ON RADIAL CURVES.*

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V.

1. My previous article upon *Radial Curves* will be found in the "Mathematical Reprints from the *Educational Times*," Vol. I., pp. 16—19; Vol. II., pp. 27—38; and Vol. IV., pp. 22—28.

Occasional reference is made to these (under the titles *A*, *B*, *C*, respectively) in the following remarks.

2. It will be sufficient for my present purpose to consider the radial as generated in the following manner:—

Let *P* and *Q* be corresponding points on a curve and its evolute; then, *O* being the origin, if the parallelogram of which *OP*, *PQ* are adjacent sides be completed, the locus of the remaining angular point (*R*) will be the radial.

Hence, given the curves on which two of the three points lie, the locus of the third angular point will be the remaining curve.

3. The area between a curve, its evolute, and the corresponding radii of curvature, is easily shown to be

$$\frac{1}{2} \int_a^b r^2 d\theta.$$

4. It may readily be seen that my method can be applied to the determination of properties of circles of curvature which are orthogonal, thus, by Art. 18 (*A*), it is seen that, in the case of the *cycloid*, the distance between the centres of two such circles is constant and equal to twice the diameter of the generating circle of the primitive curve. Hence also we can obtain properties of such circles in the case of the conic sections.

* Read Dec. 18, 1866. Professor De Morgan in the Chair.

5. If we take a curve such that its abscissa and ordinate are equal respectively to the arc of the primitive curve and its evolute, and denote the corresponding arc of the new curve by S , we have, by equation (μ) , (B) —

$$\left(\frac{dS}{ds}\right)^2 = 1 + \left(\frac{ds'}{ds}\right)^2 = \left(\frac{ds''}{ds}\right)^2,$$

and, therefore, $dS = ds''$. Hence it appears that differences of arc of this curve equal corresponding differences of arc of the radial.

6. Taking the well-known formula (Todhunter, *Diff. Calc.*, § 323), and employing the notation of Arts. 60, 63 (B) , we have, if R' stand for the radius of curvature of the evolute of the evolute,

$$\rho_1 = \frac{(\rho^2 + R^2)^{\frac{3}{2}}}{\rho^3 + 2R^2 \mp \rho R'} \dots\dots\dots (\phi).$$

This result may be put under another form,

$$\text{since} \quad \rho_1 \frac{d\phi_1}{d\phi} = \sqrt{\rho^2 + R^2} \dots\dots\dots (\xi) \text{ in } (B),$$

$$\therefore \quad \frac{d\phi_1}{d\phi} = 1 + \frac{R^2 \mp \rho R'}{\rho^2 + R^2} = \frac{\rho^3 + 2R^2 \mp \rho R'}{\rho^2 + R^2} \dots\dots\dots (\phi').$$

7. From equation (τ) (C) , we get

$$2 \frac{d\phi}{d\phi_1} = \frac{\rho R_1 + R \rho_1}{\rho_1 R}.$$

Combining this with (ϕ') , there results

$$2\rho_1 R (\rho^2 + R^2) = (\rho^2 + 2R^2 \mp \rho R') (\rho R_1 + R \rho_1),$$

$$\text{or} \quad \rho_1 R (\rho \pm R') = R_1 (\rho^2 + 2R^2 \mp \rho R') \dots\dots\dots (\psi).$$

8. In this and following Articles it is proposed to find the radials for the general equations of the second and third degrees.

The most general form of the equation of the second degree is

$$a_1 x^2 + 2b_2 xy + c_3 y^2 + 2d_1 x + 2e_1 y + f_1 = 0 \dots\dots\dots (i.)$$

Differentiating twice, and writing p, q for $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, there results

$$A_1 x + B_1 y + C_1 = 0 \dots\dots\dots (ii.),$$

$$A_2 x + B_2 y + C_2 = 0 \dots\dots\dots (iii.),$$

$$\text{where} \quad A_1 = b_2 p + a_1, \quad B_1 = c_3 p + b_2, \quad C_1 = e_1 p + d_1,$$

$$A_2 = b_2 q, \quad B_2 = c_3 q, \quad C_2 = 2b_2 p + c_3 p^2 + e_1 q + a_1;$$

$$\text{from (iii.),} \quad y = \lambda x + \mu \dots\dots\dots (iv.),$$

$$\text{if} \quad \lambda = -\frac{A_2}{B_2}, \quad \mu = -\frac{C_2}{B_2}.$$

Substituting (iv.) in (i.), the result is

$$Ax^3 + Bx + C = 0 \dots\dots\dots (v.),$$

if $A = a_1 + 2b_2\lambda + c_3\lambda^2, \quad B = 2b_2\mu + 2c_3\lambda\mu + 2d_1 + 2e_1\lambda,$

$$C = c_3\mu^2 + 2e_1\mu + f_1;$$

and (ii.) similarly becomes $A'x + C' = 0 \dots\dots\dots (vi.),$

where $A' = A_1 + B_1\lambda, \quad C' = B_1\mu + C_1.$

Eliminating x from (v.) and (vi.), the resulting equation is

$$AC'^2 - A'BC' + A'^2C = 0,$$

or $(3b_2c_3p^2 + 2b_2^2p + c_3^2p^3 + b_2e_1q + a_1c_3p + a_1b_2 - c_3d_1q)^2$
 $+ 2q(c_3d_1 - b_2e_1)(3b_2c_3p^2 + 2b_2^2p + c_3^2p^3 + b_2e_1q + a_1c_3p + a_1b_2 - c_3d_1q)$
 $+ (a_1c_3 - b_2^2)\{c_3^2p^4 + 4b_2c_3p^3 + (4b_2^2 + 2a_1c_3)p^2 + 4a_1b_2p + a_1^2$
 $+ (f_1c_3 - e_1^2)q^2\} = 0.$

This reduces to

$$c_3^3p^6 + 6b_2c_3^2p^5 + 3c_3(a_1c_3 + 4b_2^2)p^4 + 4b_2(3a_1c_3 + 2b_2^2)p^3$$

$$+ a_1(3a_1c_3 + 8b_2^2)p^2 + 6a_1^2b_2p$$

$$+ a_1^3 + \{f_1(a_1c_3 - b_2^2) - a_1e_1^2 - c_3d_1^2 + 2b_2d_1e_1\}q^2 = 0.$$

9. Hence the radial in this case is a curve of the sixth degree, and may be found from the above by writing $\frac{x}{y}$ for p , and $\frac{x^2 + y^2}{y^3}$ for q .

If the given curve be a parabola, or $a_1c_3 = b_2^2$, the resulting equation, by (8), is

$$c_3^2p^3 + 3b_2c_3p^2 + 3b_2^2p + a_1b_2 = \pm q(c_3d_1 - b_2e_1).$$

10. Taking now the most general form of the equation of the third degree

$$a_1x^3 + b_2y^3 + 3a_2x^2y + 3b_1xy^2 + 3a_3x^2 + 6dxy + 3b_3y^2 + 3c_1x + 3c_2y + c_3 = 0 \dots\dots(i.),$$

and proceeding as before, the resulting equations are

$$Ay^3 + Bxy + Cx^3 + Dy + Ex + F = 0 \dots\dots\dots (ii.),$$

$$A'y^3 + B'xy + C'x^3 + D'y + E'x + F' = 0 \dots\dots\dots (iii.),$$

where $\begin{cases} A = 3b_2p + 3b_1, & B = 6b_1p + 6a_2, & C = 3a_2p + 3a_1, \\ D = 6b_3p + 6d, & E = 6d_1p + 6a_3, & F = 3c_2p + 3c_1, \end{cases}$

and $\begin{cases} A' = 3b_2q, & B' = 6b_1q, & C' = 3a_2q, \\ D' = 2Ap + B + 6b_2q, & E' = Bp + 2C + 6d_1q, & F' = Dp + E + 3c_2q; \end{cases}$

from (ii.) and (iii.) there results

$$(A'B)xy + (A'C)x^3 + (A'D)y + (A'E)x + (A'F) = 0,$$

$$\therefore -y = \frac{\lambda x^3 + \mu x + \nu}{\mu' x + \nu} \dots\dots\dots (a),$$

$$\text{if } \left. \begin{array}{l} \lambda = (A'C), \quad \mu = (A'E), \quad \nu = (A'F) \\ \mu' = (A'B), \quad \nu' = (A'D) \end{array} \right\}.$$

Substitute (a) in (i.), and the resulting equation is

$$A''x^5 + B''x^4 + C''x^3 + D''x^2 + E''x + F'' + G'' = 0 \dots\dots\dots (iv.),$$

where

$$A'' = a_1\mu'^3 - \lambda^3b_3 - 3a_3\mu'^3\lambda + 3b_1\lambda^3\mu',$$

$$B'' = 3a_1\mu'^2\nu' - 3b_3\lambda^3\mu - 3a_3(\mu'^2\mu + 2\lambda\mu'\nu') + 3a_3\mu'^3 \\ + 3b_1(\lambda^3\nu' + 2\lambda\mu\mu') - 6d\lambda\mu'^2 + 3b_3\lambda^3\mu',$$

$$C'' = 3a_1\mu'\nu'^3 - 3b_3(\lambda^3\nu + \mu^3\lambda) - 3a_3(\mu'^2\nu + 2\mu\mu'\nu' + \lambda\nu'^3) \\ + 3b_1(\mu^3\mu' + 2\lambda\mu\nu' + 2\lambda\mu'\nu) + 9a_3\mu'^2\nu' + 3c_1\mu'^3 - 3c_3\lambda\mu'^3 \\ - 6d(\mu\mu'^2 + 2\lambda\mu'\nu') + 3b_3(\lambda^2\nu' + 2\lambda\mu\mu'),$$

$$D'' = a_1\nu'^3 - b_3(\mu^3 + 6\lambda\mu\nu) - 3a_3(\mu\nu'^2 + 2\mu'\nu\nu') + 9a_3\mu'\nu'^3 \\ + 3b_1(\mu^3\nu' + 2\lambda\nu\nu' + 2\mu\mu'\nu) + 3b_3(2\lambda\mu\nu' + \mu^3\mu' + 2\lambda\mu'\nu) \\ - 6d(\lambda\nu'^2 + 2\mu\mu'\nu' + \mu'^2\nu) + 9c_1\mu'^3\nu' + c_3\mu'^3 \\ - 3c_3(\mu\mu'^2 + 2\lambda\mu'\nu'),$$

$$E'' = -3b_3(\lambda\nu^3 + \mu^3\nu) - 3a_3\nu\nu'^2 + 3b_1(\mu'\nu^3 + 2\mu\nu\nu') + 3a_3\nu'^3 \\ - 6d(\mu\nu'^2 + 2\mu'\nu\nu') + 9c_1\mu'\nu'^3 + 9c_3\mu'^2\nu' \\ + 3b_3(\mu^3\nu' + 2\lambda\nu\nu' + 2\mu\mu'\nu) - 3c_3(\lambda\nu^3 + 2\mu\mu'\nu' + \mu'^3\nu),$$

$$F'' = -3b_3\mu\nu^3 + 3b_1\nu^3\nu' - 6d\nu\nu'^2 + 3c_1\nu'^3 + 3c_3\mu'\nu'^3 \\ + 3b_3(\mu'\nu^3 + 2\mu\nu\nu') - 3c_3(\mu\nu'^2 + 2\mu'\nu\nu'),$$

$$G'' = -b_3\nu^3 + 3b_3\nu^2\nu' - 3c_3\nu\nu'^2 + c_3\nu'^3.$$

Also, the result of substituting (a) in (ii.), (iii.) is

$$A_1x^4 + B_1x^3 + C_1x^2 + D_1x + E_1 = 0 \dots\dots\dots (v.),$$

$$a'x^4 + b'x^3 + c'x^2 + d'x + e' = 0 \dots\dots\dots (vi.);$$

$$\text{if } A_1 = A\lambda^3 - B\lambda\mu' + C\mu'^3,$$

$$B_1 = 2A\lambda\mu - B(\lambda\nu' + \mu\mu') + 2C\mu'\nu' - D\lambda\mu' + E\mu'^3,$$

$$C_1 = A(\mu^3 + 2\lambda\nu) - B(\mu\nu' + \mu'\nu) + C\nu'^2 - D(\lambda\nu' + \mu\mu') + 2E\mu'\nu' + F\mu'^3,$$

$$D_1 = 2A\mu\nu - B\nu\nu' - D(\mu\nu' + \mu'\nu) + E\nu'^2 + 2F\mu'\nu',$$

$$E_1 = A\nu^3 - D\nu\nu' + F\nu'^2.$$

Similarly, a' , b' , &c. are immediately obtained from the above by changing A into A' , &c.

Now, multiplying (v.) successively by x^3 and x , and eliminating from (iv.) and the resulting equation, we arrive at the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \dots\dots\dots (vii.),$$

where

$$\left. \begin{aligned} a &= A_1 (A_1 C'') - B_1 (A_1 B'') \\ b &= A_1 (A_1 D'') - C_1 (A_1 B'') \\ c &= A_1 (A_1 E'') - D_1 (A_1 B'') \\ d &= A_1^2 F'' - E_1 (A_1 B'') \\ e &= A_1^2 G'' \end{aligned} \right\};$$

whence, eliminating between (vi.) and (vii.), we get the known result

$$\begin{array}{cccc} (ab'), & (ac'), & (ad'), & (ae') \\ (ac'), & (ad') + (bc'), & (ae') + (bd'), & (be') \\ (ad'), & (ae') + (bd'), & + (be') + (cd'), & (ce') \\ (ae'), & (be'), & (ce'), & (de'). \end{array}$$

which is therefore the equation to the radial required.

11. The degree of the radial in this case will be found not to exceed 4×32 , or the 128th. I was induced to examine the degrees of the radials in this Article and Article 9, in consequence of a suggestion to this effect kindly made by Professor Cayley.

12. It is readily seen how to proceed to find the radials for higher curves.