VI.—SYMBOLIC REASONING (VI.).¹

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1. THERE is no question on which logicians are so divided as that of the "Existential Import of Propositions". When we make any affirmation A^B , or any ²denial A^{-B} , do we, at the same time, implicitly assert that the subject A really exists? Do we assert that the predicate B really exists? Do the four technical propositions of the traditional formal logic, namely, "Every (or all) A is B," "No A is B," "Some A is B," "Some A is not B," necessarily imply the actual existence of the class A? Do they necessarily imply the actual existence of the class B? These are questions upon which logicians have expended a great deal of thought and no small amount of ink; yet they appear to be as far from coming to an agreement upon them as ever. A simple theory of the subject, a theory to which they could all subscribe, should therefore be welcomed as a real boon. Such a theory I hope to be able to offer in what follows.

2. Let e_1, e_2, e_3 , etc. (up to any number of individuals mentioned in our argument or investigation) denote our universe of *real existences*. Let $0_1, 0_2, 0_3$, etc., denote our universe of *non-existences*, that is to say, of unrealities, such as *centaurs, nectar, ambrosia, fairies*, with self-contradictions, such as *round squares, square circles, flat spheres*, etc., including, I fear, the non-Euclidean geometry of four dimensions and other hyper-spatial geometries. Finally, let S_1, S_2, S_3 , etc., denote our *Symbolic Universe*, or "Universe of Discourse," composed of all things real or unreal that are named or expressed by words or other symbols in our argument or investigation. By this definition we assume our Symbolic Universe (or "Universe of discourse") to consist of our universe of realities, e_1, e_2, e_3 , etc., together with our universe of unrealities, $0_1, 0_2, 0_3$, etc., *when both these enter into our argument*.

¹ For V., see MIND, July, 1908.

² The symbol A^{-B} is here used as a convenient symbol for $(A^B)'$, the denial of the proposition A^B (see the *Athenæum*, 5th September, 1908).

But when our argument deals only with *realities*, then our Symbolic Universe S_1, S_2, S_3 , etc., and our Universe of realities, e_1, e_2, e_3 , etc., will be the same; there will be no universe of unrealities $0_1, 0_2, 0_3$, etc. Similarly, our Symbolic Universe may conceivably, but hardly ever in reality, coincide with our universe of unrealities.

3. Now, suppose we have a class A . The individuals, A_1, A_2, A_3 , etc., compassing it must necessarily all belong to the Symbolic Universe S ; but whether they all belong to the universe of realities e , or all to the universe of unrealities 0 , or some to the universe e and the rest to the universe 0 , depends upon the particular circumstances of our argument or investigation. When a class A belongs *wholly* to the universe e , or *wholly* to the universe 0 , we may call it a *pure* class; when it belongs *partly* to the class (or universe) e and *partly* to the class 0 , we may call it a *mixed* class. The *negative* class ' A ' (with a grave accent) consisting of the individuals ' A_1 ', ' A_2 ', ' A_3 ', etc., contains all the individuals of our symbolic universe which do not belong to the positive class A . Hence, the class ' e ' is synonymous with the class 0 ; and the class ' 0 ' with the class e . The class ' A ' may be called the *complement* of the class A , because both together make up the Symbolic Universe.

4. The subject A of any affirmative proposition A^B , or of any denial A^{-B} , is always understood to denote a *single individual*. If A happens to be the name of a class, then, in any proposition A^B or A^{-B} , the subject A is understood to denote a certain known, or previously indicated, individual of the series A_1, A_2, A_3 , etc., whose special name or number it is unnecessary to state. For example, suppose A denotes *American*, and B *barrister*, the proposition A^B will then assert that "the *American* is a *barrister*". It does not say *which* American out of the whole series A_1, A_2, A_3 , etc., is referred to; that is supposed to be known. When it is necessary to state which, then, instead of A^B , we must write A_1^B , or A_2^B , or A_3^B , as the case may be.

5. Let S be any individual taken at random out of our Symbolic Universe S , and let a, a', b, b' , etc., be synonymous respectively with S^A, S^{-A}, S^B, S^{-B} , etc. We then get the following self-evident or easily proved formulæ, which we will name F_1, F_2, F_3 , etc.

- (1) A^A ; (2) $(S^A)^{-A}$; (3) a^{-a} ; (4) a^a ;
(5) $(A)^S$; (6) $(S)^S$; (7) $(A)^{-A} + (A \equiv S)$.

The formula F_3 looks somewhat paradoxical; but it can be easily proved. By our definitions (see §§ 2, 3), the symbol ' S ' denotes an individual that does not belong to the class S .

But, by definition, the class S denotes the *whole Symbolic Universe* (or "universe of discourse") to which every individual (real, unreal, or self-contradictory) named in our argument *must* belong. Hence, ' S is a self-contradiction. But, by our definition, all self-contradictions belong to the class 0 . Hence ' S must belong to the class 0 . But, by definition, the Symbolic Universe S contains all the individuals $0_1, 0_2, 0_3$, etc., of the class 0 , as well as all the individuals e_1, e_2, e_3 , etc., which belong to the class of realities e . Hence, ' S must belong to the class S . In other words, the formula ' (S) ', denoted by F_6 , is always true. The preceding reasoning is a syllogism of the Barbara type, which may be expressed briefly as follows: "Every ' S is 0 , and every 0 is S ; therefore every ' S is S ". The last formula F_7 asserts that an individual of the negative class ' A does not belong to the positive class A , except when A is synonymous with S and therefore denotes the whole Symbolic Universe. This Symbolic Universe, or "Universe of Discourse," may enlarge as the argument proceeds, seizing, appropriating, and firmly retaining every new entity (not excepting self-contradictory entities like ' S ') which we designate by a symbol. Suppose for example, that in the course of our argument we have had to speak of several classes, *pure* or *mixed* (see § 3), and that all the individuals of all these classes amount to 82, of which 80 belong to the class e , and 2 to the class 0 . Then, up till now, our Symbolic Universe S contains 82 individuals, so that we have

$$S = (S_1, S_2, \dots S_{82}) = (e_1, e_2, \dots e_{80}, 0_1, 0_2).$$

A fresh arrival ' S enters our Symbolic Universe, which immediately widens to make room for it; but the question has to be decided whether the stranger is to enter the class e or the class 0 , just as parents have to decide the sex of a fresh addition to their family before they know whether to call it *Eva* or *Oscar*. The question presents as little difficulty in the one case as in the other; the new comer ' S (or S_{83}) is immediately recognised as belonging to the class 0 , so that now we have

$$S = (S_1, S_2, \dots S_{83}) = (e_1, e_2, \dots e_{80}, 0_1, 0_2, 0_3),$$

the new comer ' S (or S_{83}) being synonymous with the new comer 0_3 .

6. If every individual of a class A (whether a *pure* or a *mixed* class) belongs also to another class B ; then, and then only, we say that "Every (or all) A is B ". If this is not the case—if even a single A is excluded from the class B ; then we say that "Some A is not B ". If every individual in the class A be excluded from the class B ; then, and then

only, we say that "No A is B". If this is not the case—if a single individual of the class A belongs also to the class B; then we say that "Some A is B". For example, let the class A consist of the five individuals S_1, S_2, S_3, S_4, S_5 ; let the class B consist of the eight individuals made up of the preceding five individuals together with the three S_6, S_7, S_8 ; and let the class C consist of the three S_7, S_8, S_9 . More briefly expressed, let $A = (S_1, S_2, \dots S_5)$; let $B = (S_1, S_2, \dots S_8)$; and let $C = (S_7, S_8, S_9)$. Then, whether any of these three classes, A, B, C, be *pure* or *mixed* (see § 3), the following propositions follow necessarily from our data:—

- (1) Every (or all) A is B, (2) Some A is B,
- (3) Some B is A, (4) Some B is not A,
- (5) No A is C, (6) Some A is not C,
- (7) No C is A, (8) Some C is not A,
- (9) Some B is C, (10) Some B is not C.

Any of the three classes A, B, C may consist wholly of realities, or wholly of unrealities, or it may be a mixed class containing both; whatever hypothesis we take in that way, the preceding ten propositions are true. (See § 11.)

7. We may sum up briefly as follows: Firstly, when any symbol A denotes an *individual*; then, any intelligible statement $\phi(A)$, containing the symbol A, implies that the individual represented by A has a *symbolic* existence; but whether the statement $\phi(A)$ implies that the individual represented by A has a *real* existence depends upon the context. Secondly, when any symbol A denotes a *class*, then, any intelligible statement $\phi(A)$ containing the symbol A implies that the whole class A has a *symbolic* existence; but whether the statement $\phi(A)$ implies that the class A is *wholly real*, or *wholly unreal*, or *partly real and partly unreal*, depends upon the context.

8. For example, let M denote "the *man* whom you see in the garden"; let U denote "my *uncle*"; and let $\phi(M, U)$ denote the statement "The *man* whom you see in the garden is my *uncle*". In this case we generally have

$$\phi(M, U) : M^{\circ} U^{\circ}.$$

That is to say, the statement $\phi(M, U)$ would generally imply that both M and U really exist. Next, let B denote "a *bear*," and let $\phi(M, B)$ denote the statement "The *man* whom you see in the garden is really a *bear*". Here we should generally have

$$\phi(M, B) : M^{\circ} B^{\circ}.$$

That is to say, the statement $\phi(M, B)$ would generally imply that B really exists, but that the individual M is *imaginary*—a mere optical illusion. Now take the state-

ment $\phi' (M, B)$, which denies $\phi (M, B)$ and asserts that "The *man* whom you see in the garden is *not* a *bear*". Here we should generally have

$$\phi' (M, B) : M^* B^0.$$

That is to say, the denying statement $\phi' (M, B)$ would usually be understood to imply that *M* (the *man* seen in the garden) really exists, but that the particular *bear* spoken of is imaginary and non-existent. Lastly, take $\phi' (M, U)$ which denies $\phi (M, U)$ and asserts that "The *man* whom you see in the garden is *not* my *uncle*". Here we should generally say

$$\phi' (M, U) : M^* ;$$

but not necessarily $\phi' (M, U) : U^*$. That is to say, the denying statement $\phi' (M, U)$ would usually imply the real existence of the *man* *M*, but not necessarily the real existence of "my *uncle*"; for the negative statement $\phi' (M, U)$ might be true even on the supposition that neither my father nor mother ever had a brother, so that the supposed uncle had never existed. Similarly, we may give examples of the implied existence or non-existence of *classes*, and show that as regards *real* and not mere *symbolic* existence, no absolute rule can be laid down; that, in each case, the conclusion depends upon the particular nature of the statement and upon the general context.

9. The preceding discussion seems to me to point to a serious and fundamental error in the commonly accepted systems of symbolic logic, founded on the Boolean principle of class-inclusion. These usually denote the class of individuals common to the classes *A* and *B* by the symbol *AB*, and they employ the symbol (*A* = *AB*) to assert that the class *A* and the class *AB* are the same, every individual in either being also found in the other. Thus interpreted, they say, and say truly, that the statement of equivalence (*A* = *AB*) is equivalent to the traditional "All *A* is *B*," or "Every *A* is *B*". So far I agree with them. But when they define 0 (or any other symbol) as indicating non-existence, and then assert that the equivalence (*O* = *OA*) is always true, whatever the class *A* may be, they appear to me to make an assertion which cannot easily be reconciled with their data or definitions. For suppose the class 0 to consist of the three unrealities *O*₁, *O*₂, *O*₃, and the class *A* to consist of *O*₃, *e*₁, *e*₂, *e*₃ (one unreality and three realities), the class *OA* common to both contains but one individual, the unreality *O*₃. We cannot here say that the class 0, which contains three individuals, is the same as the class *OA*, which contains but one; neither can we say that every one of the three in-

dividuals $0_1, 0_2, 0_3$, which form the class 0 , is contained in the class A , which only contains one of them, namely, 0_1 . And, *a fortiori*, an *infinite* class ($0_1, 0_2, 0_3$, etc.), cannot be contained in a *finite* class $0A$, where $A = (A_1, A_2, \dots A_m)$.

10. If in my system of logic my formula $(\eta = \eta A)$ asserted that the class η and the class ηA contained exactly the same individuals, this formula would be open to exactly the same objections as the formula $(0 = 0A)$ just criticised. But my formula $(\eta = \eta A)$ does *not* assert this; it only asserts the truism expressed by the double implication $(\eta : \eta A) (\eta A : \eta)$, namely, that it never happens that either of the two *statements* η and ηA is true while the other is false. The formula is equally valid in the form $(\eta_1 = \eta_2 A)$, whatever be the impossibilities η_1 and η_2 , and whatever be the statement A . For, by a linguistic convention which I believe all logicians accept, any compound statement, say ABC , is considered *true* when, and only when, *all* its factors, A , B , and C , are true; but it is considered *false* if it has a *single false factor* A . Consequently, it must be *impossible* (or *always false*) if it has a single impossible factor η . Hence, $\eta_2 A$ is impossible because of the factor η_2 . We may therefore denote $\eta_2 A$ by η_3 (impossibility No. 3), so that the formula $(\eta_1 = \eta_3 A)$ will then be equivalent to $(\eta_1 = \eta_3)$. Now, by definition,

$$(\eta_1 = \eta_3) = (\eta_1 : \eta_2) (\eta_3 : \eta_1) = (\eta_1 \eta'_2)^* (\eta_3 \eta'_1)^*.$$

But $\eta'_2 = \epsilon_1$, and $\eta'_1 = \epsilon_2$; for ¹ the denial of any impossibility η_x is some certainty ϵ_x , so that the denial of η_3 is a certainty which we register as ϵ_1 , and the denial of η_1 is another certainty which we register as ϵ_2 . Hence, by substituting ϵ_1 for η'_2 , and ϵ_2 for η'_1 , we get

$$(\eta_1 = \eta_3 A) = (\eta_1 = \eta_3) = (\eta_1 \epsilon_1)^* (\eta_3 \epsilon_2)^* = \epsilon_3 \epsilon_4 = \epsilon_5.$$

11. Another disputed question which the preceding theory of the "Existential Import of Propositions" appears to decide is the validity or non-validity of the four traditional syllogisms, Darapti, Felapton, Fesapo and Bramantip. Now, as I pointed out in *MIND*, July, 1902, § 32, not one syllogism out of the whole nineteen is valid in its traditional form $PQ \therefore R$, as in this form it asserts without warrant that the two premisses P and Q are both true. In this form therefore any syllogism is false whenever either P or Q is false. To

¹ The denial of a certainty is an impossibility, the denial of an impossibility is a certainty, and the denial of a variable is a variable. If the chance of A is α , the chance of A' is $1 - \alpha$. When $\alpha = 1$, then A is a certainty and A' an impossibility. When $\alpha = 0$, then A is an impossibility and A' a certainty. When α is some fraction between 1 and 0, then $1 - \alpha$ is also a fraction between 1 and 0, so that, in this case, A and A' are both variables.

render the syllogism valid, it should be written in the form $PQ : R$ ("If P and Q are true, then R is true"). Thus written, if in any syllogism we substitute for P, Q, R its special premisses and conclusion, we shall find that $PQ : R$, which means $PQR' : \eta$, is a formal certainty, whatever syllogism out of the nineteen we take as an example. Take Darapti, one of the four considered doubtful. Darapti, in its corrected or conditional form, says this, "If every B is C, and every B is A, then some A is C". This is supposed to fail when B is non-existent while A and C are existent but mutually exclusive. Let us see. Suppose

$$B = (0_1, 0_2, 0_3), C = (e_1, e_2, e_3), A = (e_4, e_5, e_6).$$

Here we have

$$P = \text{Every B is C} = \eta_1$$

$$Q = \text{Every B is A} = \eta_2$$

$$R = \text{Some A is C} = \eta_3,$$

three statements each of which contradicts our data, since, by our data in this case, the three classes A, B, C are mutually exclusive (see § 6). Hence, in this case, we have

$$(PQ : R) = (\eta_1 \eta_2 : \eta_3) = (\eta_4 : \eta_3) = (\eta_4 \eta_3)' = \epsilon_1;$$

so that Darapti, in its corrected form $PQ : R$, does *not* fail in the case supposed.

12. The fallacious reasoning by which the Boolean logicians have arrived at the conclusion that Darapti, even in its corrected form $PQ : R$, is not valid, is founded on the assumption that their definitions of their symbols lead to the conclusion that the statement $(0 = 0A)$ is a formal certainty; whereas, consistently with their definitions, this statement may be either true or false. For example, in the case $B^0 A^e$, given in § 11, the statement $(0 = 0A)$ is false.

13. It is curious that, by fallacious reasoning of a totally different kind, I formerly arrived at the same erroneous conclusion as the Boolean logicians about Darapti and the other three doubtful syllogisms. Finding, firstly, that the implication of the second degree

$$(b : c) (b : a) : (a : c)'$$

which, I may denote by $F(a, b, c)$, expresses Darapti on the assumption that the propositions a, b, c have all three the same subject, namely, an individual taken at random out of our "universe of discourse"; and finding, secondly, that this formula, considered as a *general* formula, with no necessary reference to Darapti or any other syllogism, fails in the case $b^v (ac)^v$, I concluded, a little too hastily, that Darapti must also fail in this case. In this I overlooked the fact that, though the case of failure $b^v (ac)^v$ may arise in the *general* formula $F(a, b, c)$, when a, b, c are understood to be wholly *unrestricted*,

the case need not arise, and, as a matter of fact, cannot arise when the propositions a, b, c are subject to the restrictions which render $F(a, b, c)$ equivalent to Darapti. For these restrictions necessarily imply $a^{-\gamma} b^{-\gamma} c^{-\gamma}$ (see § 5, Formulæ 2, 3).

14. Bearing upon this and similar pitfalls which waylay the too hasty investigator, in whatever branch of science, when he ventures to stretch his formulæ beyond their proper limits, the following rules and cautions may be found useful. Let ϕ_u denote any formula $\phi(x, y, z, \text{etc.})$ when the variables, $x, y, z, \text{etc.}$, have an *unrestricted* (or very wide) range of values; and let ϕ_r denote the same formula when the variables have a *restricted* range (or a narrower range within the same limits). Then, employing the symbol ϕ^* to assert that ϕ is true for *all* admissible values of its constituents, $x, y, z, \text{etc.}$, we have the true formula $\phi_r^* : \phi_r^*$; but *we have no right to assume the converse formula $\phi_r^* : \phi_u^*$, nor its equivalent, the formula $\phi_u^{-*} : \phi_r^{-*}$* . The assumption $\phi_r^* : \phi_u^*$ is the common fallacy in scientific researches of a too hasty induction, which erroneously supposes that the validity¹ of ϕ_r implies the validity of ϕ_u . The assumption $\phi_u^{-*} : \phi_r^{-*}$ which erroneously supposes that the non-validity of ϕ_u implies the non-validity of ϕ_r , is the fallacy which formerly led me into the error referred to in § 13.

¹ Any formula $\phi(x, y, z, \text{etc.})$ is called *valid* when we have ϕ^* ; that is to say, when the formula is true whatever values, within the limits of our data, we assign to the variables $x, y, z, \text{etc.}$ The statement ϕ^{-*} , that ϕ is *not* valid, does not imply ϕ^γ , that it is *never* true.