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CONSTANT CURRENT MERCURY ARC RECTIFIER.

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I. General.

The operation of the mercury arc rectifier is based on the phenomenon of the electric arc, to be a good conductor in the direction of the arc blast, but a non-conductor in the opposite direction, and so to pass only unidirectional currents.

In an electric arc the current is carried across the gap between the terminals by a bridge of conducting vapor consisting of the material of the negative or the cathode, which is produced and constantly replenished by the cathode blast, a high velocity blast issuing from the cathode.

An electric arc, therefore, can not spontaneously establish itself. Before current can flow as an arc across the gap between two terminals, the arc flame or vapor bridge must exist, *i.e.*, energy must have been expended in establishing this vapor bridge. This can be done either by bringing the terminals into contact and so starting the flow of current, and then by gradually withdrawing the terminals derive the energy of the arc flame from the current, as done in practically all arc lamps. Or by increasing the voltage across the gap between the terminals so much that the electrostatic stress in the gap represents sufficient energy to establish a path for the current, *i.e.*, by jumping an electrostatic spark across the gap, which is followed by the arc flame, as occasionally happens, against the wishes of the engineer, for instance, in lightning arresters. Or by supplying the arc flame from another arc, an arc can be estab-

lished between two terminals, as also occasionally happens when a switch opens or burns up, and its arc flame envelops the blades of another switch.

The arc must therefore be continuous at the cathode, but may be shifted from anode to anode. Any interruption of the cathode blast puts out the arc by interrupting the supply of conducting vapor, and a reversal of the arc stream means stopping the cathode blast and producing a reverse cathode blast, which requires a voltage higher than the electrostatic striking voltage (at arc temperature) between the electrodes. With an alternating impressed electromotive force the arc if established will go out at the end of the half wave, or if a cathode blast is maintained continuously by a second arc (excited by direct current or overlapping sufficiently with the first arc) only alternate half waves can pass, those for which that terminal is negative from which the continuous blast issues. The arc, with an alternating impressed voltage, therefore rectifies, and the voltage range of rectification is the range between the arc voltage and the electrostatic spark voltage, hence highest with the mercury arc, due to its low temperature.

II. Description of Rectifier System.

The constant current mercury are rectifier system as it is used for operating direct current series arc circuits, of carbon arc lamps, magnetite or mercury arc lamps, from an alternating constant potential supply of 60, 40, or 25 cycles, is sketched diagrammatically in Fig. 1. It consists of:

A constant current transformer, with a tap C brought out from the middle of the secondary coil $A\ B$. A rectifier tube having two graphite anodes $a\ b$, and a mercury cathode c, and usually two auxiliary mercury anodes near the cathode c (not shown in diagram Fig. 1), which are used for excitation, mainly in starting, by establishing between the cathode c and the two auxiliary mercury anodes, from a small low voltage constant potential transformer, a pair of low current rectifying arcs in the same manner as in the well known constant potential mercury arc rectifier. Two reactive coils are inserted between the outside terminals of the transformer and rectifier tube respectively, for the purpose of producing an overlap between the two rectifying arcs, $c\ a$ and $c\ b$, and thereby the required continuity of the arc stream at c. A reactive coil is inserted into the rectified or arc circuit, which connects between transformer

neutral C and rectifier neutral c, for the purpose of reducing the fluctuation of the rectified current to the desired amount.

The rectified or direct current voltage is somewhat less than one-half of the alternating voltage supplied by the transformer secondary $A\ B$, the rectified or direct current somewhat more than double the effective alternating current supplied by the transformer.

III. Mode of Operation.

In Figs. 2 and 3 let the impressed voltage between the secondary terminals $A\ B$ of an alternating current transformer be

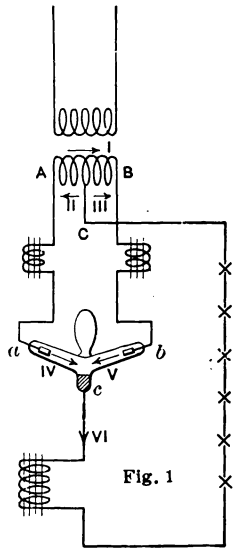


Fig. 1

shown by curve I. Let C be the middle or center of the transformer secondary $A\ B$. The voltages from C to A and from C to B then are given by curve II and III.

If now $A\ B\ C$ are connected with the corresponding rectifier terminals $a\ b\ c$ and at c a cathode blast maintained, those currents will pass for which c is negative or cathode, *i.e.*, the current flowing through the rectifier from a to c and from b to c, under the impressed electromotive forces II and III, are given by curves IV and V, and the current issuing from c will be the sum of IV and V, as shown in curve VI.

Such a rectifier as shown diagrammatically in Fig. 2, would, however, not be self-exciting, it requires some outside means

for maintaining the cathode blast at c, since the current in the half wave 1 in curve VI goes down to zero at the zero value of electromotive force III, before the current of the next half wave 2 starts, by the electromotive force II.

It is therefore necessary to maintain the current of the half wave 1 beyond the zero value of its propelling impressed electromotive force III, until the current of the next half wave 2 has started, *i.e.*, to overlap the currents of the successive half waves. This is done by inserting reactances into the leads from the transformer to the rectifier, *i.e.*, between A and a, respectively B and b, as shown in Fig. 1. The effect of this reactance is that the current of half wave 1, V, continues to

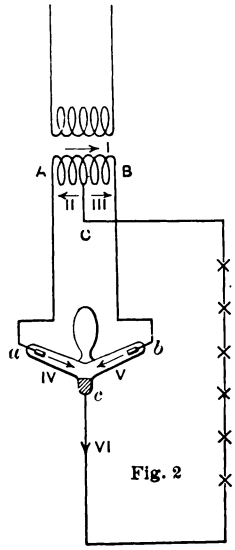
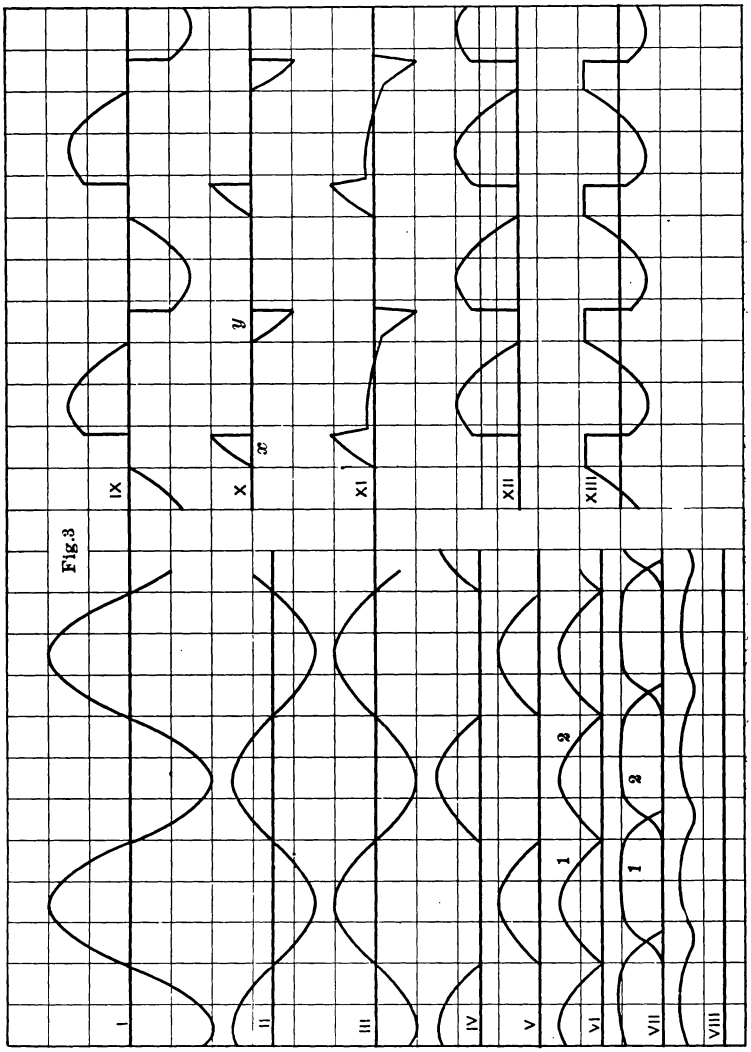


Fig. 2

flow beyond the zero of its impressed electromotive force III, *i.e.*, until the electromotive force III has died out and reversed, and the current of the half wave 2, IV, started by electromotive force II. That is, the two half waves of the current overlap, and each half wave lasts for more than half a period or 180 degrees.

The current waves then are shown in curve VII; the current half wave 1 starts at the zero value of its electromotive force III, but rises slower than it would without reactance, following essentially the exponential curve of a starting current, and the energy which is so consumed by the reactance as counter electro-

motive force, is returned by maintaining the current half wave 1 beyond the electromotive force wave, *i.e.*, beyond 180 degrees, by ω degrees (44 degrees in the tests reported in the following) so that it overlaps the next half wave 2 by ω degrees.



Hereby the rectifier becomes self-exciting, *i.e.*, each half wave of current, by overlapping with the next, maintains the cathode blast until the next half wave is started.

The successive current half waves added gives the rectified or unidirectional current curve VIII.

During a certain period of time in each half wave from the zero value of electromotive force, both arcs $c\,a$ and $c\,b$ flow. During the existance of both arcs, there can be no potential difference between the rectifier terminals a and b, and the impressed electromotive force between the rectifier terminals $a\,b$ therefore has the form shown in curve IX, Fig. 3, *i.e.*, remains zero for ω degrees, and then with the breaking of the arc of the preceding half wave, jumps up to its normal value.

The induced electromotive force of the transformer secondary, however, must more or less completely follow the primary impressed electromotive force wave, that is, has a shape as shown in curve I, and the difference between IX and I must be taken up by the reactance. That is, during the time when both arcs flow in the rectifier, the alternating current reactive coils consume the induced electromotive force of the transformer secondary, and the voltage across these reactive coils, therefore, is as shown in curve X. That is, the reactive coil consumes voltage at the start of the current of each half wave, at x in curve X, and produces voltage near the end of the current flow, at y. Between this time, the reactive coil has practically no effect and its voltage is low, corresponding to the variation of the rectified alternating current, as shown in curve XI. That is, during the intermediary time the alternating reactive coils merely assist the direct current reactive coil.

Since the voltage at the alternating terminals of the rectifier, $a\,b$, has two periods of zero value during each cycle, the rectified voltage, between c and C, must also have the same zero periods, and is indeed the same curve as IX, but reversed, as shown in curve XII.

Such an electromotive force wave cannot satisfactorily operate arcs, since during the zero period of voltage XII the arcs go out. The voltage on the direct current line must never fall below the "counter electromotive force" of the arcs, and since the resistance of this circuit is low, frequently less than 10%, it follows that the total variation of direct current line voltage must be below 10%, *i.e.*, the voltage practically constant, as shown by straight line in curve XII. Hence a high reactance is inserted into the direct current circuit, which consumes the excess voltage during that part of curve XII, where the rectified voltage is above line voltage, and supplies the line voltage during the period of zero rectified voltage. The voltage across this reactive coil therefore is as shown by curve XIII. Short circuiting

this reactance coil on an arc load therefore immediately puts out the rectifier, except if the external circuit contains sufficient reactance.

Regarding calculation of the rectifier:

The angle of overlap ω of the two arcs is determined by the desired stability of the system. By the angle ω and the impressed electromotive force is determined the sum total of electromotive forces which has to be consumed, and returned, by the alternating current reactive coil, and herefrom the size of the alternating current reactive coil.

From the angle ω also follows the wave shape of the rectified voltage, and therefrom the sum total of electromotive force which has to be given by the direct current reactive coil, and hereby the size of the direct current reactive coil required to maintain the direct current fluctuation within certain given limits.

The given factors of the problem therefore are: the resistance of the circuit, the counter electromotive force of the direct current circuit, the permissible fluctuation of the direct current, and the chosen angle of overlap of the rectifying arcs.

IV. Characteristics of System.

The efficiency, power factor, regulation, etc., of the mercury arc rectifier system are essentially those of the constant current transformer feeding the rectifier tube:

The losses in the system between constant alternating impressed voltage and constant direct current are: (1) The loss in the transformer which changes from constant alternating potential to constant alternating current. This amounts to 5–10%, according to the size of the transformer. (2) The loss by $I^2 R$ and hysteresis, in the reactive coils inserted into the alternating leads of the rectifier, the "alternating reactive coils," and in the reactive coil inserted in the rectified circuit, the "direct current reactive coil." (3) The loss of power in the rectifier tube, which in a series arc circuit is negligible. There occurs a constant drop of voltage, about 18 volts, irrespective of load or current in the tube; the mercury arc voltage. With a 75 light rectifier, at 6000 volts full load, this loss in the tube is 0.3%. The loss in the reactive coils obviously can be reduced as far as the customer is willing to pay for copper and iron.

Let N = frequency of the alternating current supply, i_0 = mean value of the rectified direct current, and a the pulsation of

the rectified current from the mean value, *i.e.*, $i_0\ (1+a)$ the maximum and $i_0\ (1-a)$, the minimum value of direct current. Experience has shown that a pulsation from a mean of 20 to 25% is not only permissible in any type of arc, but advantageous in increasing the steadiness of the arc. The total variation of the rectified current then is $2\ a\ i_0$, *i.e.*, the alternating component of the direct current has the maximum value $a\ i_0$, hence the effective value $\frac{a}{\sqrt{2}}\ i_0$ (or, for $a=2$, $0.141\ i_0$), and the frequency $2\ N$. Hysteresis and eddy losses in the direct current reactive coil, therefore, corresponds to an alternating current of frequency $2\ N$, and effective value $\frac{a}{\sqrt{2}}\ i_0$ or about $0.141\ i_0$, *i.e.*, is insignificant even at relatively high densities.

In the alternating reactive coils, the current varies, unidirectional, between 0 and $i_0\ (1+a)$, *i.e.*, its alternating component has the maximum value $\frac{1+a}{2}\ i_0$, and the effective value $\frac{1+a}{2\sqrt{2}}\ i_0$, (or, for $a = +.2$: $.0425\ i_0$), and the frequency N, The hysteresis loss therefore corresponds to an alternating current of frequency N, and effective value $\frac{1+a}{2\sqrt{2}}\ i_0$, or about $0.425\ i_0$.

Regarding the power factor of the mercury arc rectifier system, the non-inductive character of the direct current load increases, but the use of reactive coils in the alternating loads slightly decreases the power factor, so that the power factor of the load on the secondary terminals of the constant current transformer is 90 to 95%, *i.e.*, the same or slightly higher than that of an alternating series arc circuit, and the power factor of the whole system is therefore about the same or slightly higher than that of the same constant current transformer operating series alternating arc lamps at the same percentage of load, *i.e.*, with the same position of transformer coils.

With decreasing load, at constant alternating current supply, the rectified direct current slightly increases, due to the increasing overlap of the rectifying arcs, and to give constant direct current, the transformer is therefore adjusted so as to regulate for a slight decrease of alternating current output with decrease of load. Obviously, just as in constant current transformers for

series alternating arc lighting, in larger units, the regulation range for constant current is extended from full load down to $\frac{1}{3}$ load only, and not down to short circuit.

The maximum voltage which can be rectified, is unknown, since we have not been able yet to get sufficiently high voltage power supply to break down a rectifier of proper design and vacuum; and since a high frequency oscillator giving an 8-in. spark between 1 in. spheres, does not send a discharge through the vacuum of such a rectifier, it is not probable that a voltage limit of rectification will be reached in the range of voltages coming into commercial consideration. We have rectified 36 000 volts impressed upon the rectifier terminals, with small current. In the following some tests are given on the operation of a rectifier tube at 25 200 volts alternating impressed upon its terminals giving an output of 10 000 volts at 4.6 amperes direct current, and at 24 300 volts impressed, an output of 9500 volts at 6.25 amperes, or 59.4 kilowatts, from a single rectifier tube just a little larger than can conveniently be put in a coat pocket.

V. Tests of System.

A constant current mercury arc rectifier system has been in regular service for over a year, supplying 3.8 amperes constant current to 25 mercury arc lamps, for lighting streets and parks in the neighborhood of my private residence in Schenectady. The constant current transformer is a so-called "6 light testing transformer," with the secondary coil wound for two amperes and operating at somewhat higher magnetic density; therefore, the regulation range extends only down to 10 lamps. The records of test of efficiency, power factor, regulation, etc., of this system is given in Fig. 4. As seen, in this small unit, of only 3.9 kilowatts output, an efficiency of 80% and a power factor of 70% is reached, between constant potential alternating current received from the primary distributing mains, and constant direct current send into the arc circuit.

Oscillograms of this constant current rectifier, at full load of 25 mercury arc lamps, are given in Figs. 5 to 14. These curves show: Fig. 5, primary supply voltage. Fig. 6, secondary terminal voltage of transformer; Fig. 7, potential difference of alternating current reactive coils; Fig. 8, alternating voltage impressed upon rectifier tube; Fig. 9, unidirectional voltage produced between rectifier neutral and transformer neutral; Fig. 10, potential difference of direct current reactive coil; Fig. 11,

rectified voltage supplied to arc circuit; Fig. 12, primary supply current; Fig. 13, current in rectifying arcs; Fig. 14, rectified current supply to arc circuit. As seen, the angle of overlap of the two rectifying arcs is 44°. It is interesting to compare these curves with the corresponding curves, Fig. 3.

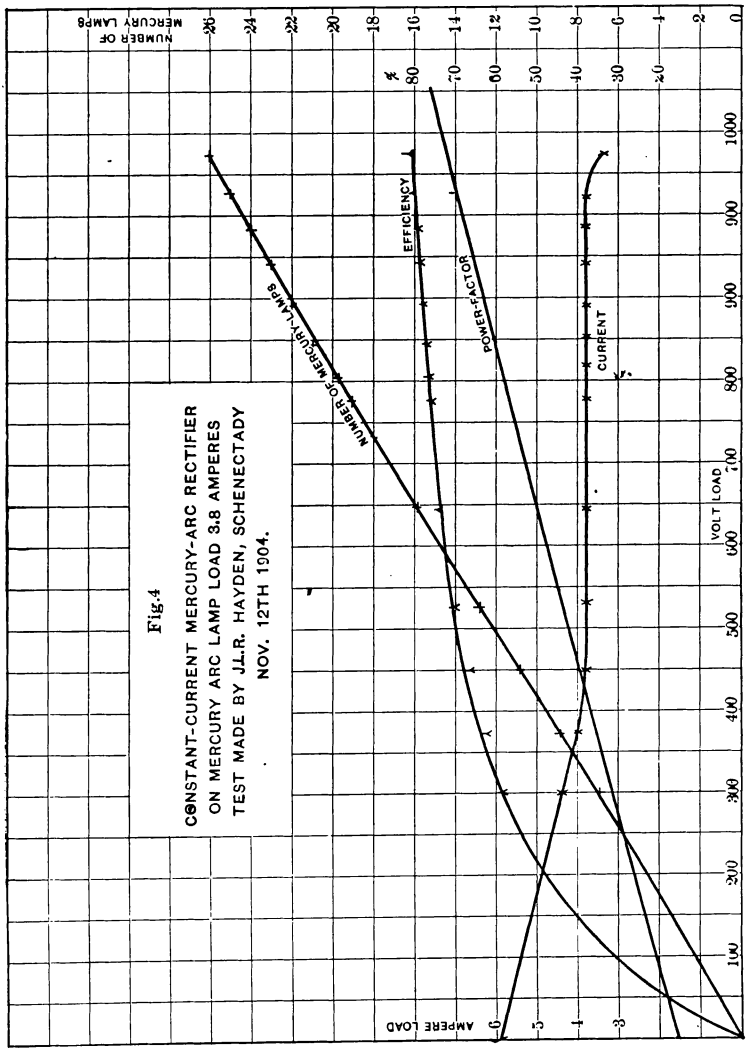


Fig. 15 gives secondary terminal voltage, efficiency, regulation, and load of a larger rectifier set operating magnetite arc lamps. At the highest load carried in test, of 57 magnetite

FIG. 5.

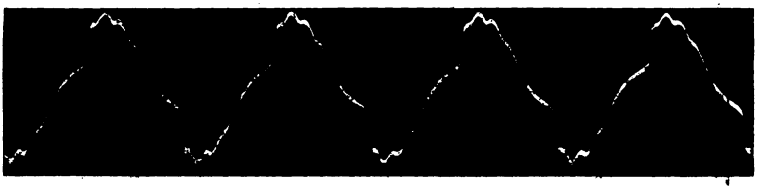


FIG. 6.

FIG. 7.

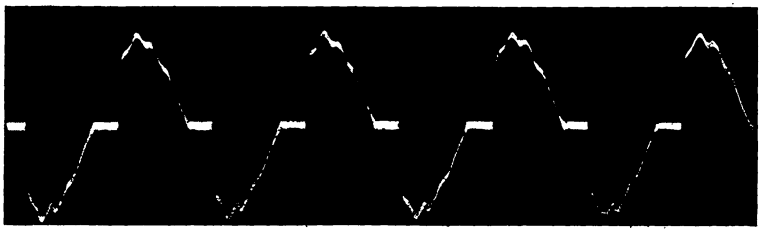


FIG. 8.

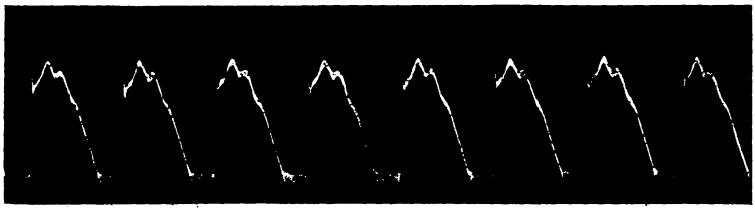


FIG. 9.

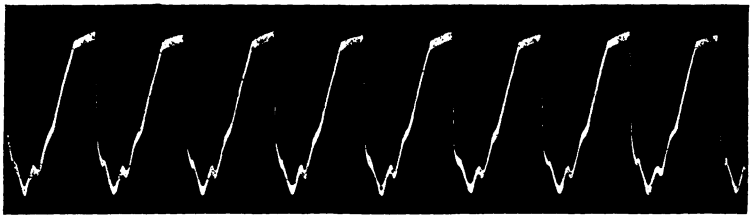


FIG. 10.

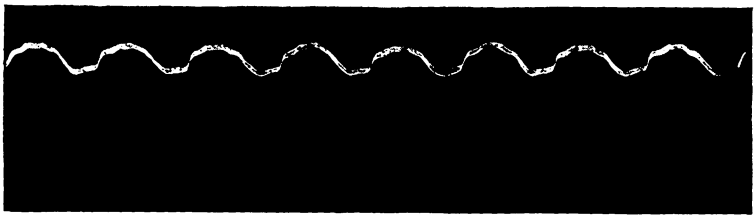


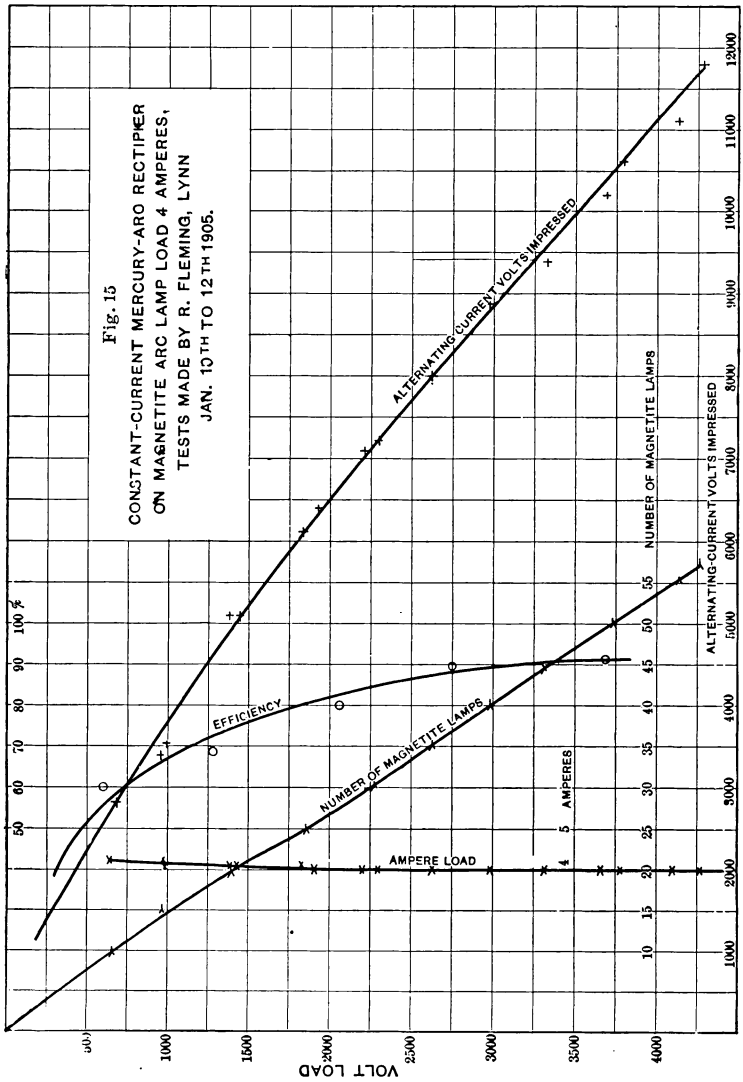
FIG. 11.

FIG. 12.

FIG. 13.

FIG. 14.

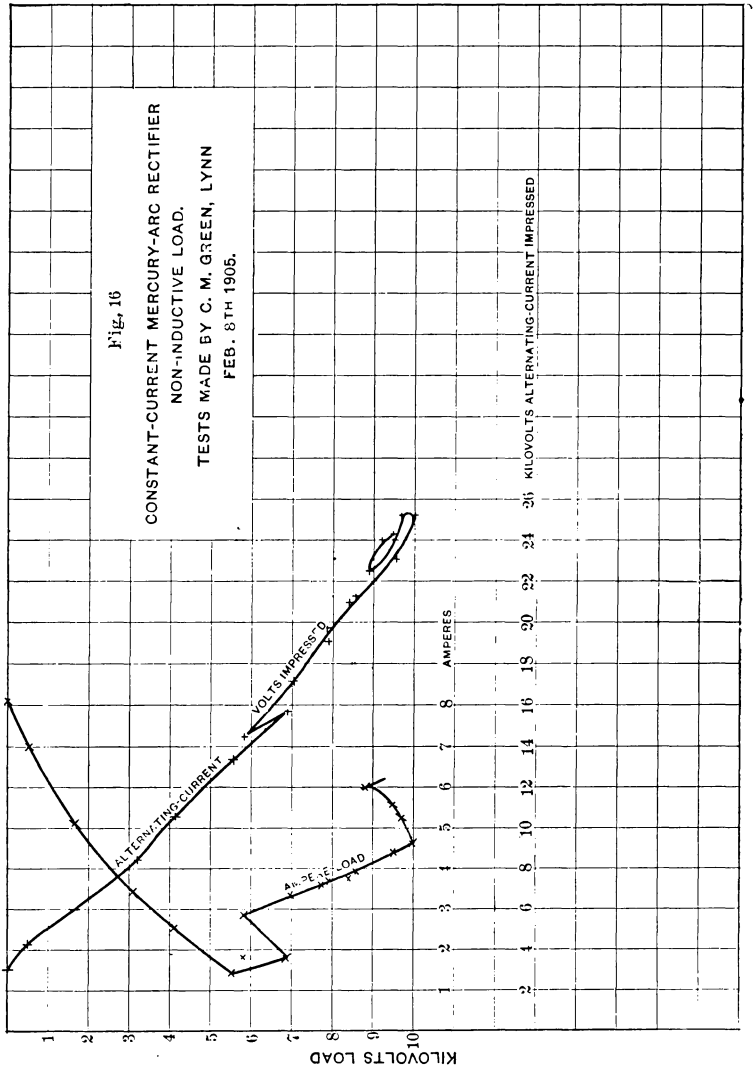
arc lamps, the rectified voltage supplied to the arc circuit at four amperes was 4260, at a voltage of 11 800 impressed upon the rectifier tube by the transformer secondary, the efficiency above 90%. This set is now in service in lighting some of the streets



of Schenectady, in the down-town district, with magnetite arc lamps.

Still higher voltages are shown in the test on Fig. 16. Here the power was supplied directly from an alternator, by step-up

transformer, to the rectifier tube, and a water resistance used as load, since neither a constant current transformer nor sufficient number of lamps were available at that time to take care of the load. The maximum voltage was 10 kilovolts at 4.6



amperes, with an impressed alternating voltage of 25.2 kilovolt, and the maximum output was 9.5 kilovolts at 6.25 amperes, or about 60 kilowatts.

VI. Theory and Calculation.

In the constant-current mercury-arc rectifier shown diagrammatically in Fig. 17, let:

$e \sin \phi$ = sinewave of electromotive force impressed between neutral and outside of alternating current supply to the rectifier, that is:

$2\, e \sin \phi$ = total secondary induced electromotive force of constant current transformer.

$Z_1 = r_1 - j\, x_1$ = impedance of reactive coil in each anode circuit of the rectifier (" alternating-current reactive coil "), in-

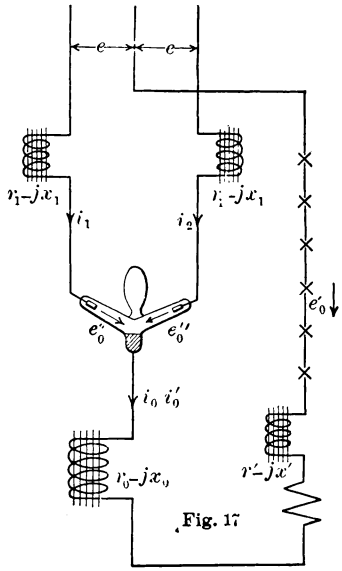


Fig. 17

clusive the internal self-inductive impedance between the two halves of the transformer secondary coil.

i_1 and i_2 = anode currents, counted in the direction from anode to cathode.

e_0'' = counter electromotive force of rectifying arc, which is constant.

$Z_0 = r_0 - j\, x_0$ = impedance of reactive coil in rectified circuit (" direct-current reactive coil ").

$Z^1 = r^1 - j\, x^1$ = impedance of load or arc lamp circuit.

e^1 = counter electromotive force in rectified circuit, which is constant (equal to the sum of the counter electromotive forces of the arcs in the lamp circuit).

ω = angle of overlap of the two rectifying arcs, or overlap of the currents i_1 and i_2.

i_0 = rectified current during the period: $0 < \phi < \omega$, where both rectifying arcs exist.

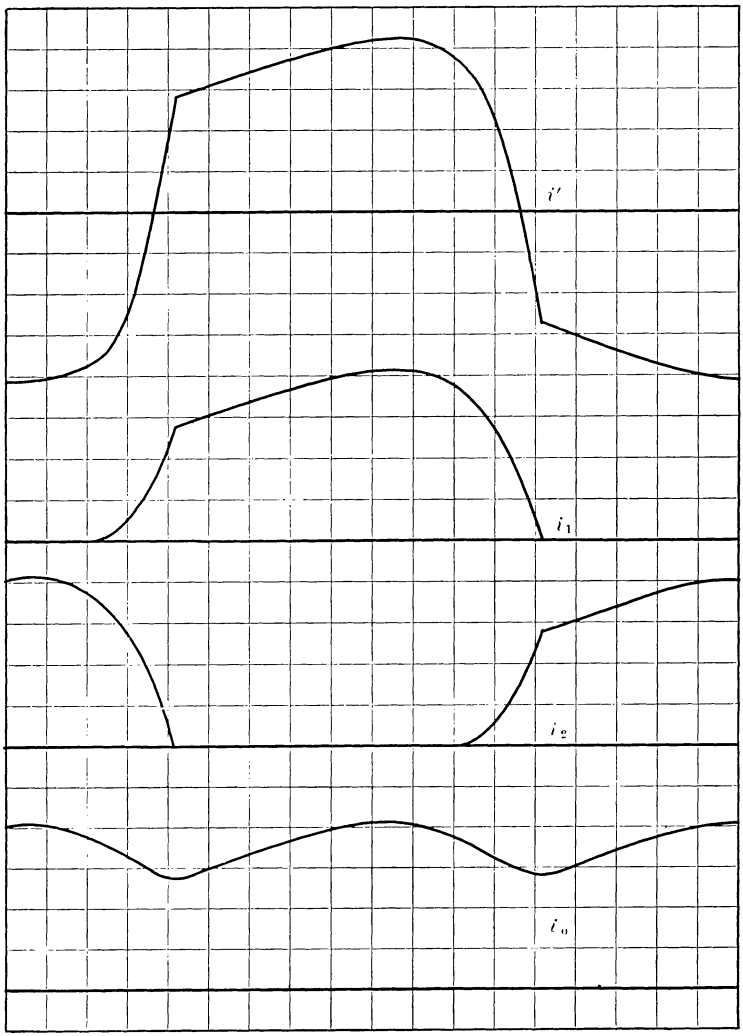


FIG. 18.

i_0' = rectified current during the period: $\omega < \phi < \pi$, where only one arc, or one anode current i_1 exists.

Let:

$e_0 = e_0' + e_0''$ = total counter electromotive force in the rectified circuit.

$Z = r - j\,x = (r_1 + r_0 + r') - j\,(x_1 + x_0 + x')$ = total impedance per circuit.

It is then:

a. During the period where both rectifying arcs exist:

$$0 < \phi < \omega$$
$$i_0 = i_1 + i_2 \tag{1}$$

In the circuit between the electromotive force $2\,e \sin \phi$, the rectifier tube, and the currents i_1 and i_2, it is, by Kirchhoff's law:

$$2\,e \sin \phi - r\,i_1 - x\,\frac{d\,i_1}{d\,\phi} + r\,i_2 + x\,\frac{d\,i_2}{d\,\phi} = 0 \tag{2}$$

It is, in the circuit from transformer neutral over electromotive force $e \sin \phi$, current i_1, rectifier arc e_0'' and rectified circuit i_0, back to the transformer neutral:

$$e \sin \phi - r_1\,i_1 - x_1\,\frac{d\,i_1}{d\,\phi} - e_0'' - r_0\,i_0 - x_0\,\frac{d\,i_0}{d\,\phi} - r^1\,i_0 - x^1\,\frac{d\,i_0}{d\,\phi} - e_0^1 = 0$$

or:

$$e \sin \phi - r_1\,i_1 - x_1\,\frac{d\,i_1}{d\,\phi} - (r_0 + r^1)\,i_0 - (x_0 + x^1)\,\frac{d\,i_0}{d\,\phi} - e_0 = 0 \tag{3}$$

b. During the period where only one rectifying arc exists:

$$\omega < \phi < \pi$$
$$i_1 = i_0'$$

hence, in this circuit:

$$e \sin \phi - r_1\,i_0' - x_1\,\frac{d\,i_0'}{d\,\phi} - (r_0 + r')\,i_0' - (x_0 + x^1)\,\frac{d\,i_0'}{d\,\phi} - e_0 = 0 \tag{4}$$

Substituting (1) in (2) and combining the result (5) of this substitution with (3), gives the

Differential Equations of the Rectifier:

$$2\,e \sin \phi + r_1\,(i_0 - 2\,i_1) + x_1\,\frac{d}{d\,\phi}\,(i_0 - 2\,i_1) = 0 \tag{5}$$

$$2\,e_0 + (2\,r - r_1)\,i_0 + (2\,x - x_1)\,\frac{d\,i_0}{d\,\phi} = 0 \tag{6}$$

$$e \sin \phi - e_0 - r\,i_0' - x\,\frac{d\,i_0'}{d\,\phi} = 0 \tag{7}$$

In these equations, i_0 and i_1 apply for the time: $0 < \phi < \omega$, i_0' for the time: $\omega < \phi < \pi$.

These differential equations are integrated by the functions:

$$i_0 - 2\,i_1 = A\,\varepsilon^{-a\phi} + A' \sin(\phi - \beta) \tag{8}$$

$$i_0 = B\,\varepsilon^{-b\phi} + B' \tag{9}$$

$$i_0' = C\,\varepsilon^{-c\phi} + C' + C'' \sin(\phi - \gamma) \tag{10}$$

Substituting (8), (9), and (10) into (5), (6), and (7) gives three identities:

$$\left.\begin{aligned}
&2\,e \sin\phi + A'[r \sin(\phi - \beta) + x \cos(\phi - \beta)] + A\varepsilon^{-a\phi}\,(r_1 - a x_1) = 0\\
&2\,e_0 + B'\,(2\,r - r_1) + B\,\varepsilon^{-b\phi}\,[(2\,r - r_1) - b\,(2\,x - x_1)] = 0\\
&e \sin\phi - e_0 - C''[r \sin(\phi - \gamma) + x \cos(\phi - \gamma)] - C'r - C\varepsilon^{-c\phi}\,(r - c\,x) = 0
\end{aligned}\right\}$$

hence:

$$\left.\begin{aligned}
r_1 - a\,x_1 &= 0\\
(2\,r - r_1) - b\,(2\,x - x_1) &= 0\\
r - c\,x &= 0
\end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned}
2\,e_0 + B'\,(2\,r - r_1) &= 0\\
e_0 + C'\,r &= 0\\
2\,e + A'\,(r \cos\beta + x \sin\beta) &= 0\\
A'\,(r \sin\beta - x \cos\beta) &= 0\\
e - C''\,(r \cos\gamma + x \sin\gamma) &= 0\\
C''\,(r \sin\gamma - x \cos\gamma) &= 0
\end{aligned}\right\} \tag{11}$$

Denoting:

$$\left.\begin{aligned}
z_1 &= \sqrt{r_1^2 + x_1^2}\\
\tan\alpha_1 &= x_1/r_1
\end{aligned}\right\} \tag{12}$$

and:

$$\left.\begin{aligned}
z &= \sqrt{r^2 + x^2}\\
\tan\alpha &= x/r
\end{aligned}\right\} \tag{13}$$

substituting (12) and (13), gives by resolving the 9 equations (11) the values of the coefficients a, b, c, A', B', C', C'', β, j:

$$\left.\begin{aligned} a &= r_1/x_1 \\ b &= \frac{2r - r_1}{2x - x_1} \\ c &= r/x \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} \beta &= \alpha_1 \\ \gamma &= \alpha \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} A' &= -\frac{2e}{z_1} \\ B' &= -\frac{2e_0}{2r - r_1} \\ C' &= -e_0/r \end{aligned}\right\} \tag{16}$$

$$C'' = +e/z \tag{17}$$

and thus the

Integral Equations of the Rectifier.

$$i_0 - 2i_1 = A\,\varepsilon^{-a\phi} - \frac{2e}{z_1}\sin(\phi - \alpha_1) \tag{18}$$

$$i_0 = B\,\varepsilon^{-b\phi} - \frac{2e_0}{2r - r_1} \tag{19}$$

$$i_0' = C\,\varepsilon^{-c\phi} - \frac{e_0}{r} + \frac{e}{z}\sin(\phi - \alpha) \tag{20}$$

where: a, b, c are given by equations (14)

α and α_1 by equations (12) and (13)

and A, B, C are integration constants given by the terminal conditions of the problem.

These terminal conditions are

$$\left.\begin{aligned} |i_1|_{\phi=0} &= 0 \\ |i_0|_{\phi=0} &= |i_0'|_{\phi=\pi} \\ |i_1|_{\phi=\omega} &= |i_0|_{\phi=\omega} = |i_0'|_{\phi=\omega} \end{aligned}\right\} \tag{21}$$

That is: At $\phi = 0$, the anode current $i_1 = 0$. After half a period, or $\pi = 180°$, the rectified current repeats the same value. At $\phi = \omega$, all three currents i_1, i_0, i_0' are identical.

The four equations (21) determine four constants: A, B, C, ω.

Substituting these constants in equations (18), (19), (20), gives the equations of the rectified current i_0, i_0', and of the anode currents i_1 and $i_2 = i_0 - i_1$, determined by the constants of the system: Z, Z_1, e_0, and by the impressed electromotive force e.

In the constant current mercury arc rectifier system of arc lighting, e, the secondary induced voltage of the constant current transformer, varies with the load, by the regulation of the transformer, and the rectified current, i_0, i_0', is required to remain constant, or rather its average value.

Let then be given as condition of the problem the average value i of the rectified current: 4 amperes in a magnetite or mercury arc lamp circuit, 5 or 6.6 or 9.6 amperes in a carbon arc lamp circuit.

Assuming as fair approximation that the pulsating rectified current i_0, i_0', has its mean value i at the moment: $\phi = 0$. This then gives the additional equation:

$$|i_0|_{\phi=0} = i \tag{22}$$

and from the five equations (21) and (22), the five constants A, B, C, ω, e are determined.

Substituting (22), (18), (19), (20) in equations (21), gives:

$$\left.\begin{aligned} A &= i - \frac{2e}{z_1}\sin\alpha_1 \\ B &= i + \frac{2e_0}{2r - r_1} \\ C &= \varepsilon^{c\pi}\left\{ i + \frac{e_0}{r} - \frac{e}{z}\sin\alpha \right\} \end{aligned}\right\} \tag{23}$$

$$\begin{aligned} -A\,\varepsilon^{-a\omega} - \frac{2e}{r_1}\sin(\alpha_1 - \omega) &= B\,\varepsilon^{-b\omega} - \frac{2e_0}{2r - r_1} \\ &= C\,\varepsilon^{-c\omega} - \frac{e_0}{r} - \frac{e}{z}\sin(\alpha - \omega) \end{aligned} \tag{24}$$

Substituting (23) in (24) gives:

$$\frac{2e}{z_1}\left\{\varepsilon^{-a\omega}\sin\alpha_1-\sin(\alpha_1-\omega)\right\}=i\left\{\varepsilon^{-a\omega}+\varepsilon^{-b\omega}\right\}$$

$$-\frac{2e_0}{2r-r_1}\left\{1-\varepsilon^{-b\omega}\right\} \tag{25}$$

$$\frac{e}{z}\left\{\varepsilon^{c(\pi-\omega)}\sin\alpha+\sin(\alpha-\omega)\right\}=i\left\{\varepsilon^{c(\pi-\omega)}-\varepsilon^{-b\omega}\right\}$$

$$+\frac{2e_0}{2r-r_1}\left\{1-\varepsilon^{-b\omega}\right\}+\frac{e_0}{r}\left\{\varepsilon^{c(\pi-\omega)}-1\right\} \tag{26}$$

and, eliminating e from these two equations, gives:

$$\frac{\varepsilon^{c(\pi-\omega)}\sin\alpha+\sin(\alpha-\omega)}{\varepsilon^{-a\omega}\sin\alpha_1-\sin(\alpha_1-\omega)}=\frac{2z}{z_1}$$

$$\frac{\left\{\varepsilon^{c(\pi-\omega)}-\varepsilon^{-b\omega}\right\}+\frac{2e_0}{i(2r-r_1)}\left\{1-\varepsilon^{-b\omega}\right\}+\frac{e_0}{ir}\left\{\varepsilon^{c(\pi-\omega)}-1\right\}}{\left\{\varepsilon^{-a\omega}+\varepsilon^{-b\omega}\right\}-\frac{2e_0}{i(2r-r_1)}\left\{1-\varepsilon^{-b\omega}\right\}} \tag{27}$$

This equation (27) determines angle ω, and by successive substitution in (26), (23), e, A, B, C are found.

Equation (27) is transcendental, and therefore has to be solved by approximation, which however is very rapid.

As first approximation, $a = b = c = 0$; $\alpha_1 = \alpha_2 = 90°$ or $\pi/2$, and, substituting these values in (27), gives:

$$\frac{\varepsilon^{c\pi}+\cos\omega_1}{1-\cos\omega_1}=\frac{2z}{z_1}\frac{\left(\varepsilon^{c\pi}-1\right)\left(1+\frac{e_0}{ir}\right)}{2}$$

$$\cos\omega_1=\frac{\frac{z}{z_1}\left(\varepsilon^{c\pi}-1\right)\left(1+\frac{e_0}{ir}\right)-\varepsilon^{c\pi}}{\frac{z}{z_1}\left(\varepsilon^{c\pi}-1\right)\left(1+\frac{e_0}{ir}\right)+1} \tag{28}$$

This value of ω_1 substituted in the exponential terms of equation (27), gives a simple trigonometric equation in ω, from which follows the second approximation ω_2, and, by interpolation, the final value:

$$\omega = \omega_2 + \frac{(\omega_2 - \omega_1)^2}{\omega_1} \tag{29}$$

For instance, in the rectifier system of which tests are given in Fig. 4, oscillograms in Figs. 5 to 14, it is, at full load of 25 mercury arc lamps:

$$e_0 = 950$$
$$i = 3.8$$

The constants of the circuit are:

$$Z_1 = 10 - 185\,j$$
$$Z = 50 - 1000\,j$$

Herefrom follows:

$$\left.\begin{aligned} a &= 0.054 \\ b &= 0.050 \\ c &= 0.050 \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} \alpha_1 &= 86.9^\circ \\ \alpha &= 87.1^\circ \end{aligned}\right\} \tag{15}$$

From equation (28), follows as first approximation: $\omega_1 = 47.8^\circ$; as second approximation: $\omega_2 = 44.2^\circ$.

Hence, by (29):

$$\omega = 44.4^\circ$$

observed in $44 \sim 45^\circ$.

Substituting a in (26) gives: $e = 2100$,
hence, the effective value of transformer secondary voltage:

$$\frac{2\,e}{\sqrt{2}} = 2980 \text{ volts}$$

and, from (23):

$$A = -\ 18.94$$
$$B = \ 24.90$$
$$C = \ 24.20$$

Therefore, the equations of the currents:

$$i_0 = 24.90\,\varepsilon^{-.050\phi} - 21.10$$

$$i_0' = 24.20\,\varepsilon^{-.050\phi} - 19.00 + 2.11 \sin(\phi - 87.1^\circ)$$

$$i_1 = 12.45\,\varepsilon^{-.050\phi} + 9.47\,\varepsilon^{-.054\phi} - 10.58 + 11.35 \sin(\phi - 86.9^\circ)$$

$$i_2 = i_0 - i_1$$

The effective or equivalent alternating secondary current of the transformer, which corresponds to the primary load current, that is, primary current minus exciting current, is:

$$i' = i_1 - i_2$$

From these equations are calculated the numerical values of rectified current i_0, i_0', of anode current i_1, and of alternating current i', and plotted as curves in Fig. 18. As seen, they perfectly agree with the oscillograms.