FORMULA FOR CALCULATING THE ELECTROMOTIVE FORCE AT ANY POINT OF A TRANSMISSION LINE FOR ALTERNATING CURRENT.

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It is a recognized fact that in long-distance transmission of electrical energy by means of alternating current, the voltage has often been found to be higher at the receiving end than at the generator end of the line. This increase in voltage may be considerable, and it is important to take it into account when making installations.

We intend to establish a formula giving the voltage at any point of an alternating current transmission line. Let us call $\rho$, $\gamma$ and $\lambda$ respectively, the resistance, capacity and self-induction per unit of length of both conductors of a line of length $l$. Let us send into it an alternating current of frequency $\beta = \frac{m}{2\pi}$ varying as a function of the time according to a sine law. Let us call $V_x$ the voltage at a point $x$ located at a distance $x$ from the generator end.

The transmission of the current is governed by the equation:

$$\frac{d^2V}{dx^2} = \gamma \lambda \frac{d^2V}{dt^2} + \gamma \rho \frac{dV}{dt}$$

Under constant load, this equation is satisfied if we make, according to Vaschy:

$$V_x = e^{a(l-x)} [A \sin m (t - x/v) + B \cos m (t - x/v)] + e^{-a(l-x)} [D \sin m (t + x/v) + F \cos m (t + x/v)]$$

$$v = \frac{1}{\sqrt{\gamma \lambda}} \sqrt{\frac{2 \lambda m}{\lambda m + \sqrt{\rho^2 + m^2 \lambda^2}}}, \quad a = \frac{1}{2} \gamma \rho v.$$
The velocity \( v \) is the speed of transmission of the electric waves along the line.

The intensity \( i_x \) at the point \( x \) being given by the equation:

\[
\rho i_x + \lambda \frac{di_x}{dt} = -\frac{dV_x}{dt},
\]

we have:

\[
(2) \quad \rho i_x + \lambda \frac{di_x}{dt} = e^{a(l-x)} \left[ (Aa - B\frac{m}{v}) \sin m \left( t - \frac{x}{v} \right) \right.
\]

\[
+ (A \frac{m}{v} + Ba) \cos m \left( t - \frac{x}{v} \right) \left. \right] + e^{-a(l-x)} \left[ (- Da + F \frac{m}{v}) \sin m \left( t + \frac{x}{v} \right) - (D \frac{m}{v} + Fa) \cos m \left( t + \frac{x}{v} \right) \right]
\]

Let us designate by the letter \( U_x \) the maximum value of the voltage \( V_x \) at the point \( x \); we deduce from equation (1):

\[
(3) \quad U_x = \sqrt{e^{2a(l-x)} \left( A^2 + B^2 \right) + e^{-2a(l-x)} \left( D^2 + F^2 \right) + 2\sqrt{(A^2 + B^2)(D^2 + F^2)} \cos 2m \left( \frac{x}{v} - u \right)}
\]

by taking:

\[
\text{Tang } 2m u = \frac{BD - AF}{AB + BF}
\]

In order to determine the quantities \( A, B, D, F \), we assume that the maximum value \( U_l \) of the voltage at the far end of the line is given, and that the line is closed on a conductor of resistance \( r \) and self-induction \( L \).

Let us make \( V_l = U_l \times \sin m t \) and \( \text{tang } m \varphi = \frac{mL}{r} \), then:

\[
\rho i_l + \lambda \frac{d i_l}{dt} = \frac{U_l}{\sqrt{r^2 + m^2L^2}} \left[ \rho \sin m \left( t - \varphi \right) + m \lambda \cos m \left( t - \varphi \right) \right]
\]

If we write that the equations (1) and (2) are satisfied at any instant of the time, for \( x = l \), we obtain the four following equations which determine \( A, B, D \) and \( F \):

\[
U_l = (A + D) \cos \frac{mL}{v} + (B - F) \sin \frac{mL}{v}
\]
\[
O = -(A - D) \sin \frac{m}{v} + (B + F) \cos \frac{ml}{v}
\]
\[\frac{U_l}{\sqrt{r^2 + m^2 l^2}} (\rho \cos m \varphi + m \lambda \sin m \varphi) =
\]
\[(A a - B \frac{m}{v} - D a + F \frac{m}{v}) \cos \frac{ml}{v} + (\frac{Am}{v} + B a + D \frac{m}{v} + F a) \sin \frac{ml}{v}
\]
\[\frac{U_l}{\sqrt{r^2 + m^2 l^2}} (m \lambda \cos m \varphi - \rho \sin m \varphi) =
\]
\[(-A a + B \frac{m}{v} - D a + F \frac{m}{v}) \sin \frac{ml}{v} + (A \frac{m}{v} + B a - D \frac{m}{v} - F a) \cos \frac{ml}{v}
\]
If we make:
\[\frac{U_l}{\sqrt{r^2 + m^2 l^2}} = I_l; \text{Tang } m \varphi = \frac{m}{\rho}; \text{Tang } m X = \frac{m \lambda}{\rho};
\]
\[(\varphi - X) = \delta
\]
As we have:
\[a^2 + \frac{m^2}{v^2} = m \gamma \sqrt{\rho^2 + m^2 \lambda^2}
\]
We can write:
\[2 A = U_l \cos \frac{ml}{v} + \sqrt{\frac{\sqrt{\rho^2 + m^2 \lambda^2}}{m \gamma}} I_l \cos m \left(\frac{l}{v} - \varphi - \delta\right)
\]
\[2 B = U_l \sin \frac{ml}{v} + \sqrt{\frac{\sqrt{\rho^2 + m^2 \lambda^2}}{m \gamma}} I_l \sin m \left(\frac{l}{v} - \varphi - \delta\right)
\]
\[2 D = U_l \cos \frac{ml}{v} - \sqrt{\frac{\sqrt{\rho^2 + m^2 \lambda^2}}{m \gamma}} I_l \cos m \left(\frac{l}{v} + \varphi + \delta\right)
\]
\[2 F = -U_l \sin \frac{ml}{v} + \sqrt{\frac{\sqrt{\rho^2 + m^2 \lambda^2}}{m \gamma}} I_l \sin m \left(\frac{l}{v} + \varphi + \delta\right)
\]
Equation (3) will be expressed as follows:
\[ U_x = \frac{1}{2} \]

\[ U_l = \sqrt{\frac{e^{2a(l-x)} + e^{-2a(l-x)} + 2 \cos \frac{2m}{v} (l-x)}{\sqrt{\frac{\rho^2 + m^2}{m} \lambda^2} I_t \left[ e^{2a(l-x)} + e^{-2a(l-x)} - 2 \cos \frac{2m}{v} (l-x) \right]}} \]

\[ + 2 \sqrt{\frac{\rho^2 + m^2}{m} \lambda^2} I_t U_l \]

\[ + \frac{(e^{\alpha(l-x)} - e^{-\alpha(l-x)}) \cos m (\varphi + \delta)}{2 \sin \frac{2m}{v} (l-x) \sin m (\varphi + \delta)} \]

if we take:

\[ \text{Tang } m \varphi = \frac{m}{\rho + \frac{\rho - \lambda}{\varphi} a} = \frac{m}{\lambda} + \frac{\rho}{\sqrt{\rho^2 + m^2} \lambda^2} \]

Such is the expression for the e.m.f. at the point \( x \), located at a distance \( x \) from the generator end, when supplying to the receiving apparatus, with a power factor equal to \( \cos m \varphi \), a current of intensity \( I_l \), of frequency \( \beta = m/2\pi \), with a voltage \( U_l \).

Let us consider the line when operated at no load.

We have \( I = 0 \), then the voltage \( U_o \) becomes

\[ U_o = U_l \sqrt{\frac{e^{2aI} + e^{-2aI} + 2 \cos \frac{2m}{i} l}{4}} \]

For a line to transmit energy efficiently, the coefficient of damping \( 2aI \) will be always very small, and the sum \( e^{2aI} + e^{-2aI} \) will be approximately equal to 2.

If the line be rather long and \( \cos 2ml/v \) differs little from unity, we have \( U_I > U_o \).

This effect will be maximum when:

\[ 2 \frac{m l}{v} = \pi, \text{ or } l = \frac{1}{4} \frac{v}{\beta} \]

Thus, the length of the line will be equal to \( \frac{1}{4} \) of the length of the transmitted wave, and the ratio \( U_I/U_o \) will be very high.

This condition will not generally occur with currents of usual frequency, but it may occur on long lines with low harmonics which will bring dangerous surging in the voltage of the line.

Therefore, we must avoid with the utmost care the production of harmonics by the generating and receiving apparatus, and use in long distance transmission currents with frequency as low as possible, in order that the length of the line may be always very short as compared with the length of the waves transmitted.
A formula similar to the preceding may be deduced giving the intensity $I_x$ at a given point $x$ of the line located at a distance $x$ from the generator end. Then we have:

$$I_x = \frac{1}{2} \sqrt{\frac{m}{\rho^2 + m^2 \lambda^2}} \left[ U_1^2 \left[ e^{2a(l-x)} - e^{-2a(l-x)} - 2 \cos 2 \frac{m l-x}{v} \right] ight.$$

$$\left. + \sqrt{\frac{\rho^2 + m^2 \lambda^2}{m \gamma}} P_1 \left[ e^{2a(l-x)} + e^{-2a(l-x)} + 2 \cos 2 \frac{m l-x}{v} \right] ight]

$$+ 2 \sqrt{\frac{\rho^2 + m^2 \lambda^2}{m \gamma}} I_1 U_1 

\left\{ \begin{array}{l}
(e^{2a(l-x)} - e^{-2a(l-x)}) \cos m (\varphi - \delta) \\
- 2 \sin 2 \frac{m l-x}{v} \sin m (\varphi - \delta)
\end{array} \right\}$$