the angle of a triangle either internally or externally be produced to meet the base, the square of the bisecting line is equal to the rectangle contained by the sum of either side and the segment of the base adjacent to it, and by the difference of the other side and the segment adjacent to it.

(γ.) Let N, n be the feet of the perpendiculars let fall from P and p upon the vanishing lines. The distances NP, np are of opposite sign, and their rectangle (see Art. 5) is equal to the rectangle of the parameters. We then have the relations

$$\frac{F_1P}{NP}=\frac{f_1p}{o'f_1},\quad \frac{F_2P}{NP}=\frac{f_2p}{o'f_2},\quad \frac{f_1p}{np}=\frac{F_1P}{OF_1},\quad \frac{f_2p}{np}=\frac{F_2P}{OF_2},$$

any one of which, combined with the equation $NP\,.\,np = OF_1\,.\,o'f_1$, and with the equation (α), gives the other three. To prove them, we have only to observe that the quadrilaterals NPF_1O, $o'f_1pn$ are not only equiangular, but also similar, because $NP\,.\,np = OF_1\,.\,o'f_1$.

If, in the construction of the point p, corresponding to a given point P, we wish to avoid the use of points lying on the further side of the vanishing line, we may either determine the ratio of f_1p to F_1P by one of the formulæ (γ); or, preferably, we may make the angle $f_1o'p$ equal to the angle OF_1N; the point p is then the intersection of $o'p$ and f_1p.

(δ.) If the two planes are in perspective, we have

$$\frac{F_1P}{f_1p}=\frac{SP}{Sp} \text{ for the similar foci,}$$

and

$$\frac{F_2P}{f_2p}=-\frac{SP}{Sp} \text{ for the dissimilar foci.}$$

(9.) *Circles changed into Circles.*

The circles of the system of which F_1, F_2 are the limiting points, and the vanishing line the radical axis, are transformed into circles of the system of which f_1, f_2 are the limiting points, and the vanishing line the radical axis; and these are the only circles in either figure which are changed into circles in the other figure. For it is evident that to the conics (P_1, P_2, Q_1, Q_2) there will correspond the conics (p_1, p_2, q_1, q_2); or again, if the locus of A be a circle of the system of which F_1, F_2 are the limiting points, the ratio $\frac{F_1A}{F_2A}$ is constant; therefore the ratio $\frac{f_1a}{f_2a}$ is also constant, (Art. 8, α); *i.e.*, the locus of a is a circle of the system of which f_1, f_2 are the limiting points. We shall, for brevity, call these two systems of circles the focal circles of the two planes.

It will be found that the radii of corresponding circles are to one another as the parameters; and that, if the figures be reduced to the same scale, and superposed so that their foci coincide, the corresponding circles will coincide (but not in respect of their corresponding points).

(10.) *The Homographic Modulus of Corresponding Pencils.*

In any two homographic pencils (A) and (a) there exists a pair of corresponding right angles (Steiner, "Systematische Entwickelung," p. 31); and if the pencils are not equiangular, there is only one such pair. We shall term these corresponding right-angles the right-angles of the pencils (A) and (a). Let A and a be any two corresponding points in the planes Ω and ω; the lines bisecting the angles F_1AF_2, f_1af_2, internally and externally, are the lines containing the right angles of the pencils at A and a. For the double rays of the pencil $A.[P_1P_2, Q_1Q_2, F_1F_2]$, (which is a pencil in involution, because P_1P_2, Q_1Q_2, F_1F_2 are the vertices of a quadrangle,) correspond to the double rays of the corresponding pencil $a.[p_1p_2, q_1q_2, f_1f_2]$. We might prove the same thing, without using imaginary points, by considering the corresponding circles which pass through A and a. And since the rectangle of the central abscissas of corresponding points is equal to the rectangle of the parameters, we see that the external bisector at either of the two points A or a answers to the internal bisector at the other.

If, in any two homographic pencils, Φ and ϕ are corresponding angles, measured from either pair of the corresponding rectangular lines, the ratio $\tan\Phi : \tan\phi$ is constant. This constant ratio we may term the homographic modulus of the two pencils. We observe (1) that the definition is applicable to homographic pencils in involution; (2) that the homographic modulus is positive or negative according as corresponding directions of rotation in the two pencils are regarded as having the same sign or opposite signs; (3) that the homographic modulus of two equiangular pencils is $+1$ or -1; (4) that the definition is relative to a given pair of the corresponding rectangular lines, and that if for this pair we substitute the other pair, the homographic modulus changes into its reciprocal.

The homographic modulus of the pencils at A and a, taken relatively to the external bisector at A and the internal bisector at a, is evidently

$$\cot \tfrac{1}{2} A \cot \tfrac{1}{2} a = \tan \tfrac{1}{2} (F_1 + F_2) \tan \tfrac{1}{2} (f_1 + f_2)$$
$$= \tan \tfrac{1}{2} (F_1 + F_2) : \tan \tfrac{1}{2} (F_1 - F_2) = R_2 + R_1 : R_2 - R_1 = r_2 - r_1 : r_2 + r_1,$$

the letters A, F_1, F_2, a, f_1, f_2 denoting the internal angles of the triangles F_1AF_2, f_1af_2, and R_1, R_2, r_1, r_2 representing the absolute values of the focal radii vectores of A and a. We have supposed that F_1 is the nearer, F_2 the further focus, so that $F_2 = f_2$, $F_1 + f_1 = \pi$.

(11.) *Angles changed into Equal or Supplementary Angles.*

In any two homographic pencils (A) and (a) there exists an infinite number of equal corresponding angles, and again an infinite number of supplementary corresponding angles (M. Chasles, "Géométrie Supérieure," Art. 147). We may add, that the angles in either pencil A, which are equal to their corresponding angles in the other pencil, form a pencil in involution, of which the right-angle is the right-angle of the pencil A, and of which the modulus is the homographic modulus of the two pencils taken positively; and, similarly, the angles in the pencil A which are supplementary to their corresponding angles form a pencil in involution which has the same right-angle as the pencil (A), and the same modulus, only taken negatively. The former involution always has real double lines, the latter never.

To find the involutions of equal angles at the corresponding points A and a; let F_1, f_1 be the nearer foci, let any circle passing through A and F cut the vanishing line in M_1, M_2, and let m_1, m_2 be the corresponding points at an infinite distance in the plane ω. The angle M_1AM_2 will be transformed into an equal angle m_1am_2; for the angles M_1AM_2, $M_1F_1M_2$ are equal (not supplementary, since the chords M_1M_2, AF_1 do not intersect); *i. e.*, the angles M_1AM_2 and $m_1f_1m_2$, or finally M_1AM_2 and m_1am_2, are equal. It will be observed that the directions AM_1, am_1; AM_2, am_2 are corresponding directions.

To find the involutions of supplementary angles at the points A and a we have only, in the preceding construction, to substitute the further focus F_2 for the nearer focus F_1. The angles M_1AM_2, $M_1F_2M_2$ will be supplementary (and not equal); so that M_1AM_2 will be transformed into a supplementary angle m_1am_2.

(12.) *Segments changed into Equal Segments.*

If H and h' are the *centres* of two homographically divided lines, (*i. e.* the points which on each line correspond to the points at an infinite distance upon the other line,) and if A, a are any two corresponding points on the two lines, the rectangle $\mathrm{HA} \times h'a$ (which we may term the rectangle of the homography) is constant.

In any two homographically divided lines there is an infinite number of segments equal to their corresponding segments and having the same sign, and again an infinite number of segments equal in absolute magnitude to their corresponding segments, but having opposite signs. Upon either line either set of segments form an involution of which the centre is the homographic centre of the line, and of which the rectangle is the rectangle of the homography, taken positively or negatively, according as the segments considered are equal to their corresponding segments with the same sign, or with opposite signs.

Thus, if 2D, $2d$ are the intercepts made by the cyclic axes on any two corresponding lines meeting the vanishing lines in H and h', the segments X_1X_2 of the involution determined by the equation $HX_1 \times HX_2 = D \times d$ are equal to the corresponding segments x_1x_2 of the involution $h'x_1 \times h'x_2 = D \times d$; and, again, the corresponding segments of the involutions $HX_1 \times HX_2 = -D \times d$, $h'x_1 \times h'x_2 = -D \times d$ are equal, but have opposite signs.

And, in general, the segments of the involution $HX_1 \times HX_2 = \mu D \times d$, where μ is any multiplier, are μ times the corresponding segments of the involution $\mu \times h'x_1 \times h'x_2 = D \times d$. The segments of the two involutions are divided externally or internally by the vanishing lines, according as μ is positive or negative.

It thus appears that there are an infinite number of triangles in either plane, similar to their corresponding triangles, and having a given ratio to them. For take any point A in the plane Ω, any straight line passing through it, and any positive ratio; there is always one triangle (and only one, if A is not a focus, and the given line not a cyclic axis), having a vertex at A, and a side in the given straight line, which is transformed into an equiangular triangle of which the sides are to the sides of the triangle in the given positive ratio. The two corresponding triangles are not intersected by the vanishing lines, so that points interior to either triangle correspond to points interior to the other. If we do not attend to the sign of the given ratio, there are in all four triangles, each having a vertex at the point A, and a side upon the given line, which are equiangular to their corresponding triangles, and which have the given ratio for their ratio of similarity. For, if we do not attend to signs, there are two segments of the given line having a common extremity at A, which are in the given ratio to their corresponding segments, and there are two lines passing through A which make angles with the given line equal to the corresponding angles. Of these triangles, that just considered is one; the other three are all intersected by the vanishing line; so that two of the sides of any one of them are to their corresponding lines in a negative ratio.

(13.) *The Confocal Conics.*

Every conic which has a focus at F_1 or F_2 is transformed into a conic having a focus at f_2 or f_1. For right-angles at F are transformed into right-angles at f, so that if the involution determined by the given conic at the point F be rectangular, the involution determined by the corresponding conic at the point f is also rectangular. And conversely, if a point in either figure, and its image in the other, be both foci of corresponding conics, the point and its image are corresponding foci in the two figures.

Two cases of this property are of special interest.

(i.) A circle having its centre at F is transformed into a conic, of which F is the focus, and the vanishing line the directrix. This follows, independently of the general property, from the equations (γ), which also show that the eccentricity of the conic is equal to the radius of the circle divided by the parameter; it thus varies directly as the radius of the circle.

(ii.) Conics in the plane Ω, of which F_1, F_2 are the foci, are transformed into conics of which f_1, f_2 are the foci, the ellipses into hyperbolas, and the hyperbolas into ellipses. We shall term these conics the confocal conics of the two homographic figures. An independent proof of the theorem is supplied by the formula (β); and conversely, the theorem may be used to establish that formula, since the rectangle contained by the major semi-axes of two corresponding confocal conics is evidently equal to the rectangle of the parameters.

The eccentricity of any confocal in either figure is the reciprocal of the eccentricity of the corresponding confocal, the asymptotes of the hyperbola containing the same angle as the focal radii vectores of the extremities of the minor axis of the ellipse. If the figures be reduced to the same scale, and the foci be superposed, corresponding confocals will intersect on the cyclic axes, and will thus have the same latus rectum.

If Σ and σ are any two corresponding confocals, the normals of Σ and the normals of σ are corresponding lines (Art. 10). Hence also the centre of curvature at any point of Σ corresponds to the centre of curvature at the corresponding point of σ; and the evolute of either is transformed into the evolute of the other. Again, any two *similar* arcs of Σ (*i. e.* arcs of which the difference is geometrically rectifiable) have for their corresponding arcs two similar arcs of σ; and the rectifying tangents in either figure (*i. e.*, the tangents of which the difference is equal to the difference of the arcs) are images of the rectifying tangents in the other figure. In the same way, the polygon of a given number of sides, and of minimum perimeter, inscribed in any arc of a confocal conic of either figure, corresponds to the polygon of the same number of sides, and of minimum perimeter, inscribed in the corresponding confocal arc. It is hardly necessary to observe, that the elliptic integrals which express the lengths of corresponding arcs of corresponding confocal curves are not themselves equal to one another, and are not transformed into one another by the homographic transformation.

(14.) *The Indicatrix, or Strain Ellipse.*

The *indicatrix* at any point A of the plane Ω is the evanescent ellipse which is the image of an evanescent circle, having its centre at the corresponding point a. The indicatrix is, in fact, the "strain ellipse," if we regard any part of the plane Ω as a deformation of the corresponding part of the plane ω, produced by a mechanical strain. It is readily seen that, if we consider the radius of the evanescent circle at a as an infinitesimal of the first order, the distance of the centre of the strain ellipse from A will be an infinitesimal of the second order. For the determination in species of the strain ellipse at the point A, we have the theorem: "The strain ellipse is similar and similarly situated to the ellipse, of which the principal axes are normal, at the point A, to the confocal ellipse and hyperbola intersecting at that point, and are respectively equal to the major axes of those curves."

This auxiliary ellipse is no other than the ellipse employed by M. Chasles, in his solution of the problem, "To determine the principal axes of an ellipse, of which one pair of conjugate diameters are given in magnitude and position" (Aperçu historique des Méthodes en Géométrie, Note 25). M. Chasles has shown that a reciprocal relation subsists between the auxiliary ellipse and the ellipse of the confocal system which passes through its centre. Thus the centre of either ellipse lies on the circumference of the other; the major axis of either is normal to the other; the asymptotes of either pass through the imaginary foci of the other; the major and minor axis of either are respectively equal to the sum and difference of the focal radii vectores of its centre considered as a point on the circumference of the other; lastly, the distance between the real foci of either is equal to that diameter of the other which is conjugate to the diameter passing through the two centres.

To prove that the strain ellipse is similar and similarly situated to the auxiliary ellipse, it is sufficient to observe, that the asymptotes of the evanescent circle at a are aq_1, aq_2; and that, consequently, the strain ellipse touches the imaginary lines AQ_1, AQ_2 at the points Q_1, Q_2. But the strain ellipse is infinitesimal; the imaginary lines AQ_1, AQ_2 are therefore its asymptotes, *i.e.*, it is similar and similarly situated to the auxiliary ellipse.

To determine, then, the strain ellipse in species, we have only to draw the focal radii vectores of the point A, and to bisect the angle contained by them internally and externally; the major and minor axes of the strain ellipse are respectively in the directions of the bisecting lines, and are proportional to the sum and difference of the radii vectores. It will be seen that the confocal hyperbolas are lines of greatest elongation (or least compression), and that the confocal ellipses are lines of least elongation (or greatest compression). The focal circles are lines of *similar distortion*, because for all points on any one of them the ratio of the two radii vectores is constant, and therefore the ratio of their sum and difference; *i. e.*, all points on the same focal circle have similar indicatrices.

It remains to find the absolute dimensions of the indicatrix. Let $d\Sigma_1$, $d\Sigma_2$ represent the elements of the arcs of the confocal hyperbola and ellipse which intersect at A; let $d\sigma_1$, $d\sigma_2$ represent the corresponding elements in the plane ω; and let Λ_1, Λ_2, λ_1, λ_2 be the semiaxes major of the curves Σ_1, Σ_2, σ_1, σ_2; so that, supposing F_1 the nearer focus, we have $\Lambda_2 = \frac{1}{2}(R_2+R_1)$, $\lambda_2 = \frac{1}{2}(r_2-r_1)$, $\Lambda_1 = \frac{1}{2}(R_2-R_1)$, $\lambda_1 = \frac{1}{2}(r_2+r_1)$, $\Lambda_1\lambda_1 = \Lambda_2\lambda_2 = Cc$. Considering two points on Σ_1 and Σ_2 indefinitely near to A, and denoting, as in Art. 10, the angles F_1AF_2, f_1af_2 by A and a, we find

$$d\Sigma_1 = \frac{d\Lambda_2}{\cos\frac{1}{2}A}, \qquad d\sigma_1 = -\frac{d\lambda_2}{\sin\frac{1}{2}a},$$

$$d\Sigma_2 = -\frac{d\Lambda_1}{\sin\frac{1}{2}A}, \quad d\sigma_2 = \frac{d\lambda_1}{\cos\frac{1}{2}a}.$$

But

$$\frac{d\Lambda_1}{\Lambda_1} + \frac{d\lambda_1}{\lambda_1} = 0, \qquad \frac{d\Lambda_2}{\Lambda_2} + \frac{d\lambda_2}{\lambda_2} = 0;$$

so that

$$d\Sigma_1 = \frac{\sin\frac{1}{2}a}{\cos\frac{1}{2}A} \cdot \frac{1}{\lambda_2} \cdot \Lambda_2 d\sigma_1,$$

$$d\Sigma_2 = \frac{\cos\frac{1}{2}a}{\sin\frac{1}{2}A} \cdot \frac{1}{\lambda_1} \cdot \Lambda_1 d\sigma_2.$$

Again, substituting for $\frac{1}{2}a$ its value $\frac{1}{2}(F_1-F_2)$, we find, from the triangles F_1AF_2, f_1af_2,

$$\frac{\sin\frac{1}{2}a}{\cos\frac{1}{2}A} = \frac{\Lambda_1}{C}, \qquad \frac{\cos\frac{1}{2}a}{\sin\frac{1}{2}A} = \frac{\Lambda_2}{C};$$

whence, if i be the radius of the evanescent circle at a, the major and minor semi-axes of the strain ellipse at A are respectively

$$(A) \ldots\ldots\ldots \frac{\Lambda_1\Lambda_2}{Cc} \cdot \frac{\Lambda_2}{C}\, i, \qquad \frac{\Lambda_1\Lambda_2}{Cc} \cdot \frac{\Lambda_1}{C}\, i.$$

From these expressions it follows that, given in magnitude the parameters of the two planes, and given in position a single pair of corresponding points A and a; given also in position and magnitude the indicatrix at one of these points, for example at A; the homography of the two planes is determined. For the coefficients,

$$\frac{\Lambda_1\Lambda_2}{Cc}\times\frac{\Lambda_2}{C}, \quad \frac{\Lambda_1\Lambda_2}{Cc}\times\frac{\Lambda_1}{C},$$

being given, the values of Λ_1 and Λ_2 may be found (by the extraction of a cube root): thus the auxiliary ellipse at A is completely determined. The vanishing line of the plane Ω is one of the four tangents of the auxiliary ellipse which are parallel to a diameter equal to 2C, and the homographic centre is the point of contact. Similarly may the vanishing line and centre be determined in the plane ω; since the indicatrix at a can be found when that at A is given.

(15.) *The Canonical and Elliptic Equations of a Plane Homography.*

If X, Y, x, y are the coordinates of two corresponding points, the focal axis and vanishing line in each plane being taken as the axes of coordinates, we have

$$Xx = Cc,$$

$$\frac{Y}{X+C} = \frac{y}{x+c},$$

the former equation being equivalent to the anharmonic equation of Art. 5, the latter expressing the property of the corresponding foci of which the abscissæ are $-C$ and $-c$. These equations, written in either of the forms

$$(B) \ldots\ldots\ldots \begin{cases} \begin{bmatrix} Xx = Cc \\ Yx = Cy \end{bmatrix} \\ \begin{bmatrix} Xx = Cc \\ Xy = cY \end{bmatrix} \end{cases}$$

may be termed the canonical equations of the homography (M. Chasles, "Géométrie Supérieure," Art. 533), and may be employed to verify analytically the preceding results. It will be remembered that the axes of y and Y are not corresponding lines, neither are the origins corresponding points. Thus the abscissæ of corresponding points are not corresponding lines, and indeed are not measured in corresponding directions; but the ordinates of corresponding points (considered as lines drawn from the extremities of the abscissæ parallel to the vanishing lines) are corresponding lines.

The elliptic coordinates Λ_1, Λ_2; λ_1, λ_2 of two corresponding points (*i. e.*, the major semi-axes of the confocal conics passing through the points) are, as we have seen, connected by the relations

$$(B') \ldots\ldots\ldots \Lambda_1\lambda_1 = \Lambda_2\lambda_2 = Cc.$$

Thus every homographic transformation of a plane figure, in which the line at an infinite distance is transformed into a line at a finite distance, is equivalent to an inverse transformation of the elliptic coordinates of the points of the plane. In this way the expressions already given for the semi-axes of the indicatrix may be immediately deduced from the elementary elliptic formulæ

$$d\Sigma_1 = \sqrt{\left(\frac{\Lambda_2^2 - \Lambda_1^2}{\Lambda_2^2 - C^2}\right)}\, d\Lambda_2, \quad d\Sigma_2 = -\sqrt{\left(\frac{\Lambda_2^2 - \Lambda_1^2}{C^2 - \Lambda_1^2}\right)}\, d\Lambda_1,$$

combined with the corresponding formulæ for the plane ω. Again, using the formulæ $\Lambda_1\Lambda_2 = CX$, $\lambda_1\lambda_2 = cx$, we may write those expressions in either of the forms

$$\text{(A)}\ldots\ldots\ldots\ldots\ldots \frac{X}{\lambda_2}\,.\,i, \quad \frac{X}{\lambda_1}\,.\,i; \quad \text{or} \quad \frac{\Lambda_2}{x}\,.\,i, \quad \frac{\Lambda_1}{x}\,.\,i.$$

Thus the ratio of corresponding elementary areas at A and a is that of X^2 to $\lambda_1\lambda_2$, or of $\Lambda_1\Lambda_2$ to x^2, or of $C^{\frac{1}{2}}X^{\frac{3}{2}}$ to $c^{\frac{1}{2}}x^{\frac{3}{2}}$; *i. e.*, it varies in the sesquiplicate ratio of the distances of the two areas from the vanishing lines. The lines $X^3 = \pm Cc^2$, $x^3 = \pm C^2c$, (of which two in each plane are real, and four imaginary,) are lines at which corresponding elementary areas are equal. More generally, the lines $kX = \pm KC$, $Kx = \pm kc$ are the real lines, in the planes Ω and ω, at which corresponding evanescent areas are to one another in the ratio of K^3C^2 to k^3c^2.

(16.) *Theorems relating to Curvature.*

Since evanescent segments, at the same point, and upon the same straight line, are altered in one and the same ratio in any homographic transformation, the curvature of all curves which touch one another at a given point is altered in one and the same ratio. Thus, if a curve touch a focal circle of the plane Ω, its radius of curvature at the point of contact is altered in the transformation in the ratio of C to c. Again, it will be found that the radius of curvature of a curve at a point at which its tangent is parallel to the vanishing line is altered in the same parametric ratio. Hence if we consider in the plane Ω any conic which passes through the imaginary points Q_1, Q_2 (and which, consequently, is transformed into a circle), it has the same curvature at the two points where it is touched by focal circles, and at the two points where it is touched by parallels to the vanishing line; for the radius of curvature at any one of these four points is to the radius of the corresponding circle in the ratio of the parameters. We thus obtain incidentally a solution of the problem, "Given a system of circles, and a conic, having the same radical axis, to determine the two circles of the system which touch the conic;" for the points of contact are at the extremities of the diameter equal to the diameter conjugate to the radical axis. In particular, the radius of curvature of the indicatrix at the points where its tangent is parallel to the vanishing line, or to the tangent of the focal circle passing through its centre, is to the radius of the corresponding evanescent circle in the ratio of the parameters: thus, if R is the radius of curvature of the auxiliary ellipse at the point O, Ai, Bi the principal semi-axes of the indicatrix, we have the equations

$$\frac{Ai}{\Lambda_2}\mathrm{R} = \frac{\mathrm{C}}{c}i, \quad \frac{Bi}{\Lambda_1}\mathrm{R} = \frac{\mathrm{C}}{c}i,$$

which are in accordance with the equations (A), since $\mathrm{R} = \dfrac{\mathrm{C}^3}{\Lambda_1\Lambda_2}$. More generally, if Di is any semi-diameter of the indicatrix at the point A, the radius of curvature of any curve touching that semi-diameter at the point A is altered in the ratio of D^3 to AB (since $\dfrac{\mathrm{D}^3}{\mathrm{AB}}i$ is the radius of curvature of the indicatrix at the extremities of the diameter conjugate to Di). It will be seen, that of all curves passing through the point A, those which touch the confocal hyperbola at A experience in the transformation the greatest augmentation (or the least diminution) of curvature, and those which touch the confocal ellipse experience the greatest diminution (or the least augmentation) of curvature; so that the confocal conics may be said to be loci of greatest and least augmentation (or diminution) of curvature. The ratio of the radius of curvature of any curve passing through the point A to the radius of curvature of the corresponding curve is thus intermediate between the ratios $\Lambda_2^3 : \mathrm{C}^2c$ and $\Lambda_1^3 : \mathrm{C}^2c$. Let $\mathrm{K}^3 : \mathrm{C}^2c$ be any ratio intermediate between these two; there are evidently two equal semi-diameters of the indicatrix at the point A such that the radii of curvature of curves touching either of them are altered in the ratio $\mathrm{K}^3 : \mathrm{C}^2c$. If Φ be the angle made by either of these semi-diameters with the major axis of the indicatrix, the equation $\dfrac{\mathrm{D}^3 i}{\mathrm{AB}} = \dfrac{\mathrm{K}^3}{\mathrm{C}^2c}i$ becomes, on substituting for A and B their values given by the equations (A),

$$\text{(C)} \;\ldots\ldots\ldots\ldots\; \frac{\cos^2\Phi}{\Lambda_2^2} + \frac{\sin^2\Phi}{\Lambda_1^2} = \frac{1}{\mathrm{K}^2},$$

or

$$\text{(C}'\text{)} \ldots\ldots\ldots\; \lambda_2^2\cos^2\Phi + \lambda_1^2\sin^2\Phi = \frac{\mathrm{C}^2c^2}{\mathrm{K}^2}.$$

From the first of these equations we learn that the two semi-diameters coincide in direction with those semi-diameters of the auxiliary ellipse which are equal to K; the second implies that the angles made by the two semi-diameters with the axis major of the indicatrix at A are equal to the angles made in the plane ω with the major axis of the indicatrix at a by the tangents drawn from the point a to that confocal conic which, in the plane ω, corresponds to the confocal conic of semi-axis major K in the plane Ω.

(17.) *Curves of Constant Alteration of Curvature.*

We may also regard the equation (C) as equivalent to the differential equation of a system of curves such that at any point on any one of them its radius of curvature is altered in the constant ratio of K^3 to C^2c. Substituting for $\tan^2\Phi$ its value,

$$-\frac{\Lambda_2^2 - C^2}{\Lambda_1^2 - C^2} \cdot \frac{d\Lambda_1^2}{d\Lambda_2^2},$$

we find for this differential equation the expression

$$\frac{d\Lambda_2}{\Lambda_2}\sqrt{\left(\frac{\Lambda_2^2 - K^2}{\Lambda_2^2 - C^2}\right)} = \pm\frac{d\Lambda_1}{\Lambda_1}\sqrt{\left(\frac{K^2-\Lambda_1^2}{C^2-\Lambda_1^2}\right)}.$$

The integral of this equation is easily obtained in a finite form; it seems, however, too cumbrous for discussion. It contains an algebraical function raised to the power $\frac{K}{C}$, but no other transcendental function. If, therefore, $\frac{K}{C}$ be rational, *i. e.*, if the given ratio $K^3 : C^2c$ is a multiple of the parametric ratio by the cube of a rational number, the curves of constant alteration of curvature are algebraic and of finite dimensions; in every other case they are transcendental. They lie entirely outside the confocal conic (K), and seem to meet it in cusps of which the tangents are normal to it. If $K = C$, *i. e.*, if the ratio is the parametric ratio, the curves of constant alteration of curvature are given by the equations

$$\frac{d\Lambda_2}{\Lambda_2} = \pm\frac{d\Lambda_1}{\Lambda_1},$$

or $$\frac{\Lambda_1}{\Lambda_2} = \text{constant}, \quad \Lambda_1\Lambda_2 = \text{constant};$$

they are thus the focal circles and the parallels to the vanishing line, as we have already seen.

The orthogonal trajectories of the curves of constant alteration of curvature are always algebraic; they have for their differential equation

$$\frac{\Lambda_2 d\Lambda_2}{\sqrt{(\Lambda_2^2 - C^2)(\Lambda_2^2 - K^2)}} = \frac{\Lambda_1 d\Lambda_1}{\sqrt{(C^2 - \Lambda_1^2)(K^2 - \Lambda_1^2)}}.$$

(18.) *Curves of Constant Elongation.*

Let $\theta C : c$ represent any given ratio; there are in general two equal semi-diameters of the indicatrix at any point A which are in that ratio to the radius i of the corresponding evanescent circle. If, however, $\theta\frac{C}{c}i$ be greater than the major axis, or less than the minor axis of the indicatrix, the two diameters are imaginary. The equations

$$\Lambda_2^2\Lambda_1 = \theta C^3 \ldots\ldots\ldots (\alpha), \qquad \Lambda_1^2\Lambda_2 = \theta C^3 \ldots\ldots\ldots (\beta),$$

represent two loci, which separate the parts of the plane Ω, in which the two diameters are real, from those parts in which they are imaginary; at points on the locus (α) the two diameters coincide with the axis major of the indicatrix; at points on the locus (β) they coincide with its axis minor. The two loci are included in the same Cartesian equation of the sixth order

$$\theta^2C^2X^2Y^2 = (X^2-\theta^2C^2)(X^2-\theta C^2)(X^2+\theta C^2),$$

which represents a curve symmetrical with respect to the vanishing line and focal axis, and having a quadruple point at Y (the point at an infinite distance on the vanishing line). The two branches, one on each side of the vanishing axis, which touch it and one another at the point Y, form the locus (α); the locus (β) consists of the two branches which have the line at an infinite distance for their common tangent at the point Y. In the space included between the two branches of the locus (α), the semi-axis major of the indicatrix is less than $\theta\dfrac{C}{c}i$; in the spaces intermediate between the loci (α) and (β) the two diameters are real, and they are again imaginary in the spaces interior to the locus (β). To determine the angle Φ, which either of the two diameters makes with the axis major of the indicatrix, we observe that these lines coincide in direction with the two diameters of the auxiliary ellipse which are equal to $2\theta\dfrac{C^3}{\Lambda_1\Lambda_2}$; we thus obtain the equation

$$(D)\ldots\ldots\ldots \Lambda_1^2\cos^2\Phi+\Lambda_2^2\sin^2\Phi = \frac{\Lambda_1^4\Lambda_2^4}{\theta^2C^6} = \frac{X^4}{\theta^2C^2},$$

which implies that the two diameters coincide in direction with the tangents drawn from the point A to the confocal conic of which the semi-axis major is $\dfrac{X^2}{\theta C}$. Substituting for $\tan^2\Phi$ its value, we find

$$\frac{\Lambda_2^4\Lambda_1^2-\theta^2C^6}{\Lambda_2^2-C^2}\cdot\frac{d\Lambda_2^2}{\Lambda_2^2} = \frac{\Lambda_1^4\Lambda_2^2-\theta^2C^6}{\Lambda_1^2-C^2}\cdot\frac{d\Lambda_1^2}{\Lambda_1^2},$$

which is the differential equation of the curves of constant elongation, and seems not to admit of integration in any finite form. Its equivalent in Cartesian coordinates is

$$\frac{dX^2+dY^2}{C^2} = \theta^2\frac{dx^2+dy^2}{c^2},$$

or

$$\frac{X^4(dX^2+dY^2)}{\theta^2C^2} = C^2dX^2+(YdX-XdY)^2.$$

It will be observed that $\Lambda_1\Lambda_2 = \theta C^2$, or $X = \theta C$, is a particular integral of the equation.

The following is an important property of the curves of constant elongation:—

"The intercept in the plane Ω on any tangent to one of these curves between the point of contact and the vanishing line is in the given ratio to the similarly defined segment on the corresponding tangent in the plane ω."

To establish this property, we have only to observe that the point of contact is one of the double points of that involution upon the tangent of which the segments are to their corresponding segments in the given ratio. Or we may infer it from the equation (D), with the help of the easily demonstrated theorem—

"If Ψ and ψ are the angles made with the vanishing line by any two corresponding tangents to the confocals (Γ) and (γ), the ratio $\frac{\sin\Psi}{\sin\psi}$ is constant and equal to $\frac{\Gamma}{C}$ or $\frac{c}{\gamma}$."

(19.) *Curves of Equal Tangential Deflexion.*

Through any point A of the plane Ω there pass two curves, such that the angle between any two tangents to either of them is equal to the angle between the two corresponding tangents of the corresponding curve. We may term these curves the curves of equal tangential deflexion. If Di be the length of the semi-diameter of the indicatrix, which touches one of these curves at the point A, and if $d\Theta = d\theta$ be the angle contained between this semi-diameter and a consecutive tangent to the curve, we find, since all areas at the point A are altered in the same ratio,

$$D^2i^2d\Theta : i^2d\theta :: ABi^2 : i^2, \quad \text{or simply} \quad D^2 = AB.$$

Denoting by Φ the angle made by the semi-diameter Di with the major axis of the indicatrix, and substituting for A, B, and D, their values, we have for the differential equation of the curves of equal tangential deflexion

$$\text{(E)} \ldots\ldots\ldots \frac{\cos^2\Phi}{\Lambda_2} = \frac{\sin^2\Phi}{\Lambda_1},$$

or

$$\frac{d\Lambda_1}{\sqrt{\Lambda_1}\sqrt{(C^2-\Lambda_1^2)}} = \pm\frac{d\Lambda_2}{\sqrt{\Lambda_2}\sqrt{(\Lambda_2^2-C^2)}}.$$

These curves are always algebraic; for, putting $\Lambda_1 = Cu_1^2$, $\Lambda_2 = \frac{C}{u_2^2}$, we have

$$\frac{du_1}{\sqrt{(1-u_1^4)}} = \pm\frac{du_2}{\sqrt{(1-u_2^4)}},$$

of which the integral is algebraic.

The equation (E) may be also obtained by observing that the curves of equal tangential deflexion which pass through the point A, must touch at that point the double lines of the pencil in involution, which is equiangular to the corresponding pencil, and that the equation (E) is the equation determining these double lines (see Arts. 10 and 11). It appears from this that the angle at which the two curves intersect is always supplementary to the corresponding angle.

(20.) *Curves similar and similarly situated to their images.*

As an additional example of the use of the formulæ (B), let us propose to determine the conic sections, which in either figure are transformed into conics similar and similarly situated with regard to the vanishing line and focal axis of the other figure. If

$$aX^2+a'Y^2+a''C^2+2bCY+2b'CX+2b''XY=0$$

be the equation of any conic in the plane Ω, the equation of the corresponding conic is

$$a''x^2+a'y^2+ac^2+2b''cy+2b'cx+2bxy=0.$$

And if these two conics are similar and similarly situated, we must have $a=a''$, $b=\pm b''$; *i. e.*, every conic for which one of the two foci is the pole of the line parallel to the vanishing axis, and passing through the other focus, (or, which is the same thing, any conic for which one of the pairs of lines joining its points at an infinite distance to its points on the vanishing axis, intersect at a focus,) is transformed into a similar and similarly situated conic, the ratio of similarity being that of the parameters. There are thus two sets of conics (each forming a triply indeterminate linear system) which satisfy the conditions of the problem; but the conics of only one set at a time can be regarded as similarly situated to the corresponding conics, because in determining the two sets different directions on the vanishing lines are taken to determine the similarity of position.

Again, the corresponding conics will have their areas in the ratio of the squares of the parameters, if $(aa'-b''^2)^3=(a'a''-b^2)^3$; *i. e.*, the areas of all conics with regard to which the lines parallel to the vanishing line and passing through the foci are self-conjugate lines are to the areas of their corresponding conics in the duplicate ratio of the parameters. The only real conics of which the area is changed in this ratio are those defined by this geometrical condition; they form a quadruply indeterminate linear tangential system. But the analytical condition is also satisfied by the imaginary conics in the plane Ω, with regard to which the imaginary lines $X=\pm\rho C$, or $X=\pm\rho^2C$, are harmonically conjugate, ρ denoting an imaginary cube root of unity. More generally, it will be found that the conics of which the area is changed in any given ratio are those which have for a pair of conjugate lines the two straight lines at which elementary areas are changed in the given ratio (Art. 15). If the corresponding conics are hyperbolas, we may substitute for the area in this result the triangle contained by the asymptotes and any tangent.

Lastly, the geometrical condition that a conic in either plane should be similar to its corresponding conic is that the pairs of points in which it intersects the vanishing line and the line at an infinite distance should subtend equal angles at a focus. But the quadruply indeterminate system determined by this condition is not a linear one.

Theorems of a similar kind to the preceding, but relating to curves of a higher order, may be obtained by observing that symmetrical functions of X, x; Y, y; or again of Λ_1, λ_1; Λ_2, λ_2 are unchanged by the transformation. Thus any curve represented by

$$F\left(X+\frac{Cc}{X}, \frac{cY^2}{X}, Y+\frac{cY}{X}\right)=0,$$

or again by

$$f\left(\Lambda_1+\frac{Cc}{\Lambda_1}, \Lambda_2+\frac{Cc}{\Lambda_2}\right)=0,$$

is transformed into a curve similar and similarly situated with regard to the focal axis.

B.—Focal Properties of Two Homographic Point-Figures.

(21.) *The Imaginary Cones corresponding to Evanescent Spheres.**

By a point-figure we shall here understand a system of straight lines and planes passing through a point which is termed the centre of the point-figure. Let S, s be the centres of two point-figures, homographically related to one another; and let P, q represent the evanescent spheres (here to be regarded as imaginary cones), which have their centres at S, s. Excluding altogether from consideration the very particular case in which these two imaginary cones correspond to one another homographically, and in which, consequently, the two figures admit of exact coincidence with one another, let us represent by p, Q the imaginary cones, which in the figures s, S correspond to the cones P, q. We observe that if either p or Q is a cone of revolution, the other is so too; for if the cones P, Q have double contact, so also have the corresponding cones p, q. We shall hereafter (Art. 38) return for a moment to this particular case, but for the present we shall suppose that neither p nor Q is a cone of revolution.

(22.) *The Principal Axes.*

On this supposition there exists in each pencil one, and only one, system of straight lines at right angles to one another, such that their corresponding lines are also at right angles to one another. These lines are the principal axes of the cones Q and p. For the principal axes of Q are the system of self-conjugate axes common to the cones P and Q; these principal axes, therefore, correspond to the system of axes self-conjugate with regard to p and q; *i. e.* to the principal axes

* See Note at end of Paper, p. 248.

of p. We shall call these two sets of rectangular axes the principal axes of the two figures, and we shall distinguish them as the axes of XYZ, xyz.

(23.) *The Focal Lines and Cyclic Planes.*

To the four imaginary lines of intersection of P and Q, and to the four imaginary tangent planes common to those two cones, there correspond the four lines of intersection, and the four common tangent planes, of p and q. Hence to C_1 and C_2, the two real cyclic planes of Q, and to F_1, F_2, the two real focal lines of Q, there correspond c_1, c_2, the cyclic planes, and f_1, f_2, the focal lines of p. It will be observed that the real focal lines of an imaginary cone (differing in this respect from the focal lines of a real cone) lie in that principal plane of the cone to which the cyclic planes are perpendicular. We shall call the axis, in which the cyclic planes intersect, and which is perpendicular to the focal plane (*i. e.* to the plane containing the focal lines), the mean axis; of the two axes in the focal plane, we shall term that the major axis which makes with either cyclic plane, and with either focal line, acute angles together less than a right angle.

(24.) *The Reciprocity of the Imaginary Cones.*

The imaginary cones Q and p are reciprocal. Let Y, y be their mean axes; XZ, xz, their focal planes; P_1P_2, Q_1Q_2, and p_1p_2, q_1q_2, the imaginary lines in which these planes meet the cones P, Q and p, q respectively. From the anharmonic equation

$$(1)\ldots\ldots[P_1, P_2, Q_1, Q_2, X, Z] = [p_1, p_2, q_1, q_2, x, z]$$

we infer the equation

$$[P_1, P_2, Q_1, Q_2] = [q_1, q_2, p_1, p_2],$$

which implies that the imaginary angles Q_1SQ_2, p_1sp_2 are equal, because P_1, P_2 and q_1, q_2 are pairs of lines representing evanescent circles. Again, X, Z and x, z are harmonic conjugates of the pairs P_1P_2, Q_1Q_2, and p_1p_2, q_1q_2 respectively. Hence we must have either the equation

$$[P_1, P_2, Q_1, Q_2, X, Z] = [q_1, q_2, p_1, p_2, x, z],$$

or else the equation

$$[P_1, P_2, Q_1, Q_2, X, Z] = [q_1, q_2, p_1, p_2, z, x].$$

But the former equation is inadmissible; for, on combining it with (1), we obtain

$$[p_1, q_1, x, z,] = [q_1, p_1, x, z],$$

which is untrue, since p_2, and not q_1, is the harmonic conjugate of p_1 with regard to xz. It is, therefore, the latter equation which subsists; it implies that Q_1 or Q_2 makes the same angles with X, that p_1 or p_2 makes with z; *i. e.*, that the angle Q_1SX is the complement of p_1sx. Similarly, the angle which the axis of X makes with either of the lines of Q which lie in the plane XY, is the complement of the corresponding angle in the plane xy; that is to say, the two cones are reciprocal.

It is evident that the mean axis of Q corresponds to the mean axis of p. Let C, c be the acute angles F_1SX, f_1sx; then the acute angle contained by YX and either cyclic plane of Q is the complement of c, and the acute angle contained by yx and either cyclic plane of p is the complement of C. Hence X is the major or minor axis of Q, according as $C < c$, or $C > c$; and to the major axis of Q the minor axis of p corresponds, and *vice versâ*. Neither of the angles C, c can be zero, nor a right angle, nor can they be equal to one another.

The reciprocity of the cones Q and p gives rise to a reciprocal relation between the two homographic figures, which may be thus stated. Conceive the two figures placed with their corresponding principal axes coincident. Let A, a be any two corresponding planes in the figures S and s; let b be the normal to A at the common centre of the figures; and B the normal to a at the same point; then B and b are corresponding lines in the figures S and s. If, therefore, we consider any two corresponding systems of planes and lines in S and s, the *reciprocal* systems of lines and planes will also be corresponding systems in s and S. Thus all the properties (metrical as well as descriptive) of two homographic point-figures are double, and we have an uniform method for passing from any property to its correlative.

(25.) *The Correspondence of Directions.*

The angles contained by planes intersecting in a focal line of S are equal to the corresponding angles contained by planes intersecting in a focal line of s; and, correlatively, the angles contained by lines intersecting at S in one of the cyclic planes of S, are equal to the corresponding angles in a cyclic plane of s. These theorems are evident, because the imaginary tangent planes of P, which intersect in F_1, correspond to the imaginary tangent planes of q, which intersect in f_1; and similarly, the lines in which P is intersected by either cyclic plane of Q, correspond to the lines in which q is intersected by either cyclic plane of p.

To fix the correspondence of the directions of rotation round either pair of corresponding focal lines, or in either pair of corresponding cyclic planes, we consider the intersections of the planes and lines of S and s by the surfaces of two spheres of radius unity having their centres at S and s. Let A, B, C be three points on the sphere S, forming a spherical triangle; it will be remembered that three points, not in the same great circle, always form one, and only one, spherical triangle, if by a spherical triangle we understand (as is usually done), a triangle formed by arcs of great circles, each of which is less than two right angles. As corresponding point to any point A on the sphere S, we might take either of two diametrically opposite points a, a' on the sphere s. But for one of these points (for example a) the corresponding directions of rotation round A and a are similar (*i. e.*, both right handed or both left handed, when viewed from the centres of the spheres); while for the other point a' the corresponding directions of rotation are dissimilar. Let then a, b, c be the three points which on the sphere s correspond with similar rotations to the points A, B, C. These three points are thus determined without any ambiguity, and we shall now show that to points in the interior of the triangle ABC there correspond, with similar rotations, points in the interior of abc. The proof of this important theorem depends on the two principles: (i.), that if a point move continuously on either sphere, and traverse any curve on that sphere, its corresponding point on the other sphere simultaneously traverses the corresponding curve; (ii.), that if A and a are corresponding points with similar rotations, and if, while A moves continuously to B, a moves continuously to b, then a and b are also corresponding points with similar rotations. The first of these principles may be considered as evident; to establish the second, it will suffice to consider A and B as consecutive positions of A, so that while A describes the element AB, a describes the element ab. Let E be any great circle not intersecting AB, then the corresponding great circle e does not intersect ab, and if these two great circles be described by corresponding points V and v, the vector arcs AV, av will by hypothesis revolve in similar directions. But the arcs AV, BV evidently revolve in similar directions, and so do the arcs av, bv; *i. e.*, the corresponding rotations round B and b are similar. Let us now suppose that a point sets out from B, and describes the side BC of the triangle ABC; the corresponding point will at the same time describe the side bc of the triangle abc; for as it must not traverse either of the great circles ab, ac, it cannot describe an arc greater than a semicircle. Thus, to the points of any side of ABC there correspond, with similar rotations, the points of the corresponding side of abc. Let V be any point internal to ABC, let AV cut BC in A_1, and let a_1 on bc correspond to A_1 on BC; then AA_1B, aa_1c are corresponding spherical triangles, with similar rotations at their corresponding points; therefore the points of aa_1 correspond, with similar rotations, to the points of AA_1; *i. e.*, the point v, which corresponds with similar rotation to V, lies on aa_1 in the interior of the triangle abc.

The great circles which form the triangles ABC, abc, divide the spheres S and s each into eight spherical triangles, which correspond to one another one by one, with similar rotations at their corresponding vertices, just as the triangles ABC, abc. Thus each sphere is divided into eight regions, corresponding to the eight regions of the other sphere, in such a manner, that, if any point be taken on either sphere, the point which corresponds to it with similar rotation lies in the corresponding region of the other sphere.

We shall now take for ABC one of the eight octantal triangles XYZ, and for abc the corresponding octant xyz; we shall denote by F_1, f_1; $Y\Omega_1$, $y\omega_1$, the foci and cyclic arcs which lie in the octants XYZ, xyz; and by F_2, f_2; $Y\Omega_2$, $y\omega_2$; the foci and cyclic arcs which lie in the octants $XY\bar{Z}$, $xy\bar{z}$; so that the directions of rotation round the corresponding foci F_1, f_1 and F_2, f_2, will be similar, and the directions $Y\Omega_1$, $y\omega_1$; $Y\Omega_2$, $y\omega_2$, will be corresponding directions on the cyclic arcs. It will be convenient to consider only the hemispheres of which the points X, x are the spheric centres, and the planes YZ, yz the bases. Thus, to any given point on the hemisphere S (not lying on the base circle itself), there corresponds one point, and only one, on the hemisphere s; and again any two great circles upon either hemisphere (neither of which is the base circle) intersect one another only in one point. To find the point a of the hemisphere s, which corresponds to a given point A of the hemisphere S, we draw the vector arcs F_1A, F_2A, and make the angles $f_2 f_1 a$, $f_1 f_2 a$ equal in sign and magnitude to the angles F_2F_1A, F_1F_2A; the point of intersection of the arcs $f_1 a$, $f_2 a$ is the point a required. Similarly, to find the great circle a of the hemisphere s, which corresponds to a given great circle A of the hemisphere S, we find the points D_1, D_2, in which A intersects the cyclic arcs of S, and we make the arcs $\omega_1 d_1$, $\omega_2 d_2$, equal in sign and magnitude to the arcs $\Omega_1 D_1$, $\Omega_2 D_2$; the arc $d_1 d_2$ is the arc required.

(26.) *The Confocal Spherical Conics.*

The spherical conics of which F_1, F_2 are the foci are transformed into the spherical conics of which f_1, f_2 are the foci. This is evident from the equiangular property of the foci; or, again, if R_1, R_2, r_1, r_2 are the focal radii vectores of the corresponding points Δ and δ, the spherical triangles $F_1\Delta F_2$, $f_1\delta f_2$ give the equations

$$\frac{\tan\frac{1}{2}(R_1+R_2)}{\tan\frac{1}{2}(r_1+r_2)}=\frac{\tan\frac{1}{2}(R_1-R_2)}{\tan\frac{1}{2}(r_1-r_2)}=\frac{\tan C}{\tan c},$$

which imply that if $R_1 \pm R_2$ is constant, $r_1 \pm r_2$ is also constant. Thus the ellipses are transformed into ellipses, and the hyperbolas into hyperbolas, these denominations being relative to the two foci lying on each of the hemispheres S and s. If $\Lambda=\frac{1}{2}(R_1+R_2)$, or $=\frac{1}{2}(R_1-R_2)$, is the focal semi-axis of one of the confocal conics of S, the quotient $\frac{\tan \Lambda}{\tan C}$, which is one of the spherical eccentricities of the conic, remains unchanged in the transformation; for, if λ be the semi-axis of the corresponding conic, we have the equation

$$\frac{\tan \Lambda}{\tan \lambda}=\frac{\tan C}{\tan c},$$

which results from the homography of corresponding points of the great circles XZ, xz. It is also evident that if we consider any two corresponding conics of the two confocal systems, there will correspond to one another in the two figures—the normal arcs of the two curves, their spherical centres of curvature, their evolutes, their *similar* arcs, as also the spherical polygons of minimum perimeter circumscribing corresponding arcs, and the spherical polygons of maximum perimeter inscribed in corresponding arcs.

(27.) *The Concyclic Spherical Conics.*

Correlatively, the system of concyclic conics of which $Y\Omega_1$, $Y\Omega_2$ are the cyclic arcs are transformed into concyclic conics of which $y\omega_1$, $y\omega_2$ are the cyclic arcs, the ellipses into ellipses, and the conics of the third species into conics of the third species; these denominations being again relative to the hemispheres which we are considering. (See M. Chasles "Sur les propriétés générales des coniques sphériques," art. 1—4). If D_1D_2, d_1d_2 are corresponding arcs, cutting the cyclic arcs in D_1, D_2, d_1, d_2, the spherical triangles D_1YD_2, d_1yd_2, in which $D_1Y = d_1y$, $D_2Y = d_2y$, supply the equations

$$\frac{\tan\frac{1}{2}(D_1+D_2)}{\tan\frac{1}{2}(d_1+d_2)} = \frac{\tan\frac{1}{2}(D_1-D_2)}{\tan\frac{1}{2}(d_1-d_2)} = \frac{\tan c}{\tan C}.$$

Let E, e be the areas of the spherical quadrilaterals $\Omega_1D_1D_2\Omega_2$, $\omega_1d_1d_2\omega_2$, we find $E = \pi - D_1 - D_2$, $e = \pi - d_1 - d_2$, whence

$$\frac{\tan\frac{1}{2}E}{\tan\frac{1}{2}e} = \frac{\tan C}{\tan c},$$

a formula which expresses a remarkable property of the cyclic arcs.

To chords of any conic (Φ) of the concyclic system of S, which cut off equal spherical areas from that conic, there will correspond chords cutting off from the corresponding conic (ϕ) areas equal to one another. To a spherical polygon of maximum area inscribed in any arc of (Φ), or to a polygon of minimum area circumscribing any arc of (Φ), there will correspond polygons possessing a similar maximum or minimum property with regard to the corresponding arc of (ϕ). These results follow from the known properties of concyclic spherical conics; or they may be deduced by reciprocation from the properties of the confocal conics of the two homographic systems.

(28.) *Arcs and Angles changed into equal Arcs and Angles.*

On any great circle A of S there are two points at right angles to one another, such that their corresponding points, on the corresponding great circle a, are also at right angles. These points are the external and internal points of bisection of the intercept made on the great circles by the cyclic arcs; they are also the points at which the great circles are touched by conics of the concyclic systems. Let 2D, $2d$ be the intercepts; the homographic modulus of the two great circles (relative to the internal points of bisection) is $\frac{\tan D}{\tan d}$; the arcs of the involution

$$\tan H_1 \tan H_2 = \frac{\tan D}{\tan d}$$

are equal to the corresponding arcs of the involution

$$\tan h_1 \tan h_2 = \frac{\tan d}{\tan D};$$

and the arcs of the involution

$$\tan H_1 \tan H_2 = -\frac{\tan D}{\tan d}$$

are equal to the supplements of the corresponding arcs of the involution

$$\tan h_1 \tan h_2 = -\frac{\tan d}{\tan D}.$$

The determination of the angles which at any point Δ are transformed into equal or supplementary angles at the point δ is correlative to the preceding. The external and internal bisectors of the angles between the radii vectores at Δ and δ are the right angles of the homographic pencils at A and a, and if $F_1\Delta F_2 = 2\Delta$, $f_1 \delta f_2 = 2\delta$, the homographic modulus of the pencils, relative to the internal bisectors, is $\frac{\tan \Delta}{\tan \delta}$. The equiangular and supplementary involutions are respectively

$$\tan H_1 \tan H_2 = \frac{\tan \Delta}{\tan \delta}, \qquad \tan h_1 \tan h_2 = \frac{\tan \delta}{\tan \Delta},$$

and $$\tan H_1 \tan H_2 = -\frac{\tan \Delta}{\tan \delta}, \qquad \tan h_1 \tan h_2 = -\frac{\tan \delta}{\tan \Delta}.$$

Combining the results relating to equal arcs and to equal angles, we see that, given any arc of a great circle in either figure, and a point upon it, there is always a spherical triangle having a vertex at the given point, and a second vertex upon the given arc, which is transformed into an equal and superposable spherical triangle.

The homographic modulus of the pencils at Δ and δ may be also expressed in terms of the radii vectores of the points Δ and δ, since from the triangles $F_1\Delta F_2$, $f_1\delta f_2$ we find

$$\frac{\tan \Delta}{\tan \delta} = \frac{\sin \frac{1}{2}(R_1 - R_2)}{\sin \frac{1}{2}(R_1 + R_2)} : \frac{\sin \frac{1}{2}(r_1 - r_2)}{\sin \frac{1}{2}(r_1 + r_2)}$$

$$= \frac{\cos \frac{1}{2}(R_1 - R_2)}{\cos \frac{1}{2}(R_1 + R_2)} : \frac{\cos \frac{1}{2}(r_1 - r_2)}{\cos \frac{1}{2}(r_1 + r_2)}.$$

(29.) *The Equations of the Homography in Spherical Coordinates.*

The equation of the cone Q, referred to its principal axes, is

$$\frac{\sin^2 C}{\sin^2 c}X^2+Y^2+\frac{\cos^2 C}{\cos^2 c}Z^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (Q);$$

the equations of its cyclic planes, and of its focal lines, are respectively

$$Y^2+Z^2\sec^2 c=0,$$
$$Z^2-X^2\cot^2 c=0,$$
$$X^2+Y^2\sin^2 c=0;$$

and

$$Y^2+Z^2\cos^2 C=0,\quad X=0,$$
$$Z^2-X^2\tan^2 C=0,\quad Y=0,$$
$$X^2+Y^2\operatorname{cosec}^2 C=0,\quad Z=0.$$

The equations of the cone p, and of its cyclic planes and focal lines, are obtained by interchanging C and c.

Let Δ, δ be corresponding points on the two spheres, and let the arcs $X\Delta$, $Y\Delta$, $Z\Delta$, $x\delta$, $y\delta$, $z\delta$ meet the arcs YZ, ZX, XY, yz, zx, xy in the points A, B, C, a, b, c respectively. If we take the ratios of the cosines

$$X=\cos\Delta X,\quad Y=\cos\Delta Y,\quad Z=\cos\Delta Z,$$
$$x=\cos\delta x,\quad y=\cos\delta y,\quad z=\cos\delta z,$$

as the spherical coordinates of the points Ω and ω respectively, the homographic relation of the two figures is expressed by the equations

$$\frac{\sin C\cos X}{\sin c\cos x}=\frac{\cos Y}{\cos y}=\frac{\cos C\cos Z}{\cos c\cos z}.$$

Or again, if we take one of the following systems of tangents as the coordinates of the points Δ and δ,

(1) ...... $Y=\tan XB$, $Z=\tan XC$; $y=\tan xb$, $z=\tan xc$;
(2) ...... $Z=\tan YC$, $X=\tan YA$; $z=\tan yc$, $x=\tan ya$;
(3) ...... $X=\tan ZA$, $Y=\tan ZB$; $x=\tan za$, $y=\tan zb$;

the homographic relation is expressed by the equations

$$(1)\ \ldots\ldots\ Y=y\frac{\tan C}{\tan c},\quad Z=z\frac{\sin C}{\sin c};$$

$$(2)\ \ldots\ldots\ Z=z\frac{\sin c}{\sin C},\quad X=x\frac{\cos c}{\cos C};$$

$$(3)\ \ldots\ldots\ X=x\frac{\cos C}{\cos c},\quad Y=y\frac{\tan c}{\tan C}.$$

(30.) *The Parameters of the Confocal and Concyclic Cones.*

Instead of the equation (Q), it will be convenient to employ the equation

$$\frac{X^2}{A^2}+\frac{Y^2}{B^2}+\frac{Z^2}{\Gamma^2}=0$$

to represent the cone Q; so that

$$A : B : \Gamma :: \frac{\sin c}{\sin C} : 1 : \frac{\cos c}{\cos C},$$

$$\tan C = \sqrt{\left(\frac{B^2-\Gamma^2}{A^2-B^2}\right)}, \quad \tan c = \frac{A}{\Gamma}\tan C = \frac{A}{\Gamma}\sqrt{\left(\frac{B^2-\Gamma^2}{A^2-B^2}\right)}.$$

We suppose A, B, Γ all positive, and $A > B > \Gamma$, *i.e.*, $c > C$. The figure S is then transformed into *s* by the equations

$$(1) \ldots\ldots\ldots\ldots\ X = Ax, \quad Y = By, \quad Z = \Gamma z,$$

or, writing $\alpha = \frac{1}{A}$, $\beta = \frac{1}{B}$, $\gamma = \frac{1}{\Gamma}$, by the equations

$$X = \frac{x}{\alpha}, \quad Y = \frac{y}{\beta}, \quad Z = \frac{z}{\gamma}.$$

We shall term the quantities Ψ and Φ the *parameters* of the confocal cone

$$(2) \ldots\ldots\ldots\ \frac{X^2}{A^2-\Psi^2} + \frac{Y^2}{B^2-\Psi^2} + \frac{Z^2}{\Gamma^2-\Psi^2} = 0,$$

and of the concyclic cone

$$(3) \ldots\ldots\ X^2\left(\frac{1}{A^2}-\frac{1}{\Phi^2}\right)+Y^2\left(\frac{1}{B^2}-\frac{1}{\Phi^2}\right)+Z^2\left(\frac{1}{\Gamma^2}-\frac{1}{\Phi^2}\right)=0,$$

respectively. These quantities are of frequent use in the theory, as will appear from the following observations:—

(α.) If ψ and φ are the parameters of the cones corresponding to (Ψ) and Φ, we have $\psi = \frac{1}{\Psi}$, $\phi = \frac{1}{\Phi}$; for the cones (2) and (3) are transformed by the equations (1) into the cones

$$\frac{x^2}{\alpha^2-\psi^2} + \frac{y^2}{\beta^2-\psi^2} + \frac{z^2}{\gamma^2-\psi^2} = 0,$$

$$x^2\left(\frac{1}{\alpha^2}-\frac{1}{\phi^2}\right)+y^2\left(\frac{1}{\beta^2}-\frac{1}{\phi^2}\right)+z^2\left(\frac{1}{\gamma^2}-\frac{1}{\phi^2}\right)=0,$$

of which the parameters are respectively ψ and φ.

(β.) If we imagine the principal axes of the two pencils coincident, the cone reciprocal to that confocal cone in S of which the parameter is Ψ is the concyclic cone in *s* of which the parameter is $\frac{1}{\Psi}$.

(γ.) If Ψ_1, Ψ_2 are the parameters of the two confocal spherical conics which pass through a given point, the parameter of the concyclic conic passing through that point is $\frac{AB\Gamma}{\Psi_1\Psi_2}$; and, reciprocally, if Φ_1, Φ_2 are the parameters of the concyclic conics touching a given arc, the parameter of the confocal conic touching that arc is $\frac{AB\Gamma}{\Phi_1\Phi_2}$.

(δ.) Let Λ, λ represent the focal semi-axes of the corresponding confocal conics (Ψ) and (ψ); we have

$$\sin^2 \Lambda = \frac{\Psi^2 - \Gamma^2}{A^2 - \Gamma^2}, \quad \cos^2 \Lambda = \frac{A^2 - \Psi^2}{A^2 - \Gamma^2}, \quad \tan^2 \Lambda = \frac{\Psi^2 - \Gamma^2}{A^2 - \Psi^2},$$

with similar values for $\sin^2 \lambda$, $\cos^2 \lambda$, $\tan^2 \lambda$; and hence

$$\sin \Lambda = \frac{\Psi}{A} \sin \lambda, \quad \cos \Lambda = \frac{\Psi}{\Gamma} \cos \lambda, \quad \tan \Lambda = \frac{\Gamma}{A} \tan \lambda.$$

(ϵ.) Thus, for the homographic modulus of the pencils at the corresponding points Δ, δ, we have the expression (see Art. 28)

$$\frac{\tan \Delta}{\tan \delta} = \frac{\sin \Lambda_2}{\sin \lambda_2} : \frac{\sin \Lambda_1}{\sin \lambda_1} = \frac{\Psi_2}{\Psi_1},$$

the angles being measured from the tangents to the confocals (Ψ_2), (ψ_2). And correlatively for the modulus of the homography on any corresponding arcs D, d, we have $\dfrac{\tan D}{\tan d} = \dfrac{\Phi_2}{\Phi_1}$,

the arcs D, d being measured from the points of contact of the concyclic conics (Φ_2) and (ϕ_2).

(ζ.) Lastly, if (Ψ_1), (Ψ_2) are the two confocals intersecting at Δ, (Φ) the concyclic conic passing through Δ, we have

$$\frac{\sin (\Lambda_1 + \Lambda_2)}{\sin (\lambda_1 + \lambda_2)} = \frac{\Psi_1 \Psi_2}{A\Gamma}, \quad \text{or} \quad \frac{\sin R_1}{\sin r_1} = \frac{\sin R_2}{\sin r_2} = \frac{B}{\Phi},$$

an equation which corresponds to the equations (γ) of Art. 8.

(31.) *The Indicatrix on the Sphere.*

Let $d\Sigma_1$, $d\Sigma_2$, $d\sigma_1$, $d\sigma_2$ be corresponding elements of the spherical ellipses and spherical hyperbolas which pass through the corresponding points Δ and δ; let also the arcs Λ_1, Λ_2, λ_1, λ_2 be the focal semi-diameters of these conics; and let $2\Delta = F_1 \Delta F_2$, $2\delta = f_1 \delta f_2$. Considering two consecutive corresponding points on the two ellipses, and again on the two hyperbolas, we find

$$d\Sigma_1 = \frac{d\Lambda_2}{\sin \Delta}, \quad d\Sigma_2 = \frac{d\Lambda_1}{\cos \Delta},$$

$$d\sigma_1 = \frac{d\lambda_2}{\sin \delta}, \quad d\sigma_2 = \frac{d\lambda_1}{\cos \delta}.$$

But, differentiating the equations

$$\frac{\tan \Lambda_1}{\tan \lambda_1} = \frac{\tan \Lambda_2}{\tan \lambda_2} = \frac{\tan C}{\tan c},$$

we have $$\frac{d\Lambda_1}{\sin 2\Lambda_1} = \frac{d\lambda_1}{\sin 2\lambda_1}, \quad \frac{d\Lambda_2}{\sin 2\Lambda_2} = \frac{d\lambda_2}{\sin 2\lambda_2};$$

and from the triangles F_1PF_2, f_1pf_2,

$$\frac{\sin\Delta}{\sin\delta}=\frac{\cos C}{\cos\Lambda_1}:\frac{\cos c}{\cos\lambda_1}=\frac{\sin C}{\sin\Lambda_1}:\frac{\sin c}{\sin\lambda_1},$$

$$\frac{\cos\Delta}{\cos\delta}=\frac{\cos C}{\cos\Lambda_2}:\frac{\cos c}{\cos\lambda_2}=\frac{\sin C}{\sin\Lambda_2}:\frac{\sin c}{\sin\lambda_2};$$

whence

$$\frac{d\Sigma_1}{d\sigma_1}=\frac{\sin c}{\sin C}\times\frac{\sin\Lambda_1}{\sin\lambda_1}\times\frac{\sin 2\Lambda_2}{\sin 2\lambda_2},$$

$$\frac{d\Sigma_2}{d\sigma_2}=\frac{\sin c}{\sin C}\times\frac{\sin\Lambda_2}{\sin\lambda_2}\times\frac{\sin 2\Lambda_1}{\sin 2\lambda_1};$$

or, substituting from the equations (δ) and (γ), Art. 30,

$$\frac{d\Sigma_1}{d\sigma_1}=\frac{\Psi_1\Psi_2^2}{\mathrm{AB}\Gamma}=\frac{\Psi_2}{\Phi},$$

$$\frac{d\Sigma_2}{d\sigma_2}=\frac{\Psi_2\Psi_1^2}{\mathrm{AB}\Gamma}=\frac{\Psi_1}{\Phi}.$$

If in these formulæ we put $d\sigma_1=d\sigma_2=i$, the corresponding values of $d\Sigma_1$ and $d\Sigma_2$ are the principal semi-axes of the evanescent ellipse corresponding to the circle of which the centre is δ, and i the infinitesimal radius.

(32.) *Curves of Equal Tangential Deflexion and of Constant Elongation.*

Since $d\Sigma_1$ is the circular measure of the infinitesimal angle contained between the two lines in which (Ψ_1) is cut by (Ψ_2) and ($\Psi_2+d\Psi_2$), we have

$$d\Sigma_1^2=\frac{(\Psi_1^2-\Psi_2^2)\,\Psi_2^2\,d\Psi_2^2}{(\mathrm{A}^2-\Psi_2^2)\,(\mathrm{B}^2-\Psi_2^2)\,(\Psi_2^2-\Gamma^2)},$$

$$d\Sigma_2^2=\frac{(\Psi_1^2-\Psi_2^2)\,\Psi_1^2\,d\Psi_1^2}{(\mathrm{A}^2-\Psi_1^2)\,(\Psi_1^2-\mathrm{B}^2)\,(\Psi_1^2-\Gamma^2)},$$

which may also be deduced from the ordinary formulæ of elliptic coordinates in space. We may use these expressions to obtain the differential equations of certain loci analogous to those considered in Arts. 17, 18, and 19. Thus, observing that the homographic modulus of the pencil at Δ is $\frac{\Psi_2}{\Psi_1}$, we have for the curves of equal tangential deflexion the differential equation

$$(4)\ \ldots\ldots\ \frac{\sqrt{\Psi_2}\,.\,d\Psi_2}{\sqrt{[(\mathrm{A}^2-\Psi_2^2)\,(\mathrm{B}^2-\Psi_2^2)\,(\Psi_2^2-\Gamma^2)]}}$$

$$=-\frac{\sqrt{\Psi_1}\,.\,d\Psi_1}{\sqrt{[(\mathrm{A}^2-\Psi_1^2)\,(\Psi_1^2-\mathrm{B}^2)\,(\Psi_1^2-\Gamma^2)]}}.$$

The curves of "constant elongation" are defined by the equation

$$d\Sigma_1^2 + d\Sigma_2^2 = K^2 (d\sigma_1^2 + d\sigma_2^2),$$

or

$$\left(1 - \frac{K^2 \Psi_1^2 \Psi_2^4}{A^2 B^2 \Gamma^2}\right) d\Sigma_1^2 + \left(1 - \frac{K^2 \Psi_1^2 \Psi_2^4}{A^2 B^2 \Gamma^2}\right) d\Sigma_2^2 = 0,$$

in which the variables are not separated. If, however, we attend only to the curves of no elongation, and consider any tangent to one of them as determined by the parameters Φ_1 and Φ_2 of the two concyclic conics which it touches, its differential equation, in this system of tangential coordinates, is obtained by writing Φ_1 and Φ_2 for Ψ_1 and Ψ_2 in the equation (4). For, substituting $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\psi_1}, \frac{1}{\psi_2}$ for A, B, Γ, Ψ_1, Ψ_2, in that equation, we have an equation between ψ_1 and ψ_2 of the same form as (4), which represents a curve of no elongation on the hemisphere s; ψ_1 and ψ_2 being the parameters of the concyclic conics which touch any tangent of that curve.

It is evident that an infinitesimal spherical area at any point of the concyclic conic (Φ) is altered in the ratio of $AB\Gamma : \Phi^3$; and, in particular, that this is the ratio of the area contained between (Φ) and ($\Phi + d\Phi$) on S, to the area contained between the corresponding curves (ϕ) and ($\phi + d\phi$) on s. The concyclic conics, as curves of constant alteration of adjacent infinitesimal areas, resemble the parallels to the vanishing line in the theory of two homographic plane figures.

(33.) *Circles of which a Focus is the Centre.*

Since, in general, spherical conics, of which F_1 or F_2 is a focus, are transformed into spherical conics of which f_1 or f_2 is a focus, and the director arcs of the corresponding curves are corresponding arcs, it follows that circles of which F_2 is the spherical centre, are transformed into conics of which f_2 is a focus, and of which the director arc is an arc $y\theta$ perpendicular to xz at a distance θ from x defined by the equation

$$\tan \theta = \frac{\tan c}{\tan^2 C}.$$

If R is the spherical radius of one of the given circles, and if r and δ denote the spherical distances from the focus and from the director arc of any point on the corresponding conic, we shall have the equation

$$\frac{\sin r}{\sin \delta} = \frac{\sin c}{\sin \theta} \times \frac{\tan R}{\tan C}.$$

Similarly, spherical conics, of which $\Upsilon\Omega_1$ or $\Upsilon\Omega_2$ is a cyclic arc, are transformed into spherical conics of which $y\omega_1$ or $y\omega_2$ is a cyclic arc; and the cyclic poles of corresponding curves are corresponding points. In particular, circles parallel to $\Upsilon\Omega_2$ are transformed into conics of which

$y\omega_2$ is a cyclic arc, and of which the cyclic pole is a point (ϕ) on xz at a distance φ from x defined by the equation

$$\tan\phi=\frac{\tan^2 c}{\tan C}.$$

If R is the radius of a circle parallel to $Y\Omega_2$, p the spherical perpendicular let fall from the cyclic pole on any tangent arc to the corresponding conic, ρ the angle contained between the tangent arc and the cyclic arc, we shall have the equation

$$\frac{\sin p}{\sin \rho}=\frac{\sin\phi}{\cos C}\times\frac{\tan R}{\tan c}.$$

The two arcs $y\theta$ may be termed the director arcs, and the two points (ϕ) the cyclic poles, of the figure s. It is evident that we shall have the relation $\tan\theta\tan\Phi=\tan\phi\tan\Theta=1$, or $\Phi+\theta=\frac{1}{2}\pi=\phi+\Theta$.

(34.) *Circles changed into Circles.*

To determine the small circles of the sphere S which are transformed into small circles of the sphere s, we make use of the principle that a small circle of a sphere is a spherical conic having double contact with the imaginary asymptotic circle; the chord (or arc) of contact being the parallel great circle. Hence, the circles required are the spherical conics which have double contact with both P and Q. Of these circles there are three series corresponding to the three pairs of chords of intersection of P and Q. For the chords of contact of any one of the circles with P and Q are a pair of harmonic conjugates of one of the pairs of chords of intersection of P and Q; and, conversely, any such pair of harmonic conjugates may be taken for the chords of contact of a circle with P and Q, or again with Q and P. But the circles of only one of these series are real; their chords of contact being harmonic conjugates of the cyclic arcs, and their centres being on the great circle of the foci. Let R be the radius of one of these circles, Φ the distance of its centre (Φ) from X. The harmonic conjugate of the great circle, of which (Φ) is the spherical pole, with regard to the cyclic arcs, must have the same pole with regard to the imaginary conic Q and with regard to the circle. This condition supplies the equation

$$\tan^2 R=-\frac{\sin(\Phi-C)\sin(\Phi+C)}{\sin(\Phi-c)\sin(\Phi+c)},$$

which determines the radius of the circle when the position of its centre is given; and shows that the circle is real only when Φ is intermediate between C and c.

If r be the radius of the corresponding circle, and ϕ' the distance of its centre from x, we shall have the equations

$$\frac{\tan R}{\tan r} = \frac{\sin 2C}{\sin 2c},$$

$$\tan \Phi \tan \phi' = \tan C \tan c.$$

The corresponding formulæ for the circles of the imaginary systems whose centres lie on XY, xy are

$$\tan^2 R = -\frac{\tan^2 C + \sec^2 C \tan^2 \Phi}{\tan^2 c + \sec^2 c \tan^2 \Phi},$$

$$\tan \Phi \tan \phi' = -\sin C \sin c,$$

$$\frac{\tan^2 R}{\tan^2 r} = \frac{\cos^2 c}{\cos^2 C} \cdot \frac{\tan^2 C}{\tan^2 c},$$

where Φ and ϕ' are the distances of the centres of the corresponding circles from X and x. Changing in these formulæ c and C into their complements, we have the formulæ for the corresponding imaginary circles of which the centres lie on ZY, zy.

(35.) *Theorems relating to Curvature.*

If two curves on either sphere touch one another at any point, the ratio of the tangents of their spherical radii of curvature remains unchanged in the transformation. This is evident from the corresponding theorem relating to plane homographical figures, because the two planes touching the spheres at two corresponding points are homographic. Thus the ellipses and hyperbolas on either sphere are lines of greatest or least alteration of curvature, as well as lines of greatest or least elongation or contraction. The circles which are transformed into circles are of course loci of points at which the tangent of the radius of curvature is altered in a constant ratio.

(36.) *Connexion with the Plane Theory.*

If we suppose the arcs C and c to become infinitely small, retaining a finite ratio to one another, the parts of the two spherical figures which lie infinitely near to X and x will ultimately become two plane similar figures. But we can also regard two dissimilar homographic plane figures as a limiting case of two homographic figures upon a sphere. The points of the two hemispheres, which we have hitherto considered, correspond to one another throughout the whole of each surface with similar directions of rotation. But if, in the hemisphere S, we substitute for the quadrant containing F_1 the opposite quadrant, so as to consider the hemisphere of which Z is the spheric centre, and the great circle XY the base, we shall obtain a figure of which one quadrant (F_2) answers with similar rotation to the corresponding quadrant (f_2), and the other quadrant (F_1) answers with dissimilar rotation to the corresponding quadrant (f_1). If, for example, in the formulæ of Arts. 26—28, we change C, D, $\frac{1}{2}D_1$, or $\frac{1}{2}D_2$, H_1, H_2, Δ, $\frac{1}{2}R_1$, or $\frac{1}{2}R_2$, into their complements, we shall have the equations which express the metrical relations of the two figures, considered in this particular manner. In these new formulæ, $2c$ and $2C$ are the angles contained between the foci, and between the cyclic planes of s; or, again, they are the angles contained between the cyclic planes and the foci of S. The new arcs D, H_1, H_2, are not measured from the points corresponding to the original points of arcs d, h_1, h_2, but from points distant by a quadrant from the points corresponding to those original points; they are also measured backward—*i.e.*, in the direction opposite to that which corresponds to the direction in which the arcs d, h_1, h_2 are measured. And a correlative statement is true for the angles Δ, H_1, H_2. It will be observed that R_1 or R_2 is changed into its supplement according as the points considered lie in the regions of similar or dissimilar rotation. Again, it is immaterial whether we change D_2 or D_1 into its supplement; in the former case, we consider (in the figure S) the triangle D_1YD_2, in the latter the triangle $D_1\bar{Y}D_2$.

If we now suppose the arcs c and C to become evanescent, the parts of the two figures adjacent to x and Z respectively will become two dissimilar homographic plane figures, and we may pass from the spherical formulæ to the corresponding formulæ of the plane theory.

(37.) *Point-Figures in Perspective.*

When two homographic point-figures are in a perspective position, (*i.e.*, when the corresponding planes and lines of the two figures intersect upon the same plane,) one of the focal lines of each pencil is, evidently, the line joining the centres S and s of the two pencils. To find the other focal lines, let Ss meet the plane of intersection in O_1, let O be the point harmonically conjugate to O_1, with regard to Ss, and O_2 the orthogonal projection of O on the plane of intersection; SO_2, and sO_2, are the focal lines required.

To place two given homographic point-figures in a perspective position, we first of all place a pair of corresponding focal lines in the same straight line, the vertices of the two pencils not coinciding, but corresponding vectorial planes coinciding. Let O_2 be the point of intersection of the two remaining focal lines; let V_1 and V_2 be the planes which bisect the angle SO_2s externally and internally. According as the corresponding directions of rotation round SO_2 and so_2 are similar or dissimilar, V_1 or V_2 is the plane of intersection of the two homographic figures. It is evident that the two figures continue in perspective if their centres be moved nearer to or further from one another in the coincident focal lines; or, again, if either of them be rotated through an angle of 180° round these coincident lines. Of the cyclic planes, one pair are parallel to the plane of intersection, the other pair intersect in that plane, and in the plane bisecting S*s* at right-angles.

(38.) *Case when the Homography is Spheroidal.*

The theory of the particular case in which the transformation is *spheroidal—i.e.*, in which the imaginary cones Q and p, corresponding to the evanescent sphere-cones q and P, are cones of revolution—presents no difficulty whatever. If X and x are the centres of the imaginary small circles Q and p, the *azimuths* of any two corresponding points A and a are equal, and their *zenith-distances* are connected by the relation $\frac{\tan XA}{\tan xa} = \text{constant}.$

This constant ratio we may term the modulus of the transformation.

C.—Focal Properties of Two Homographic Spaces.

(39.) *The Imaginary Conics, and the Parameters.*

We proceed, in the last place, to consider two spaces S and s, homographically related to one another. Let Ω and σ be the imaginary circles at an infinite distance in which all spheres in the two spaces intersect one another; ω and Σ the imaginary conics corresponding to them. As we shall suppose that the planes at an infinite distance in the two spaces are not corresponding planes, the imaginary conics Ω and Σ, ω and σ, are certainly different. If either ω or Σ is an imaginary circle, the other is so too; for if Ω and Σ have a common chord, ω and σ must also have a common chord, and *vice versâ*. We shall, however, for the present, exclude this important particular case, and shall suppose that neither ω nor Σ is an imaginary circle. Let O, o' be the centres of Σ and ω respectively (these conics have no real tangents, and therefore are not parabolas); X, Y, x', y' the points at an infinite distance on their principal axes; Z, z' the points at an infinite distance on the normals to their planes; Σ_1, Σ_2, ω_1, ω_2 the asymptotic points of Σ, ω, lying on the lines XY, $x'y'$, which are the lines at an infinite distance in the planes of the two conics, and which we shall suppose to meet the imaginary circles Ω and σ in the points $\Omega_1\Omega_2$ and $\sigma_1\sigma_2$. The lines XY, $x'y'$ are evidently corresponding lines; and because the poles of XY, with regard to Ω and Σ, correspond to the poles of $x'y'$ with regard to ω and σ, the points Z, o' and the points O, z' are corresponding points. The anharmonic equation $[\Sigma_1, \Sigma_2, \Omega_1, \Omega_2] = [\sigma_1, \sigma_2, \omega_1, \omega_2]$, which is implied by the homographic relation of the figures, may also be written $[\Omega_1, \Omega_2, \Sigma_1, \Sigma_2] = [\sigma_1, \sigma_2, \omega_1, \omega_2]$, and expresses, in this form, that the imaginary angles $\Sigma_1' O \Sigma_2$, $\omega_1 \sigma' \omega_2$ are superposable; *i. e.*, that the imaginary conics Σ and ω are similar. Again, because X, Y are harmonic conjugates of $\Omega_1\Omega_2$ and $\Sigma_1\Sigma_2$, while x, y are harmonic conjugates of $\omega_1\omega_2$ and $\sigma_1\sigma_2$, x, y correspond to X, Y; and we may suppose the correspondence fixed by the equation

$$[\Omega_1, \Omega_2, \Sigma_1, \Sigma_2, X, Y] = [\omega_1, \omega_2, \sigma_1, \sigma_2, x, y].$$

This equation implies one or other of the equations

$$[\Omega_1, \Omega_2, \Sigma_1, \Sigma_2, X, Y] = [\sigma_1, \sigma_2, \omega_1, \omega_2, x, y],$$

or
$$[\Omega_1, \Omega_2, \Sigma_1, \Sigma_2, X, Y] = [\sigma_1, \sigma_2, \omega_1, \omega_2, y, x].$$

Of these, the former is inadmissible, as it would imply that $[\sigma_1, \omega_1, x, y] = [\omega_1, \sigma_1, x, y]$, which is impossible, since ω_2, and not σ_1, is the harmonic conjugate of ω_1 with regard to xy. We infer, therefore, that the point at infinity on the major axis of Σ corresponds to the point at infinity on the minor axis of ω, and *vice versâ*. Let $A\sqrt{(-1)}$, $B\sqrt{(-1)}$, $a\sqrt{(-1)}$, $b\sqrt{(-1)}$ be the principal semi-axes of Σ and ω; A, B are the parameters of S, and a, b of s; they are connected by the equation $Aa = Bb$, which results from the similarity of Σ and ω.

(40.) *The Correspondence of Directions—the Principal Axes.*

From the homographic relation of the two figures, it follows that to each direction on any straight line in either figure there corresponds a definite direction on the corresponding line. And again, to each direction of rotation round any line there corresponds a definite direction of rotation round the corresponding line. It is easily shown (by considering in each figure two infinitesimally near positions of a straight line in relation to a line at a finite distance) that the two figures are either similar in respect of all rotations, or dissimilar in respect of all rotations; *i.e.*, that corresponding rotations round corresponding directions are either always similar, or else always dissimilar. For clearness we may suppose that corresponding rotations in the two figures are similar. We shall call the lines OX, OY, OZ, $o'x$, $o'y$, $o'z$ the principal axes of the two figures; and the planes OYZ, OXZ, OXY, $o'yz$, $o'xz$, $o'xy$ the principal planes; the axes OZ, $o'z$, which alone are corresponding lines, we shall call the focal axes; and the planes of XY, xy the vanishing planes.

Each space is divided by its three principal planes into eight octants, corresponding respectively to the eight octants of the other space. Considering these octants as tetrahedra, of which the plane at an infinite distance is one boundary, and observing that in either space the plane at an infinite distance corresponds to the vanishing plane of the other space, we find that to adjacent octants on the same side of the vanishing plane in either space there correspond in the other space adjacent octants on the same side of the vanishing plane, but that adjacent octants on opposite sides of the vanishing plane in either space correspond to octants diametrically opposite in the other space; so that if the correspondence of two octants is given, that of the remaining octants is immediately ascertained. Again, if P, Q are any two points on the same side of the vanishing plane of S, and if p, q are the points corresponding to P, Q, the directions PQ, pq are corresponding directions in the two spaces; and similarly the corresponding directions of rotation round any two corresponding lines may be ascertained. We may add that if V is any closed figure in S, which lies wholly on one side of the vanishing plane, points in the interior of V will correspond to points in the interior of a closed figure v corresponding in the space s to V.

(41.) *Determination of the Principal Axes and Parameters.*

The geometrical construction for the determination of the principal axes in each figure, and of the parameters A, B, a, b, is as follows. We first obtain the vanishing plane of each figure; *i. e.*, we determine in each figure three points corresponding to three points at an infinite distance in the other figure; the points at an infinite distance in the directions normal to the vanishing planes are the points Z and z', and the points corresponding to these are the centres o' and O of the imaginary conics ω and Σ; thus the focal axes OZ, and $o'z'$ are known. At the point O, in the vanishing plane of S, take two pairs of lines corresponding to two pairs of rectangular lines intersecting at z' in the plane at an infinite distance in s. The axes OX and OY are the pair of lines at right angles to one another in the involution determined by the two pairs so constructed; x and y, which determine $o'x$ and $o'y$, are the points corresponding to X and Y. Lastly, to find the parameters, we observe that if in any two corresponding planes the chords intercepted by Σ and ω are $2D\sqrt{(-1)}$, and $2d\sqrt{(-1)}$ respectively, the parameters of the two homographic plane figures are D and d; their homographic centres are the points of bisection of the chords, and their focal axes are the perpendiculars to the chords at their points of bisection. Hence we obtain the four parameters A, B, a, b by constructing the homographic foci of the principal planes XZ, YZ, xz, yz.

(42.) *The Confocal Quadrics.*

The imaginary conic Σ, in which we may suppose $A > B$, determines a system of confocal quadrics, of which it is the imaginary focal conic. The two real focal conics are an ellipse in the plane of YZ, of which the foci (in the axis of Z) are the homographic foci of the plane YZ, and of which the vertices, in the same axis, are the homographic foci of the plane XZ. The focal hyperbola lies in the plane of XZ, and has of course the vertices of the ellipse for foci, and its foci for vertices. The system of confocal quadrics of which ω is the imaginary focal conic, correspond homographically to the confocal quadrics of the system S. For since the conics σ and ω correspond to the conics Σ and Ω, the imaginary developable circumscribing the two former conics corresponds to the imaginary developable circumscribing the two latter conics, and therefore the quadrics inscribed in these corresponding developables are themselves corresponding surfaces. In particular, to the focal ellipse of S there corresponds the focal hyperbola of s, and *vice versâ*; the extremities of the focal axes of the ellipses being transformed into the extremities of the focal axes of the hyperbolas, and the extremities of the minor axes of the ellipses into the asymptotic points of the hyperbolas. Again, the ellipsoids of either confocal system are changed into the hyperboloids of two sheets of the other system; and the hyperboloids of one sheet into the hyperboloids of one sheet. And by considering the two pairs of homographic planes XZ, xz, YZ, yz, we see immediately that the eccentricities of the sections of corresponding confocals made by corresponding principal planes are reciprocal, and that the rectangle of their major semi-axes is equal to the rectangle of the parameters $A \times a$ or $B \times b$. Again, to the normals of any confocal there correspond the normals of the corresponding confocal; the lines of curvature of the two surfaces, their umbilics, the two systems of orthogonal developables formed by the normals of each of them, their centres of curvature, and the surfaces which are the loci of those centres, all correspond homographically; the cuspidal lines of the normal developables are corresponding geodesic lines upon the surfaces of centres, and the lines of contact of two corresponding developables with those sheets of the surfaces of centres upon which their cuspidal lines do not lie, are in like manner corresponding lines. Further, since the normals of corresponding confocals are corresponding lines, the geodesics of either surface correspond to the geodesics of the other; and the confocals enveloped by the developables of two corresponding geodesics are corresponding confocals, and the lines of contact are corresponding lines. To the various modes of description of the lines of curvature of either system of confocals by means of a thread stretched upon surfaces of the system, there will correspond similar modes of description of the lines of curvature of the other system of confocals. For an example, we may take the general theorem of M. Chasles,

"If an inextensible thread, of which the extremities are fastened to two fixed points upon one of two confocal surfaces of different kinds, is strained by the point of a pencil which moves upon the second surface, so that the thread consists (in general) of six portions, two of which are geodesics of the first surface, two are geodesics of the second surface, while the other two are the portions of common tangents to the two surfaces included between the points of contact, the point of the pencil will describe a line of curvature of the second surface."

While an inextensible thread moves in either space in the manner described in this enunciation, an inextensible thread will move in the same manner in the other space; and the six portions of the first thread will correspond homographically to the six portions of the second. But it is to be observed that the constant lengths of the two threads will be related transcendentally to one another; as also will the lengths of the corresponding curvilinear portions of the two threads.

We may add that to two geodesic arcs of which the difference is rectifiable, there will correspond two geodesic arcs of which the difference is rectifiable. And when the difference of two arcs of a line of curvature can be expressed by geodesic lines in either figure, the corresponding difference can be similarly expressed in the other figure.

(43.) *The Point-Figures at Corresponding Points—their Focal Lines.*

We shall next consider any two corresponding points P and p in the two spaces. At these two points we have two homographic point-figures, of which the relations to one another are readily ascertained. To the cones which from P envelope the conics Ω and Σ, there will correspond the cones which from p envelope ω and σ. Thus the principal axes of the point-figures at P and p are the normals to the surfaces of the confocal system which pass through P and p; and the focal lines of the figures are the generators of the hyperboloids of a single sheet which pass through P and p. We thus have the theorem :

"Any two corresponding generators of two hyperboloids of the two confocal systems are the axes of pencils of planes of which the correspondence is equiangular."

If the points P and p be taken on corresponding focal conics, the two generators coincide. Thus, "the focal conics are the loci of points at which the correspondence of the homographic point-figures is spheroidal."

It is evident that, given in one of the two spaces a point and three generators (of the same or different hyperboloids), and the corresponding things in the other space, we can immediately, by means of the equiangular pencils of planes, determine the point p in either space which corresponds to a given point P in the other. We might take for the three generators in each space any three tangents to a focal conic; the simplest construction being perhaps that in which the tangents at the vertices of the focal conics are employed as the axes of equiangular pencils.

We thus obtain the following rule, which is well adapted to the methods of descriptive geometry:—"Project the given point P orthogonally on the planes of XZ, YZ, and using the focal radii vectores of the projections, as in Art. 5, determine the points corresponding to them in the planes of xz and yz: these points are the orthogonal projections of the point p."

(44.) *The Strain Ellipsoid—its Cyclic Planes and Focal Asymptotes.*

The position of the cyclic planes of the homographic figure at P may be ascertained by means of the focal lines of the figure at p. But these cyclic planes are also the cyclic planes of the "strain ellipsoid" at P; *i.e.*, of the evanescent ellipsoid which has its centre at P, and corresponds to an evanescent sphere having its centre at p. For this evanescent ellipsoid has for its asymptotic cone the imaginary cone which from P envelopes Σ, and is thus concentric, similar, and similarly situated with the auxiliary ellipsoid of M. Chasles, *i.e.*, with the ellipsoid of which the principal axes are equal to the axes major of the three confocal surfaces passing through P, and are normal to those three surfaces respectively (Aperçu historique des Méthodes en Géométrie, Note 25). It appears at the same time that the asymptotes of the focal conic of the auxiliary ellipsoid, or of the strain ellipsoid, coincide with the focal lines of the point P.

The cyclic planes at P and p are the "planes of no distortion" at those corresponding points; *i.e.* (1) evanescent lines passing through P and lying in either cyclic plane, are altered in a constant ratio; (2) angles in a cyclic plane at P are transformed into equal angles in the corresponding cyclic plane; so that P, p are homographic foci of either pair of cyclic planes. The second property is analogous to the property that the focal lines are the axes of equal homographic pencils of planes. If we observe that the focal asymptotes of a quadric are the axes of its circumscribing right cylinders, we may enunciate a property of the focal lines analogous to the first property of the cyclic planes:—

"Planes parallel to either focal line, and infinitely near to P, are transformed into planes, which may ultimately be regarded as parallel to the corresponding focal line, and of which the distances from p are in a constant ratio to the distances of the first planes from P."

We may express this by saying that a generating line of a confocal hyperboloid is, at any point of it, a line of equal transverse elongation. And since the right cylinder, of which the focal line at P is the axis, and which circumscribes the strain ellipsoid at P, is transformed into a right cone of which the vertex lies on the vanishing plane of s, we see that if the point P vary its position on a given hyperbolic generator, the ratio of transverse elongation varies inversely as the distance of p from the vanishing plane of s, or directly as the distance of P from the vanishing plane of S.

(45.) *The Canonical and Elliptic Equations.*

If we represent by X, Y, Z, x, y, z, the coordinates of corresponding points in the two spaces referred to their principal axes, the canonical equations of the homography will be

$$Zz = Aa = Bb$$

$$\left.\begin{matrix} Xz = Ax \\ Yz = By \end{matrix}\right] \text{ or } \left.\begin{matrix} xZ = aX \\ yZ = bY \end{matrix}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (A).$$

If, again, we denote the elliptic coordinates of corresponding points in either space (referred to the corresponding confocal systems) by $\Lambda_1, \Lambda_2, \Lambda_3, \lambda_1, \lambda_2, \lambda_3$, the homographic equations are

$$\Lambda_1\lambda_1 = \Lambda_2\lambda_2 = \Lambda_3\lambda_3 = Aa = Bb \ldots\ldots\ldots\ldots\ldots\ldots (B);$$

so that every general homographic transformation may be represented as a transformation of the elliptic coordinates of a point into their reciprocals.

(46.) *Determination of the Strain Ellipsoid.*

Either of these sets of formulæ will serve to determine the ratios of the axes of the strain ellipsoid at P to the radius of the evanescent sphere at p. The rectangular formulæ show that the ratio of an evanescent volume at P to the corresponding volume at p is that of Z^4 to $Aa \times ab$; whence, if $\theta\Lambda_1$, $\theta\Lambda_2$, $\theta\Lambda_3$ are the semi-axes of the strain ellipsoid at P, and i the radius of the evanescent sphere at p,

$$\theta^3 \frac{\Lambda_1\Lambda_2\Lambda_3}{i^3} = \frac{Z^4}{Aa \times ab},$$

or, since $\Lambda_1\Lambda_2\Lambda_3 = ABZ$, and $Zz = Aa$, $\theta = \frac{i}{z}$.

Or again, transforming by the equations (B) the elliptic formula

$$d\Sigma_1 = \sqrt{\left[\frac{(\Lambda_1^2 - \Lambda_2^2)(\Lambda_1^2 - \Lambda_3^2)}{(\Lambda_1^2 - A^2)(\Lambda_1^2 - B^2)}\right]}\, d\Lambda_1,$$

we find
$$d\sigma_1 = \frac{Aa \times AB}{\Lambda_1\Lambda_2\Lambda_3} \frac{d\Sigma_1}{\Lambda_1},$$

or
$$\frac{d\Sigma_1}{\Lambda_1} = \frac{d\sigma_1}{z},$$

which agrees with the preceding determination of θ, the symbols $d\Sigma_1$ and $d\sigma_1$ representing corresponding elementary arcs, normal to (Λ_1) and (λ_1).

Our limits prevent us from applying these formulæ to the determination of the loci corresponding to those considered in Arts. 17—19. For the same reason, we omit the elementary theorems relating to the curvature and torsion of curve lines, and the curvature of curve surfaces.

(47.) *The Parameters of the Confocal and Concyclic Cones, at any Point.*

The equation of the imaginary cone which from the point P envelopes Σ is

$$\frac{X^2}{\Lambda_1^2} + \frac{Y^2}{\Lambda_2^2} + \frac{Z^2}{\Lambda_3^2} = 0,$$

and the cone which from the same point envelopes the confocal surface, of which Ψ is the semi-axis major, is

$$\frac{X^2}{\Lambda_1^2 - \Psi^2} + \frac{Y^2}{\Lambda_2^2 - \Psi^2} + \frac{Z^2}{\Lambda_3^2 - \Psi^2} = 0;$$

so that the coefficients (designated by A, B, Γ, α, β, γ in Art. 30), which determine the homography of the point-figures at P and p, are, in fact, the elliptic coordinates of those points; and the parameters of the confocal cones of the point-figures are the same as the parameters of the confocal quadrics which they envelope. Thus, the formulæ of Arts. 30, 31 are immediately applicable to the figures at P and p. And, if Φ is the parameter of a concyclic cone at P, so that $\Phi = \frac{\Lambda_1\Lambda_2\Lambda_3}{\Psi_1\Psi_2}$, where Ψ_1, Ψ_2 are the parameters of two confocal quadrics touching any line of (Φ), the elongation at P in the direction of any line of (Φ) is given by any one of the formulæ

$$\frac{T}{r} = \frac{\Lambda_1\Lambda_2\Lambda_3}{\mathrm{A}a \times \mathrm{AB}}\Phi = \frac{\Lambda_1^2\Lambda_2^2\Lambda_3^2}{\mathrm{A}a \times \mathrm{AB}} \times \frac{1}{\Psi_1\Psi_2} = \frac{Z\Phi}{\mathrm{A}a} = \frac{B}{a} \times \frac{Z^2}{\Psi_1\Psi_2} = \frac{\Phi}{z},$$

T and r representing corresponding elements at P and p. It will be observed that, at equal distances from the vanishing plane, the elongation is the same on all lines touching the same two confocal quadrics.

(48.) *Lines Tangent to two Confocal Quadrics.*

Let L_1 and L_2 be two straight lines in the space S, each of which touches the two confocals (Ψ_1) and (Ψ_2); let also l_1, l_2, (ψ_1), (ψ_2) be the corresponding lines and confocal quadrics in the space s. The tangent planes $L_1\Psi_1$, $L_1\Psi_2$ (*i. e.*, the tangent planes to (Ψ_1), (Ψ_2), at their points of contact with L_1) are at right-angles to one another, and are transformed into two planes which are at right-angles to one another. Again, the pair of planes, tangent to any third confocal surface (Ψ_3), which intersect in L_1, make the same angles with the bisecting planes $L_1\Psi_1$, $L_1\Psi_2$ that the pair of planes, tangent to the same confocal surface (Ψ_3), and intersecting in L_2, make with the bisecting planes $L_2\Psi_1$, $L_2\Psi_2$. For the involutions of pairs of planes determined by the confocal system at the lines L_1 and L_2 are necessarily equiangular in respect of all their corresponding pairs, because they are equiangular in respect of the coincident pairs of planes determined by (Ψ_1) and (Ψ_2), and of the imaginary pair of cyclic planes determined by the imaginary circle at an infinite distance. From this theorem (of which M. Chasles has given a different demonstration; see Liouville, Vol. XI., First Series, p. 109) we infer that equal dihedral angles, similarly placed in the pencils of planes at L_1 and L_2, are transformed into dihedral angles equal to one another, and placed similarly to one another, in the pencils of planes at l_1 and l_2. Or again, if I and i are the angles made with $L_1\Psi_1$ and $l_1\psi_1$ by corresponding planes passing through L_1 and l_1, and if we denote the major semi-axes of the surfaces (Ψ_1), (Ψ_2), (ψ_1), (ψ_2) by Ψ_1, Ψ_2, ψ_1, ψ_2, we shall have the equation

$$\frac{\tan \mathrm{I}}{\tan i} = \frac{\Psi_1}{\Psi_2} = \frac{\psi_2}{\psi_1},$$

which results immediately from a formula given by M. Chasles (*loc. cit.* p. 106), combined with the equations of transformation (B); and which shows, in conformity with our theorem, that the ratio of tan I to tan i is the same, whatever common tangent of (Ψ_1) and (Ψ_2) we consider. We have, in fact, the still more general theorem:

"All pencils of planes, of which the axes are touched by two confocals having their major semi-axes in a given ratio, have that ratio for their modulus of transformation; and in all such pencils, the involutions which are transformed into equiangular involutions, are equiangular with one another," which is an immediate consequence from Art. 47, and 30, ϵ.

Since a generating line of a confocal hyperboloid may be regarded as a line of which the two tangent confocals coincide, this enunciation includes, as a particular case, the equiangular property of the generating lines.

We have seen that the focal conics are the loci of points at which the transformation is spheroidal. We may now add, that at any one of these points the modulus of transformation (Art. 38) is $\frac{Q}{P}$, if Q is the semi-axis major (A or B) of the focal conic on which the point is taken, and P is the semi-axis major of the confocal quadric which passes through the point.

(49.) *Ivory's Theorem.*

If on two confocal surfaces of the same kind in the space S we consider two points which *correspond* to one another in the sense in which that term is employed in Ivory's theorem, these two points will be transformed into two others in the space s, which will also *correspond* to one another in the same sense. This principle, which is immediately verified by means of the equations (A), may serve to transform some geometrical, and even physical, propositions. For example, we see that to every focal generation of a quadric according to Jacobi's method, there corresponds homographically a similar focal generation of another quadric.

(50.) *Equi-Segmental Axes and Planes.*

The equi-segmental axes of all planes in the space S, which cut the vanishing plane in straight lines parallel to a given straight line, (or, which is the same thing, of all planes which pass through a given point at an infinite distance on the vanishing plane,) lie on two planes at equal distances from the vanishing plane and parallel to it.

For, in the first place, parallel planes in the space S have their equi-segmental axes at one and the same distance from the vanishing plane, since to parallel planes in the space S there correspond planes in the space *s*, which intersect the vanishing plane of that space in the same straight line, and of which the foci are consequently at a constant distance from one another; this constant distance being equal to the distance of the equi-segmental axes of the planes in the space S. Again, planes in the space S, which intersect the vanishing plane in the same straight line, have their equal axes situated at equal distances from the central plane. For to these planes correspond parallel planes in the space *s*; and, by what has just been proved, the equi-segmental axes of these planes lie in two planes parallel to the vanishing plane; therefore the equi-segmental axes of the planes in S lie in two corresponding planes, *i. e.* in two planes parallel to the vanishing plane. The theorem itself results from the combination of these two particular cases of it.

It may be worth while to verify the theorem analytically. If

$$\frac{X}{p}+\frac{Y}{q}+\frac{Z}{r}=1$$

is the equation of any plane of S, the equation of the corresponding plane of *s* is

$$\frac{Ax}{pz}+\frac{By}{qz}+\frac{Aa}{rz}=1,$$

which meets the vanishing plane of *s* in the line

$$z=0,\quad \frac{Ax}{p}+\frac{By}{q}=-\frac{Aa}{r},$$

or

$$z=0,\quad \frac{x}{a\frac{p}{r}}+\frac{y}{b\frac{q}{r}}=-1.$$

The square of the semichord determined on this line by the imaginary conic ω, or

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+1=0,$$

is
$$-\frac{(a^2p^2+b^2q^2)(q^2r^2+p^2r^2+p^2q^2)}{r^2(p^2+q^2)^2};$$
and this square, multiplied by the square of the sine of the angle between the given plane and the vanishing plane of S, becomes
$$-\frac{a^2p^2+b^2q^2}{p^2+q^2},$$
an expression of which the value depends only on the ratio of p to q.

It follows from this theorem, that to ascertain the position in the space S of the equi-segmental axes of any plane whatever, it will suffice to attend to the principal equi-segmental axes, *i.e.* to the equi-segmental axes of planes which pass through the focal axis. Let P be such a plane, and let $D\sqrt{(-1)}$ be the semi-diameter of the imaginary focal conic lying in that plane, $d\sqrt{(-1)}$ the semi-diameter of the imaginary focal of s, determined by the corresponding plane. It will be found that $Dd=Aa$; so that we have for the semi-distance d of the equi-segmental axes of the plane P, the expression $d=\frac{Aa}{D}$. Thus all the planes, loci of real equi-segmental lines, or, as we shall term them, all the equi-segmental planes of S, are comprised between two planes, at distances a and b on the positive side of the vanishing plane, and between two planes symmetrically situated on the negative side of the same plane. Again, since the semi-diameters of S, which are equally inclined to its principal axes, are equal to one another, each equi-segmental plane contains two distinct series of equi-segmental parallels, the two series being equally inclined to the plane of ZX, or ZY; in the two extreme pairs of equi-segmental planes these two series coincide with one another, and their common direction is that of one of the principal axes OX or OY.

If we consider two planes intersecting in the axis of Z, and inclined at an angle I to the plane of ZX, we have for the square of the semi-distance of their equi-segmental lines the expression,
$$d^2=a^2\cos^2 I+b^2\sin^2 I.$$
The corresponding inclination i is given by the equation
$$\tan I=\frac{B}{A}\tan i=\frac{a}{b}\tan i,$$
and the corresponding value of D^2 is
$$D^2=A^2\cos^2 i+B^2\sin^2 i.$$
These equations show that if we imagine the spaces S and s so placed that their focal axes coincide, while the axes of X and Y lie in the axes of y and x respectively, the principal equi-segmental axes of either space will be those generating lines of hyperboloids of the other space, which lie in planes parallel to the central plane.

The homographic relation of any two corresponding equi-segmental planes is very simple. If we conceive of the points of each plane as referred to its principal equi-segmental axes, the corresponding co-ordinates of corresponding points will be equal, and only the angle between the axes will be different in each plane. These angles are never equal to one another (except in the excluded case $A=B$, $a=b$); they are, however, supplementary to one another in the *principal* equi-segmental planes, *i.e.* in the planes defined by the equations $Z = \pm\sqrt{(ab)}$, $z = \pm\sqrt{(AB)}$, since in these planes we have

$$\tan^2 I = \frac{B}{A}, \quad \tan^2 i = \frac{b}{a}, \quad \tan I \tan i = 1.$$

To obtain the straight line in s, which corresponds to any given straight line in S, we may either determine its projections on the two focal planes, by means of the equi-segmental axes in those planes; or we may, instead, consider the intersections of the given straight line with any pair of equi-segmental planes of S, and obtain the corresponding points in the corresponding equi-segmental planes of s. We have, however, in every case to measure the equal corresponding segments in corresponding directions; and these can always be ascertained by inspection, if we have first fixed the correspondence of the eight octants of each space to the eight octants of the other.

(51.) *Properties of the Hyperbolic Generators.*

The generating lines of the confocal hyperboloids possess a metrical property with regard to the equi-segmental planes, which may be very variously expressed, according to the equi-segmental planes considered. Thus:

"The intercept made in the space S upon any generator of a confocal hyperboloid by the tangent planes to that hyperboloid, which are parallel to the vanishing plane, is to the corresponding intercept in the space s in the constant ratio of $\sqrt{(AB)}$ to $\sqrt{(ab)}$."

Or, again:

"The intercept made on any hyperbolic generator of S by the two equi-segmental planes $Z = \pm\sqrt{(ab)}$, is to the corresponding intercept in the space s in the inverse ratio of the major-axes of the hyperboloids to which the two generators belong."

In connexion with this property we may mention the following; which, however, does not depend on the general homographic transformation we are considering:—

"If one of two confocal hyperboloids be transformed into the other by the transformation of Ivory, segments on any generator of the one are transformed into equal segments on the generator of the other."

(52.) *Homographic Spaces placed Symmetrically.*

It is in general impossible to place two homographic spaces S and s in the same space, so that any given point of that space shall have the same corresponding point, to whichever of the two spaces it is considered to belong. The conditions that this reciprocal relation of the two spaces should be possible are, that corresponding rotations in the two figures should be similar, and that

$$(C) \ldots\ldots \mathrm{A} = b, \quad \mathrm{B} = a,$$

either of these equations, of course, implying the other. For, if these equations be satisfied, and if corresponding rotations be similar, we may place the axes of OZ, OX, OY upon the axes of oz, $\pm oy$, $\pm ox$, inasmuch as the positive directions of OZ and oz are not corresponding directions. Writing, as we may then do, $\pm$ X for Y and $\pm$ Y for X in the equations (A), we find

$$z\mathrm{Z} = ab, \quad z\mathrm{X} = \pm\, by, \quad z\mathrm{Y} = \pm\, ax,$$

and these equations are not altered by interchanging simultaneously X, x; Y, y; Z, z. The points which coincide with their conjugates are the points of the lines

$$z = +\sqrt{(ab)}, \quad y = \pm\sqrt{\left(\frac{a}{b}\right)}\, x,$$

$$z = -\sqrt{(ab)}, \quad y = \mp\sqrt{\left(\frac{a}{b}\right)}\, x;$$

i.e., the principal equi-segmental planes of S coincide with their corresponding planes, and in each of these planes the points of one of the principal equi-segmental axes coincide with their corresponding points. Every plane which passes through either of these lines, corresponds to itself, and so does every line which meets both of them. Again, we may also place the axes of OZ, OX, OY upon the axes of $-oz$, $\pm oy$, $\mp ox$; in this case, the equations (A) become

$$z\mathrm{Z} = -ab, \quad z\mathrm{X} = \mp by, \quad z\mathrm{Y} = \pm ax,$$

which are still symmetrical, but which give imaginary loci of coincident points. Either the upper signs, or else the lower signs, may be taken in each case; so that the two spaces admit of four different symmetrical positions.

We may arrive at the preceding results without using the equations (A); for it is readily seen that the necessary and sufficient conditions for the reciprocity of the two homographic systems are that the imaginary conics Σ and ω should coincide, and that those points on the two conics should be coincident, which correspond to the same points of the imaginary circle at an infinite distance. The equations (C) are the conditions that the two conics should be equal in all respects; if these equations are satisfied, the two conics can be brought into coincidence in four different ways, and in each of these four ways the points which ought to coincide will coincide, if corresponding rotations in the two spaces are similar.

(53.) *Case of a Spheroidal Homography.*

It is hardly necessary to do more than mention the case of a spheroidal homography, in which $A=B$, $a=b$. All meridian planes of the space S have the same foci at a distance $\pm A$ from the equatorial (or vanishing plane), and their equi-segmental axes lie in the same two parallel planes at a distance $\pm a$ from the equatorial plane.

The angle contained by any two meridian planes is unchanged in the transformation; and the homographic relation is the same for all pairs of corresponding meridian planes. Thus, all angles between planes and lines intersecting at either focus remain unchanged in the transformation, and the pencils in space at corresponding foci are superposable. Similarly, each equi-segmental plane is superposable upon the plane corresponding to it. The two spaces may, in fact, be conceived as generated by the equiangular rotation of two homographic planes round their focal axes. The condition that they should be capable of occupying a reciprocal position, is that the distances between the foci in each space should be equal.

(54.) *Historical Note.*

The existence of two pairs of parallel equi-segmental axes in any two homographic plane figures was established by M. Moebius in 1827. ("Barycentrische Calcul," p. 320, sect. 230.) M. Moebius also showed that, if the corresponding points of two corresponding equi-segmental axes coincide in the line of intersection of two homographic planes, the two planes are in perspective. Magnus ("Sammlung von Aufgaben und Lehrsätzen aus der Analytischen Geometrie," Berlin, 1833, p. 41, sect. 12) proved that in two homographic plane figures there exists a pair of corresponding points at which the corresponding pencils are equiangular; and that, if the figures be placed in the same plane with these "centres of collineation" coincident, and either of them rotate in its own plane round the centre of collineation, it will become homological with the other in two diametrically opposite positions, in one of which positions one pair of equi-segmental axes will coincide, while the other pair will coincide in the other position. Magnus expressly says that "of two collinearly-related systems" [*i.e.*, two homographic plane figures in which the straight lines at an infinite distance are not corresponding lines] "each has, in general, only one centre of collineation." As Magnus tacitly supposes that the figures are not in any position whatever with regard to one another, but are already placed in the same plane, this statement is not untrue; but it is only part of the truth, and the analysis by which Magnus obtains one centre of collineation in each figure, will also supply a second pair, if we change the sign of the constant p in the equations (1) of p. 42

loc. cit. It is of course quite true that if the two figures are once placed in the same plane, there is only one point in each which can be regarded as a centre of collineation; and this, which Magnus has proved analytically, Dr. Salmon has also shown geometrically ("Higher Plane Curves," Art. 230, p. 246). But it is to be remembered that two planes can be made to coincide in two different ways according as they are placed face to face, or both facing the same way, and, in one of these positions of coincidence, one of the pairs of foci are the centres of collineation, and the other pair in the other position. It is worth while to add that though, as Dr. Salmon has observed, the position of the imaginary circular points at an infinite distance is unaffected by any motion of translation or rotation of a plane figure in its own plane, those two imaginary points are interchanged with one another if the figure be rotated, through an angle of 180°, round any axis in its own plane. And the change of the centre of collineation, which takes place when one of two homographic figures, of which the planes are coincident, is thus rotated, is a necessary consequence of the interchange of the imaginary cyclic points in the rotated figure.

In the "Traité de Géométrie Supérieure," only one pair of equi-segmental axes and one pair of foci are expressly mentioned. But the omission is only accidental, as the methods by which one pair of foci and one pair of equi-segmental axes are obtained would equally supply the other pair. The theorem, that "if two planes are in perspective, the foci are the points in which they are intersected by the perpendiculars let fall from the centre of perspective on the planes bisecting the angles contained by the two planes," is an immediate inference from a principle, first given by M. Chasles ("Aperçu de l'Histoire des Méthodes en Géométrie," note iv.), and subsequently employed by Mr. Mulcahy ("Principles of Modern Geometry," cap. VIII., art. 115).

Subsequently to the communication of this memoir to the London Mathematical Society, but (it is unnecessary to say) quite independently of it, three papers have appeared, relating in part to the same subject. (1) In the May number of the "Nouvelles Annales de Mathématiques," M. Abel Transon obtains the theorem of the two pair of foci by the application of a very general analytical method; he accurately describes the similarity and dissimilarity of the foci, and speaks of the theorem itself as "une propriété de l'homographie qui n'avait peut-être pas encore été remarquée." (2) M. Richelot, of Königsberg, in a paper dated Oct. 29, 1868, and published in the second part of the 70th volume of Crelle's Journal, has considered the analytical theory of homographic figures in space, and has been led to the consideration of their focal properties. It would seem, however, that M. Richelot supposes the tangents of the focal conics to be the only axes of equiangular pencils of planes; whereas, as we have shown, this property is possessed by every generating line of any confocal hyperboloid.

The cause of the oversight (if it is one) appears in the words : "Es muss, in der That, eine Axe im obigen Sinne [i.e., if we understand M. Richelot correctly, a line which is the axis of a pencil of planes equiangular with its corresponding pencil] die Eigenschaft besitzen, dass unter den unendlich vielen auf ihr senkrechten Ebenen eine existirt, deren entsprechende Ebene auf der der Axe entsprechenden Gerade senkrecht steht" (p. 141). This property, however, is not possessed by every axis of a pencil of planes equiangular with its corresponding pencil, but only by those which lie in one of the principal planes. M. Richelot speaks of a forthcoming work of a pupil ofhis own, M. Maegis, as intended to contain a complete analytical theory of homography in space. (3) In the November number of the Nouvelles Annales de Mathématiques, M. Housel enuntiates the theorem : "En deplaçant sans déformation deux figures homographiques dans l'espace, on peut les rendre homologiques." This theorem is not in accordance with Art. 50 of the present Paper, because in that article we have in effect shown that corresponding equi-segmental planes are never superposable except in the case of a spheroidal homography. But the analysis of M. Housel seems insufficient to establish his conclusion, since it is not shown that the values ultimately obtained of the ten unknown quantities of Art. XIII. of M. Housel's Memoir actually satisfy the twelve equations of that article. [The values of the unknown quantities are not obtained in an explicit form, and there are only ten of them, and not eleven, because p depends on X, Y, Z.] And, considered in itself, the conclusion is inadmissible; for any homological transformation of space must change the imaginary circle, in which all spheres intersect, into a circle, whereas in general that circle is changed into an imaginary ellipse by a homographic transformation. Again, the homographic relation depends on fifteen constants, the homological relation on seven, and the six constants of displacement can only reduce the fifteen constants to nine. Thus it would seem *à priori* that two conditions must be satisfied in order that two homographic spaces should be capable of a homological position. And the equation A=B (or $a=b$) of Art. 50 is equivalent to two independent relations connecting the fifteen constants of the homography, since that equation is equivalent to the two conditions that a certain conic should be a circle.

NOTE.—The focal properties of homographic point-figures might be obtained by simple considerations of perspective (see Arts. 21 and 37). We have, however, preferred to deduce these properties from their genuine source—the properties of the imaginary circle in which all spheres intersect one another at an infinite distance. In the case of homographic plane figures, we have ventured to employ both methods successively (Arts. 1—4, and Art. 5). This has been done at some risk of repetition; but it seemed desirable to exhibit this part of the theory in its most elementary and practical, as well as in its most abstract form, in the hope that some of the simpler results may be found of use in the actual practice of perspective.

APPENDIX.

In accordance with the wishes of the Council, the second volume of the Society's Proceedings closes with the present Number. An attempt has been made to recover the shorter, and in many cases very interesting, communications which have from the outset been given in the course of discussions on the main papers of the Evening Meetings. A few of these are appended below, and references are, in some cases, given to the works in which others subsequently appeared.

Taking these communications in the order in which they were made, we have "The regular Hypocycloidal Tricusp," by M. Jenkins, B.A., (read June 19th, 1865); a portion of this paper is given in the "Educational Times" for September, 1865.* "A proof of Euclid i. 47 not involving the definition of a parallelogram," communicated by A. De Morgan, F.R.A.S., (Nov. 20th, 1865). This proof, which is given in many German editions of Euclid, is now also to be met with in Cassell's Elements. "On Motion in a Circle, and its relation to Planetary Motion," communicated by Prof. Sylvester, (Dec. 18th, 1865,) has since been given in "Nugæ Mathematicæ," extracted from the "Philosophical Magazine for 1866."

At the same meeting, Prof. Cayley gave the following simple method for finding the volume of a Tetrahedron. "If a, b be the lengths of two opposite sides, λ their inclination, h the length of their shortest distance; the section by a plane perpendicular to the line h, at a distance x from the side b, is a parallelogram, angle λ and sides $\frac{x}{h}a$, $\frac{h-x}{h}b$; whence element of volume is

$$\frac{x}{h}a \cdot \frac{h-x}{h}b \sin\lambda\, dx = \frac{ab\sin\lambda}{h^2}x(h-x)\,dx;$$

whence integrating from $x = 0$, to $x = h$, the whole volume is

$$\frac{ab\sin\lambda}{h^2}\left(\frac{h^3}{2}-\frac{h^3}{3}\right) = \tfrac{1}{6}abh\sin\lambda."$$

"The Centres of Algebraical Curves and Surfaces," by S. Roberts, M.A., (read March 19th, 1866,) was subsequently printed in the "Quarterly Journal of Mathematics," Vol. IX., No. 33, p. 25.

A fuller account of Prof. Sylvester's paper, read at the same meeting,

* Mathematical Reprint, Vol. IV., p. 58.

of which an abstract is given in No. VI., appears in the "Proceedings of the Royal Society," No. 84, 1866, where it is entitled, "On the Motion of a Rigid Body moving freely about a fixed point."

At the meeting held April 16th, 1866, Prof. Cayley called attention to the theorem, that the difference between two consecutive prime numbers may exceed any given number $N-1$ whatever. For if $a, b, c \ldots k$ are the prime numbers not greater than N, then $abc \ldots k+1$, and $abc \ldots k+1+N$ may be one or both of them prime, but all the intermediate numbers are composite; that is, the difference of the two successive primes is $= N$ at least. Mr. A. J. Ellis, at the same meeting, communicated the following constructions:—"From a pair of conjugate diameters in an ellipse, to find the foci and the axes." Let CP, CD be the conjugate semi-diameters. Through P draw MPN, bisected in P, twice the length of CD, and perpendicular to it. Draw CS bisecting the angle MCN, and a mean proportional between the lengths of CM, CN. S is one focus, whence H the other, and directions of the axes are given. Draw PN, DR perpendicular to CS, then CA, CB being the semi-axes, their lengths are given by the equations

$$CA^2 = CN^2 + CR^2,\ CB^2 = PN^2 + DR^2.$$

If, by the conjugate diameter CD in an hyperbola, we mean the diameter of the conjugate hyperbola which is parallel to the tangent at the extremity P of the first diameter, and preserve the same letters, the construction is the same, with these differences; MPN must be parallel to CD (instead of being perpendicular to it), and $CA^2 = CN^2 \backsim CR^2$, $CB^2 = PN^2 \backsim DR^2$. This construction is easier and more complete than that given for the ellipse only, in Chasles' "Sections Coniques, art. 205." The demonstration depends upon Proceedings of the Royal Society, 14th June, 1866, vol. 15, p. 200, equations d, e, (where in equation d, read $o'e$ for $o'e'$), first given by Mr. Ellis, and having a much wider application. A solution of the problem, "Given a pair of conjugate diameters of an ellipse, to find any number of points on the curve," by R. Tucker, M.A., (communicated Nov. 22nd, 1866,) has since appeared in the "Mathematical Reprint," vol. VII., p. 28. "Proof of the Rectangle of Forces," by J. J. Walker, M.A., (communicated March 28th, 1867,) was given subsequently in the "Quarterly Journal of Mathematics," Vol. IX., No. 34, p. 173.

It remains only to add a few words of explanation about the two Indices. They are intended to be mutually exclusive. The first gives the names of all the authors of papers, and nearly all their communications; the second is concerned with points which turn up in the communications themselves.

A short list of the Errata which have been detected follows; doubtless others exist. The Secretaries will be glad to receive a list of any which have escaped their notice, for insertion in a future Volume.

R. T.