Note on an Identity in Elliptic Functions. By Prof. L. J. ROGERS.

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1. In the *Fund. Nov.*, see Vol. I., p. 336, of his collected works, Jacobi establishes the identity,

$$\begin{aligned} \operatorname{sn} a \operatorname{sn} b + \operatorname{sn} x \operatorname{sn} (x+a+b) - \operatorname{sn} (x+a) \operatorname{sn} (x+b) \\ = k^2 \operatorname{sn} a \operatorname{sn} b \operatorname{sn} x \operatorname{sn} (x+a+b) \operatorname{sn} (x+a) \operatorname{sn} (x+b) \dots (1). \end{aligned}$$

By putting $x = u - \frac{m}{2}$, $a = \frac{m-l}{2}$, $b = \frac{m+l}{2}$,

this identity becomes

$$\begin{aligned} \operatorname{sn} \left(u - \frac{m}{2} \right) \operatorname{sn} \left(u + \frac{m}{2} \right) - \operatorname{sn} \left(u - \frac{l}{2} \right) \operatorname{sn} \left(u + \frac{l}{2} \right) + \operatorname{sn} \frac{m-l}{2} \operatorname{sn} \frac{m+l}{2} \\ = k^2 \operatorname{sn} \left(u - \frac{m}{2} \right) \operatorname{sn} \left(u + \frac{m}{2} \right) \operatorname{sn} \frac{u-l}{2} \operatorname{sn} \left(u + \frac{l}{2} \right) \operatorname{sn} \frac{m-l}{2} \operatorname{sn} \frac{m+l}{2} \\ \dots \dots \dots (2). \end{aligned}$$

It will be found convenient to write U_r for

$$\operatorname{sn} \left(u - \frac{rl}{2} \right) \operatorname{sn} \left(u + \frac{rl}{2} \right),$$

and $\operatorname{sn} rl$ for $\operatorname{sn} l \operatorname{sn} 2l \dots \operatorname{sn} rl$,

when r is a positive integer.

By (2), putting $m = 3l$, we get

$$k^2 U_1 U_3 \operatorname{sn} l \operatorname{sn} 2l = U_3 - U_1 + \operatorname{sn} l \operatorname{sn} 2l,$$

or, as we may write it, so as to correspond more exactly to formula to be obtained subsequently,

$$k^2 U_1 U_3 \operatorname{sn} 3l = U_3 \operatorname{sn} 3l - \frac{\operatorname{sn} 3l}{\operatorname{sn} l} U_1 \operatorname{sn} l + \operatorname{sn} 3l \dots \dots \dots (3).$$

Multiplying by U_3 , we get on the right-hand side the products $U_3 U$ and $U_1 U_3$, which, by (1), may be replaced by linear functions of U_1 , U_3 , and U_5 .

By actual calculation it will be found, after replacing

$$\frac{\operatorname{sn} 3l}{\operatorname{sn} l} - 1 + k^2 \operatorname{sn} l \operatorname{sn} 2l \operatorname{sn} 3l \operatorname{sn} 4l \text{ by } \frac{\operatorname{sn} 4l}{\operatorname{sn} 2l},$$

to which it may be reduced by (1), we get

$$k^4 U_1 U_3 U_5 \sin 5l = U_5 \sin 5l - \frac{\sin 5l}{\sin l} U_3 \sin 3l + \frac{\sin 5l \sin 4l}{\sin l \sin 2l} U_1 \sin l \dots (4).$$

It will be observed that in (3) and (4) the successive coefficients of expressions of the form $U_r \sin rl$ bear a similarity to binomial coefficients, each factor in the latter being replaced by a corresponding elliptic sine.

The object of the present paper is to establish a general formula for the product $U_1 U_3 U_5 \dots$ to n factors.

It is easily seen, by continued repetition of Jacobi's formula, that we may reduce the product $U_1 U_3 U_5 \dots$ to a linear expression in the U 's, which may contain a term independent of u , as in (3), or not, as in (4).

Thus we may assume that

$$\begin{aligned} & k^{2n} U_1 U_3 U_5 \dots U_{2n+1} \sin (2n+1) l \\ &= A_0 U_{2n+1} \sin (2n+1) l - A_1 U_{2n-1} \sin (2n-1) l + \dots + L_{2n+1} \dots (5), \end{aligned}$$

where L_{2n+1} is independent of u .

We see then, by (3) and (4), that

$$L_3 = \sin 3l, \quad \text{and} \quad L_5 = 0.$$

By changing u into $u + K'i$, $\sin u$ becomes $1 \div k \sin u$, so that (5) becomes

$$\frac{\sin (2n+1) l}{U_1 U_3 U_5 \dots} = \frac{A_0 \sin (2n+1) l}{U_{2n+1}} - \frac{A_1 \sin (2n-1) l}{U_{2n-1}} + \dots + k^2 L_{2n-1} \dots (6).$$

By making $u = \frac{l}{2}, \frac{3l}{2} \dots$ in succession, we may obtain the values of all the coefficients $A_0, A_1 \dots$.

These coefficients may best be evaluated by considering the corresponding algebraic identity

$$\frac{(2n+1)!}{\left(x^2 - \frac{1}{4}\right) \left(x^2 - \frac{3^2}{4}\right) \dots} = A_0 \frac{2n+1}{x^2 - \left(\frac{2n+1}{2}\right)^2} - A_1 \frac{2n-1}{x^2 - \left(\frac{2n-1}{2}\right)^2} + \dots + C.$$

Let

$$x = \frac{2r+1}{2};$$

then $(2n+1)!/r(r+1)(r-1)(r+2) \dots$

$$\dots 1 \cdot 2r(2r+1)(-1)(2r+2)(-2)(2r+3) \dots (-n+r)(r+n+1) \\ = (-1)^{n-r} A_{n-r};$$

therefore
$$A_{n-r} = \frac{(2n+1)!}{(n-r)!(n+r+1)!},$$

so that the A 's in (5) and (6) are fully established as corresponding to binomial coefficients, as we observed in (3) and (4), and

$$A_{n-r} = \frac{\text{sn}(2n+1)!}{\text{sn}(n-r)l! \text{sn}(n+r+1)l!} \dots\dots\dots(7).$$

It remains now to find L_{2n+1} .

Multiplying (5) by $k^3 U_{2n+3} \text{sn}(2n+2)l \text{sn}(2n+3)l$,
and remarking that, by (1),

$$k^3 U_{2n+1} U_{2n+3} = \frac{U_{2n+3} - U_{2n+1} + \text{sn}l \text{sn}(2n+2)l}{\text{sn}l \text{sn}(2n+2)l}, \text{ \&c. ...},$$

we see that the coefficient of U_{2n+3} in the new product is similar to the numerical expression

$$(2n+2)(2n+3) \left\{ \frac{2n+1}{(2n+2)1} - \binom{2n+1}{1} \frac{2n-1}{(2n+1)2} \right. \\ \left. + \binom{2n+1}{2} \frac{2n-3}{2n \cdot 3} - \dots + k^3 L_{2n+1} \right\}.$$

But we know, by (7), that this is $(2n+3)$; therefore

$$k^3 L_{2n+1} (2n+2)(2n+3) \\ = (2n+3) - \binom{2n+3}{1} (2n+1) + \binom{2n+3}{2} (2n-1) - \dots \dots\dots(8),$$

where each numerical factor has to be replaced by the corresponding elliptic sine.

Similarly calculating the absolute term L_{2n+5} in

$$k^{2n+4} U_1 U_3 \dots U_{2n+5} \text{sn}(2n+5)l!,$$

by multiplying $k^{2n+3} U_1 U_3 \dots U_{2n+3} \text{sn}(2n+3)l!$

by $k^3 U_{2n+5} \text{sn}(2n+4)l \text{sn}(2n+5)l,$

and noticing that by Jacobi's formula the absolute term, in any expression such as $k^2 U_{2n+5} U_{2r+1}$, is unity, we see that, in the corresponding algebraic form, we get

$$L_{2n+5} = (2n+4)(2n+5) \left\{ (2n+3) - \binom{2n+3}{1} (2n+1) + \dots \right\} \dots (9).$$

Hence, by (8),

$$L_{2n+5} = k^2 \operatorname{sn} (2n+2) l \operatorname{sn} (2n+3) l \operatorname{sn} (2n+4) l \operatorname{sn} (2n+5) l \cdot L_{2n+1} \dots (10).$$

From the known values of L_5 and L_6 , we see then that

$$L_{4m+1} = 0,$$

$$\text{while} \quad L_{4m+3} = k^{2m} \operatorname{sn} (4m+3) l! \dots (11).$$

We have therefore completely determined a formula

$$\begin{aligned} & k^{2n+1} U_1 U_3 \dots U_{2n+1} \operatorname{sn} (2n+1) l! \\ &= U_{2n+1} \operatorname{sn} (2n+1) l - \frac{\operatorname{sn} (2n+1) l}{\operatorname{sn} l} U_{2n-1} \operatorname{sn} (2n-1) l + \dots + L_{2n+1} \dots (12), \end{aligned}$$

where $L_{2n+1} = 0$ if n is even,

and $L_{2n+1} = k^{2n-1} \operatorname{sn} (2n+1) l!$ if n is odd,

a formula which gives an easily remembered equivalent for the product of $2n$ elliptic sines whose arguments are in arithmetic progression.

By changing u into $u+K\tau$, we moreover deduce a similar expression for the reciprocal of such a product.

So, too, by the various transformations in elliptic functions, we may deduce corresponding formulæ for tn -functions, cn -functions, dn -functions, sines, tangents, and cosines.

2. A similar formula may be derived from Jacobi's formula by writing

$$x = u+l, \quad a = -m-l, \quad b = -u-m,$$

whence $k^2 U_m \operatorname{sn} (u+l) \operatorname{sn} (m+l) \operatorname{sn} (m-l) \operatorname{sn} 2m$

$$= \operatorname{sn} (m+l) \operatorname{sn} (u+m) + \operatorname{sn} (m-l) \operatorname{sn} (u-m) - \operatorname{sn} 2m \operatorname{sn} (u+l) \dots (1).$$

$$\text{Thus} \quad k^2 \operatorname{sn} u \cdot U_l \operatorname{sn} l \operatorname{sn} 2l = \operatorname{sn} (u+l) - \frac{\operatorname{sn} 2l}{\operatorname{sn} l} \operatorname{sn} u + \operatorname{sn} (u-l) \dots (3).$$

By multiplying by U_4 and reducing such products as $U_4 \operatorname{sn}(u + rl)$ by (1), we get a linear expression in $\operatorname{sn}(u + 2l)$, $\operatorname{sn}(u + l)$... for the product $k^4 \operatorname{sn} u \cdot U_4 U_4 \operatorname{sn} 4l!$, there being no absolute term.

Similarly $k^{2n} \operatorname{sn} u \cdot U_4 U_4 \dots U_{2n} \operatorname{sn} 2nl!$

may be reduced to the form

$$A_0 \operatorname{sn}(u + nl) - A_1 \operatorname{sn}\{u + (n-1)l\} + \dots - A_n \operatorname{sn}\{u - (n-1)l\} \\ + A_0 \operatorname{sn}(u - nl),$$

when the A 's are independent of u .

By changing u into $u + K'i$, as in the last section, we can evaluate the coefficients, which will be found to correspond to those in the expansion of $(1-x)^{2n}$; thus

$$k^{2n} \operatorname{sn} u \cdot U_4 U_4 \dots U_{2n} \operatorname{sn} 2nl! \\ = \operatorname{sn}(u + nl) - \frac{\operatorname{sn} 2nl}{\operatorname{sn} l} \{\operatorname{sn} u + (n-1)l\} \\ + \frac{\operatorname{sn} 2nl \operatorname{sn}(2n-1)l}{\operatorname{sn} l \operatorname{sn} 2l} \operatorname{sn}\{u + (n-2)l\} - \dots \dots \dots (3).$$

This formula will also, by the various transformations in elliptic functions, lead to others, giving like relations connecting tn -functions, &c.

Thus k may be changed into k' , and sn into tn throughout, and further we may put $k = 1$ and write tan for sn .

If, again, in this trigonometrical identity, we put

$$e^{-u} = q^{\frac{1}{2}}, \quad \text{and} \quad n = \infty,$$

we get the formula connecting the q -series and the q -product form for

$$\operatorname{dn} \frac{2K}{\pi} u \operatorname{tn} \frac{2K}{\pi} u.$$