On a Method in the Analysis of Ternary Forms: Mr. J. J. Walker, F.R.S.

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Uni-Brocardal Triangles and their Inscribed Triangles: Mr. R. Tucker, M.A.

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Harmonic Decomposition of Functions and some Allied Expansions

By A. R. JOHNSON, M.A.

[Read December 8th, 1887.]

The general method of procedure for the solution of Green's problem in the case of the sphere necessitates integrations over the surface of the sphere, or else, when the function of the coordinates that expresses the given surface potential is rational and integral, requires the decomposition of the function into surface harmonics by

the method of indeterminate coefficients. It is the object of this paper to furnish methods of solution that shall be free from these operations, when the expression for the surface potential is a rational and integral function of the coordinates. The solution is effected, firstly in a determinantal form, secondly in the form of a finite series. When the function that expresses the surface potential is unrestricted in form, the analogous method of solution requires an infinite series, which the paper gives if certain conditions are fulfilled. One of the chief results of the paper is a formula that enables us to separate at once any rational and integral function into its harmonic components at the surface of the sphere of reference.

1. Let $f_n(x, y, z)$ denote a rational integral and homogeneous function of degree n , and let V_s denote a solid harmonic of order s . Then it is well known that $f_n(x, y, z)$ can be expressed in the form

$$f_n(x, y, z) = V_n + \left(\frac{r}{a}\right)^2 V_{n-2} + \left(\frac{r}{a}\right)^4 V_{n-4} + \dots \dots \dots (1),$$

where the last term is

$$\left(\frac{r}{a}\right)^n V_0 \text{ or } \left(\frac{r}{a}\right)^{n-1} V_1,$$

according as n is even or odd. Performing the operation ∇^2 on this identity, once, twice, thrice, etc., and remarking that

$$\nabla^2 r^m V_n = m(m+2n+1) r^{m-2} V_n,$$

there result the relations

$$\left. \begin{aligned} & a^2 \nabla^2 f_n(x, y, z) \\ &= 2(2n-1) V_{n-2} + 4(2n-3) \left(\frac{r}{a}\right)^2 V_{n-4} + 6(2n-5) \left(\frac{r}{a}\right)^4 V_{n-6} + \dots, \\ & a^4 \nabla^4 f_n(x, y, z) \\ &= 4 \cdot 2(2n-3)(2n-5) V_{n-4} + 6 \cdot 4(2n-5)(2n-7) \left(\frac{r}{a}\right)^2 V_{n-6} + \dots, \\ & a^6 \nabla^6 f_n(x, y, z) \\ &= 6 \cdot 4 \cdot 2(2n-5)(2n-7)(2n-9) V_{n-6} \\ & \quad + 8 \cdot 6 \cdot 4(2n-7)(2n-9)(2n-11) \left(\frac{r}{a}\right)^2 V_{n-8} + \dots, \\ & \dots \dots \dots (2), \end{aligned} \right\}$$

and so on, the last relation being

$$a^n \nabla^n f_n(x, y, z) = (n+1)n(n-1)(n-2) \dots V_0 = (n+1)! V_0,$$

or

$$a^{n-1} \nabla^{n-1} f_n(x, y, z)$$

$$= (n+2)(n-1) n (n-3) \dots 5 \cdot 2 \cdot V_1 = \frac{(n+2) n!}{3} V_1,$$

according as n is even or odd.

If the given potential at the surface of the sphere whose centre is the origin and whose radius is a , be $f_n(x, y, z)$, then putting $r = a$, in the relations (1) and (2), we have a system of relations which hold true at the surface of the sphere. Let the radius vector ρ to the point $\xi\eta\zeta$ meet the sphere at the point xyz . Then the potential at $\xi\eta\zeta$ within the sphere is

$$V = \left(\frac{\rho}{a}\right)^n V_n + \left(\frac{\rho}{a}\right)^{n-2} V_{n-2} + \dots,$$

where in V_n, V_{n-2} , etc., $r = a$.

Eliminating V_n, V_{n-2} , etc., between this relation and the results of putting $r = a$, in (1) and (2), we have V given by the determinantal equation

0 =

$$\begin{vmatrix} V, & \left(\frac{\rho}{a}\right)^n, & \left(\frac{\rho}{a}\right)^{n-2}, & \left(\frac{\rho}{a}\right)^{n-4}, & \left(\frac{\rho}{a}\right)^{n-6}, & \dots \\ f_n, & 1, & 1, & 1, & 1, & \dots \\ a^2 \nabla^2 f_n, & . & (2,1)(2n-1,1), & (4,1)(2n-3,1), & (6,1)(2n-5,1), & \dots \\ a^4 \nabla^4 f_n, & . & . & (4,2)(2n-3,2), & (6,2)(2n-5,2), & \dots \\ a^6 \nabla^6 f_n, & . & . & . & (6,3)(2n-5,3), & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

.....(A),

the last row being

$$a^n \nabla^n f_n, \dots, (n+1)! \text{ or } a^{n-1} \nabla^{n-1} f_n, \dots, \dots, \frac{(n+2) n!}{3},$$

according as n is even or odd; and the last column

$$1, 1, (n+1) n, (n+1) n (n-1)(n-2), \text{ etc.};$$

or

$$\frac{\rho}{a}, 1, (n+2)(n-1), (n+2)(n-1) n (n-3), \text{ etc.};$$

according as n is even or odd.

In the determinant (A), (p, s) denotes $p(p-2)(p-4) \dots$ to s factors. The expression for the potential in space external to the sphere

is obtained by writing $\left(\frac{a}{\rho}\right)^{s+1}$ for $\left(\frac{\rho}{a}\right)^s$ in (A). Consequently the required distribution is obtained by replacing V by $4\pi a\sigma$, and $\left(\frac{\rho}{a}\right)^s$ by $2s+1$, in (A), when σ will be the required density of distribution.

If the potential at the surface of the sphere be given as a rational and integral function of x, y, z of degree n , it may be expressed in the form $f_n + f_{n-1}$ by inserting powers of $\frac{x^2+y^2+z^2}{a^2}$, and then the internal and external potentials and the surface density may be at once written down by a double application of the preceding results.

This method is useful for resolving $f_n(x, y, z)$ into surface harmonics, for the coefficient of $\left(\frac{\rho}{a}\right)^s$ in the expression obtained for V from (A) is V_s .

E.g., To resolve f_4 into surface harmonics ($r = 1$).

$$\text{From (A),} \quad 0 = \begin{vmatrix} V, & \rho^4, & \rho^3, & 1 \\ f_4, & 1, & 1, & 1 \\ \nabla^2 f_4, & 0, & 14, & 20 \\ \nabla^4 f_4, & 0, & 0, & 120 \end{vmatrix}$$

whence we have

$$f_4 = S_4 + S_2 + S_0,$$

where

$$S_4 = f_4 - \frac{1}{14}\nabla^2 f_4 + \frac{1}{280}\nabla^4 f_4,$$

$$S_2 = \frac{1}{14}\nabla^2 f_4 - \frac{1}{84}\nabla^4 f_4,$$

$$S_0 = \frac{1}{120}\nabla^4 f_4.$$

The same method is applicable to p variables instead of only three; care being taken to write, in (A), $(2s, q)(2n-2s+p-2, q)$ instead of $(2s, q)(2n-2s+1, q)$ in the element belonging to the $(q+2)^{\text{th}}$ row and $(s+2)^{\text{th}}$ column.

2. But the solution may be effected in another manner, which will in general be found more convenient than the foregoing method. Put

$$V = \phi_0 f_n + \phi_2 \nabla^2 f_n + \phi_4 \nabla^4 f_n + \dots \dots \dots (B),$$

where the ϕ 's represent functions of r at present undetermined.

Then

$$\begin{aligned}\nabla^2 V &= \phi_0 \nabla^2 f_n + \phi_2 \nabla^4 f_n + \phi_4 \nabla^6 f_n + \dots \\ &+ \frac{1}{r} \frac{d^2 (r\phi_0)}{dr^2} f_n + \frac{1}{r} \frac{d^2 (r\phi_2)}{dr^2} \nabla^2 f_n + \frac{1}{r} \frac{d^2 (r\phi_4)}{dr^2} \nabla^4 f_n + \dots \\ &+ \frac{2n}{r} \frac{d\phi_0}{dr} f_n + \frac{2n-2}{r} \frac{d\phi_2}{dr} \nabla^2 f_n + \frac{2n-4}{r} \frac{d\phi_4}{dr} \nabla^4 f_n + \dots\end{aligned}$$

Consequently, if we determine the ϕ 's so that

$$0 = \frac{d^2 (r\phi_0)}{dr^2} + 2n \frac{d\phi}{dr} \dots\dots\dots (5),$$

$$\text{and} \quad 0 = \frac{d^2}{dr^2} (r\phi_{2s}) + 2(n-2s) \frac{d\phi_{2s}}{dr} + r\phi_{2s-2} \quad (s = 1, 2, 3, \text{etc.}) \dots (6),$$

the condition $\nabla^2 V = 0$ is then satisfied.

From equation (5), we have

$$\phi_0 = Or^{-2n-1} + O',$$

$$\text{and from (6),} \quad \phi_{2s} = - \int_a^r dr r^{-2n+4s-2} \int_0^r dr r^{2n-4s+2} \phi_{2s-2}.$$

The constants of integration remain still at our disposal. We choose them so that the ϕ 's are finite when $r = 0$, and each ϕ except ϕ_0 vanishes when $r = a$, and $\phi_0 = 1$, when $r = a$.

Thus $\phi_0 = 1$,

$$\phi_2 = - \int_a^r dr r^{-2n+2} \int_0^r dr r^{2n-2},$$

$$\phi_4 = \int_a^r dr r^{-2n+6} \int_0^r dr r^{2n-6} \int_a^r dr r^{-2n+2} \int_0^r dr r^{2n-2},$$

and in general

$$\begin{aligned}\phi_{2s} &= (-1)^s \int_a^r dr r^{-2n+4s-2} \int_0^r dr r^{2n-4s+2} \int_a^r dr r^{-2n+4s-6} \int_0^r dr r^{2n-4s+6} \dots \\ &\dots \int_a^r dr r^{-2n+6} \int_0^r dr r^{2n-6} \int_a^r dr r^{-2n+2} \int_0^r dr r^{2n-2} \dots\dots\dots (7).\end{aligned}$$

Substituting the values of ϕ found from (7) in (B), it is seen that V satisfies Laplace's equation, is finite within the sphere, and is equal to $f_n(x, y, z)$ at the surface. V , therefore, is the potential within the sphere.

The following are the calculated developments of the first few ϕ 's:—

$$\phi_0 = 1,$$

$$\phi_2 = \frac{1}{2(2n-1)} \{a^3 - r^3\},$$

$$\phi_4 = \frac{1}{2^3(2n-1)(2n-3)(2n-5)} \{a^4(2n-1) - 2a^2r^2(2n-3) + r^4(2n-5)\},$$

$$\phi_6 = \frac{2^3(n-1)!(2n-11)!}{3!(2n-1)!(n-6)!} \{a^6(2n-1)(2n-3) - 3a^4r^2(2n-1)(2n-7) \\ + 3a^2r^4(2n-3)(2n-9) - r^6(2n-7)(2n-9)\},$$

$$\phi_8 = \frac{2^5(n-1)!(2n-15)!}{4!(2n-1)!(n-8)!} \{a^8(2n-1)(2n-3)(2n-5) \\ - 4a^6r^2(2n-1)(2n-3)(2n-11) + 6a^4r^4(2n-1)(2n-7)(2n-13) \\ - 4a^2r^6(2n-3)(2n-11)(2n-13) + r^8(2n-9)(2n-11)(2n-13)\},$$

$$\phi_{10} = \frac{2^4(n-1)!(2n-19)!}{5!(2n-1)!(n-10)!} \{a^{10}(2n-1)(2n-3)(2n-5)(2n-7) \\ - 5a^8r^2(2n-1)(2n-3)(2n-5)(2n-15) \\ + 10a^6r^4(2n-1)(2n-3)(2n-11)(2n-17) \\ - 10a^4r^6(2n-1)(2n-7)(2n-15)(2n-17) \\ + 5a^2r^8(2n-3)(2n-13)(2n-15)(2n-17) \\ - r^{10}(2n-11)(2n-13)(2n-15)(2n-17)\},$$

$$\phi_{12} = \frac{2^5(n-1)!(2n-23)!}{6!(2n-1)!(n-12)!} \{a^{12}(2n-1)(2n-3)(2n-5)(2n-7)(2n-9) \\ - 6a^{10}r^2(2n-1)(2n-3)(2n-5)(2n-7)(2n-19) \\ + 15a^8r^4(2n-1)(2n-3)(2n-5)(2n-15)(2n-21) \\ - 20a^6r^6(2n-1)(2n-3)(2n-11)(2n-19)(2n-21) \\ + 15a^4r^8(2n-1)(2n-7)(2n-17)(2n-19)(2n-21) \\ - 6a^2r^{10}(2n-3)(2n-15)(2n-17)(2n-19)(2n-21) \\ + r^{12}(2n-13)(2n-15)(2n-17)(2n-19)(2n-21)\}.$$

From the preceding values that of ϕ_{2s} is inferred to be the product
of

$$\frac{2^{s-1}(n-1)!(2n-4s+1)!}{s!(2n-1)!(n-2s)!}$$

and a factor formed in the following manner:—Write down in order the terms of the expansion of $(a^2 - r^2)^s$. To the first term apply the multiplier

$$(2n-1)(2n-3) \dots (2n-\overline{2s-7})(2n-\overline{2s-5})(2n-\overline{2s-3});$$

to the second term apply a multiplier obtained from the preceding one by doubling the part $\overline{2s-3}$ of the last factor and adding unity, thus—

$$(2n-1)(2n-3) \dots (2n-\overline{2s-7})(2n-\overline{2s-5})(2n-\overline{4s-5});$$

to the third term apply a multiplier obtained by changing the last but one factor in the same way and subtracting 2 from the last factor, thus—

$$(2n-1)(2n-3) \dots (2n-\overline{2s-7})(2n-\overline{4s-9})(2n-\overline{4s-3});$$

to the fourth term apply a multiplier obtained from the preceding one by changing the part $\overline{2s-7}$ into twice itself + unity, and by augmenting the last but one factor so as to exceed the last factor by 2, thus—

$$(2n-1)(2n-3) \dots (2n-\overline{4s-13})(2n-\overline{4s-5})(2n-\overline{4s-3}).$$

Repeat this process and find multipliers to the other terms. Thus the next multiplier and the last two will be

$$(2n-1)(2n-3) \dots (2n-\overline{4s-7})(2n-\overline{4s-5})(2n-\overline{4s-3}),$$

$$\text{and } (2n-3)(2n-\overline{2s+3}) \dots (2n-\overline{4s-7})(2n-\overline{4s-5})(2n-\overline{4s-3}),$$

$$(2n-\overline{2s+1})(2n-\overline{2s+3}) \dots (2n-\overline{4s-7})(2n-\overline{4s-5})(2n-\overline{4s-3}).$$

We may also write ϕ_n in the form

$$\phi_n = \sum_{p=0}^{p=s} (-1)^{s-p} \frac{2(2n-4s+4p+1)}{p!(s-p)!} \frac{(2n-4s+2p-1)!(n-s+p)!}{(2n-2s+2p+1)!(n-2s+p-1)!} \\ \times a^{2s-2p} r^{2p} \dots \dots \dots (7').$$

The correctness of this inferred form can be established by means of the form given in (7), by the method of mathematical induction.

3. We might proceed to find the potential in the space external to the sphere by starting with the solution

$$\phi_0 = \left(\frac{a}{r} \right)^{2n+1}$$

But the solution by the aid of an elementary principle in the theory of the potential, may be written down at once from (B) thus:—

Let ϕ'_2 , denote the result of interchanging a and r in ϕ_{2s} . Then, in external space,

$$V = \left(\frac{a}{r}\right)^{2n+1} f_n + \left(\frac{a}{r}\right)^{2n-1} \phi'_2 \nabla^2 f_n + \left(\frac{a}{r}\right)^{2n-3} \phi'_4 \nabla^4 f_n + \text{etc.} \dots (B');$$

for to the term $Ka^{2s-2p} r^{2p} \nabla^{2s} f_n$ of $\phi_{2s} \nabla^{2s} f_n$ corresponds in the external potential a term

$$\begin{aligned} & \left(\frac{a}{r}\right)^{2(2p+n-2s)+1} Ka^{2s-2p} r^{2p} \nabla^{2s} f_n \\ &= \left(\frac{a}{r}\right)^{2n-2s+1} K \left(\frac{a}{r}\right)^{4p-2s} a^{2s-2p} r^{2p} \nabla^{2s} f_n \\ &= \left(\frac{a}{r}\right)^{2n-2s+1} K r^{2s-2p} a^{2p} \nabla^{2s} f_n. \end{aligned}$$

From (B) and (B') we deduce the density of the distribution whose potential over the surface of the sphere $r = a$ is f , to be

$$\sigma = \frac{1}{4\pi a} \{ \psi_0 f_n + \psi_2 \nabla^2 f_n + \psi_4 \nabla^4 f_n + \dots \},$$

where

$$\psi_{2s} = a \left[\frac{d\phi_{2s}}{dr} - \frac{d\phi'_{2s}}{dr} \right]_{r=a}.$$

Hence, we calculate

$$\begin{aligned} \psi_2 &= -\frac{2}{2n-1} a^3, \\ \psi_4 &= -\frac{2^3}{(2n-1)(2n-3)(2n-5)} a^5, \\ \psi_6 &= -\frac{2^7 (n-1)! (2n-11)!}{(2n-1)! (n-6)!} a^7, \\ \psi_8 &= -\frac{2^8 \cdot 5 (n-1)! (2n-15)!}{(2n-1)! (n-8)!} a^9, \\ \psi_{10} &= -\frac{2^{10} \cdot 8! (n-1)! (2n-19)!}{4! 5! (2n-1)! (n-10)!} a^{11}, \\ \psi_{12} &= -\frac{2^{12} \cdot 10! (n-1)! (2n-23)!}{5! 6! (2n-1)! (n-12)!} a^{13}. \end{aligned}$$

And generally

$$\begin{aligned} \psi_{2s} &= \frac{2^{s-1} (n-1)! (2n-4s+1)!}{s! (2n-1)! (n-2s)!} \\ &\times \{ -2s (2n-1)(2n-3) \dots (2n-\overline{2s-7})(2n-\overline{2s-5})(2n-\overline{2s-3}) \\ &+ s (2s-4)(2n-1)(2n-3) \dots (2n-\overline{2s-7})(2n-\overline{2s-5})(2n-\overline{4s-5}) + \text{etc.} \\ &+ (-1)^{s-1} s (2s-4)(2n-3)(2n-\overline{2s+3}) \dots (2n-\overline{4s-7})(2n-\overline{4s-5})(2n-\overline{4s-3}) \\ &+ (-1)^s 2s (2n-\overline{2s+1})(2n-\overline{2s+3}) \dots (2n-\overline{4s-7})(2n-\overline{4s-5})(2n-\overline{4s-3}) \}. \end{aligned}$$

From the calculations of $\psi_s, \psi_4, \psi_6 \dots \psi_{12}$, we infer that the part within parentheses is independent of $2n$. Put, therefore, in that part $2n=1$. Thus it is seen that

$$\psi_{2s} = - \frac{2^{2s} (2s-2)! (n-1)! (2n-4s+1)!}{s! (s-1)! (2n-1)! (n-2s)!} a^{2s}.$$

Hence, summarily,

$$\sigma = \frac{1}{4\pi a} \left\{ (2n+1) f_n - \frac{(n-1)!}{(2n-1)!} \sum_{s=1}^n \frac{2^{2s} (2s-2)! (2n-4s+1)!}{s! (s-1)! (n-2s)!} a^{2s} \nabla^{2s} f_n \right\} \dots\dots\dots (C).$$

If then the potential at the surface of the sphere be given as a rational and integral function of the coordinates, we may express it in the form $f_n + f_{n-1}$, and a double application of (B), (B'), and (C) gives at once the internal and external potentials, and the required density of distribution.

It is noteworthy that, by grouping the terms of the same dimensions in (B), we are enabled to resolve f_n into its component surface harmonics. To do this, we note that the coefficient of

$$r^{2p} \nabla^{2s} f_n \text{ in } \phi_{2s} \nabla^{2s} f_n$$

is

$$(-1)^{s-p} \frac{2 (2n-4s+4p+1)}{p! (s-p)!} \frac{(2n-4s+2p-1)! (n-s+p)!}{(2n-2s+2p+1)! (n-2s+p-1)!} a^{2s-2p}$$

To find, therefore, the terms of dimensions $n-2m$ in (B), we must put $n-2m = n-2s+2p$, i.e., $p = s-m$, and sum with respect to s . Hence the harmonic component of f_n of order $n-2m$, relative to the sphere $r = a$, is

$$\left(-\frac{a^2}{r^3} \right)^m \frac{2 (2n-4m+1)(n-m)!}{m! (2n-2m+1)!} \sum_{s=m}^n \frac{(2n-2s-2m-1)!}{s! (s-m)! (n-s-m-1)!} (-r^2)^s \nabla^{2s} f \dots\dots\dots (a).$$

This result enables us to decompose any rational and integral function into its harmonic components at the surface of the sphere of reference.

When n is even, we must except the constant component which comes

out as
$$\frac{1}{(n+1)!} a^n \nabla^n f_n.$$

5. We now propose to determine the solution of Laplace's equation that satisfies the conditions

$$V = f_n, \quad (r = a); \quad V = f_m, \quad (r = b).$$

Let V_1 and V_2 be solutions of Laplace's equation, such that

$$V_1 = f_n, \quad (r = a); \quad V_1 = 0, \quad (r = b);$$

$$V_2 = 0, \quad (r = a); \quad V_2 = f_m, \quad (r = b).$$

Then

$$V = V_1 + V_2.$$

Put

$$V = \chi_0 f_n + \chi_2 \nabla^2 f_n + \chi_4 \nabla^4 f_n + \chi_6 \nabla^6 f_n + \dots \dots \dots (D),$$

where the χ 's are determined by relations analogous to (5) and (6).

Take for the value of χ_0 ,

$$\chi_0 = \frac{r^{-2n-1} - b^{-2n-1}}{a^{-2n-1} - b^{-2n-1}},$$

so that

$$\chi_0 = 1, \quad (r = a); \quad \chi_0 = 0, \quad (r = b).$$

The constants of integration of the remaining χ 's must be determined so as to make them vanish when $r = a$, $r = b$. We then find, employ-

ing the notation

$$u_s \equiv \frac{r^{-s} - b^{-s}}{a^{-s} - b^{-s}},$$

$$\chi_0 = u_{2n+1},$$

$$\chi_2 = \frac{1}{2(2n-1)} \{u_{2n-3} a^2 - u_{2n+1} r^2\},$$

$$\chi_4 = \frac{1}{2^3(2n-1)(2n-3)(2n-5)}$$

$$\times \{u_{2n-7} a^4 (2n-1) - u_{2n-3} 2a^2 r^2 (2n-3) + u_{2n+1} r^4 (2n-5)\}.$$

And, in general, the value of χ_{2s} is found by prefixing the several terms of ϕ_{2s} as found by the law of formation in § 2, with $u_{2n-4s+1}$, $u_{2n-4s-3}$, $u_{2n-4s-7}$, ... u_{2n+1} taken in order. Thus V_1 is completely determined; and V_2 may be written down in the same way by making the necessary changes from a to b and from n to m . Consequently $V = V_1 + V_2$ is completely determined.

The solution of the problem of finding the distributions on two spherical surfaces, that shall produce a potential f_n at the outer ($r = a$) and a potential f_m at the inner ($r = b$), may now be written down. The required density of distribution on the surface $r = a$ is $\sigma + \sigma' + \sigma''$; where σ is given by (C), σ' is the value of $\frac{1}{4\pi a} V_1$ found by interpreting u , as $\frac{-sa^{-s}}{a^{-s}-b^{-s}}$, and σ'' is the result of substituting $\frac{-sa^{-s}}{a^{-s}-b^{-s}}$ for v , in the expression for $\frac{V_2}{4\pi a}$ (v , in the first place standing for $\frac{r^{-s}-a^{-s}}{b^{-s}-a^{-s}}$). The required density of distribution over ($r=b$) is $\sigma'_1 + \sigma''_1$; where σ'_1 is found by interpreting the values of u , and v , in the expressions for $\frac{V_1}{4\pi b}$, $\frac{V_2}{4\pi b}$ to mean $\frac{-sb^{-s}}{a^{-s}-b^{-s}}$, and $\frac{sb^{-s}}{a^{-s}-b^{-s}}$, respectively.

6. The series given above are inapplicable when the surface potential is not expressible as the sum of a finite number of rational, integral and homogeneous functions. When this impossibility exists, suppose that the surface potential is put into the form $V = f(x, y, z)$, where f denotes a homogeneous function of degree zero, such that neither it nor any of the functions $\nabla^{2s}f$ become infinite at any real point on the sphere.

We shall first investigate a series for the external potential. As before, assume

$$V = \phi_0 f + \phi_1 \nabla^2 f + \phi_2 \nabla^4 f + \text{etc.} \dots \dots \dots (\text{E}).$$

Then we have, as before, the relations

$$0 = \frac{1}{r} \frac{d^3}{dr^3} (r\phi_{2s}) - \frac{4s}{r} \frac{d\phi_{2s}}{dr} + \phi_{2s-2} \dots \dots \dots (8),$$

when $s = 1, 2, 3$, etc.; and

$$0 = \frac{1}{r} \frac{d^3}{dr^3} (r\phi_0) \dots \dots \dots (9)$$

From (9), we get $\phi_0 = \frac{a}{r},$

and from (8)

$$\begin{aligned} \phi_{2s} = & a \int_a^r dr r^{4s-2} \int_r^\infty dr r^{-4s+2} \int_a^r dr r^{4s-6} \int_r^\infty dr r^{-4s+6} \dots \\ & \dots \int_a^r dr r^6 \int_r^\infty dr r^{-6} \int_a^r dr r^2 \int_r^\infty dr r^{-2} \dots \dots \dots (10). \end{aligned}$$

Or we may say at once that

$$\phi_s = \sum_{p=0}^{s-1} (-1)^{s+p} \frac{(2s-p-1)! (2s-2p-2)!}{s! (s-p-1)! (4s-2p-1)!} (4s-4p-1) a^{2s-2p} r^{2p} \\ + \frac{1}{(2s)!} a r^{2s-1} \dots\dots\dots (11').$$

If we start with the solution $\phi_0 = 1$, and determine the constants of integration of the other ϕ 's, so that every term of the series shall be of positive dimensions in the variables, and, except the first, shall vanish when $r = a$, we get the expression for the internal potential,

$$\text{viz.,} \quad V = f + \sum_{s=1} \left(\frac{r}{a} \right)^{2s} \phi'_s \nabla^{2s} f \dots\dots\dots (E');$$

where ϕ'_s denotes the result of interchanging a and r in ϕ_{2s} . For the density of distribution, we get

$$\sigma = \frac{1}{4\pi a} \left[f - \sum_{s=1} \left\{ \frac{2}{(2s)!} - \frac{2^{2s-1} \{(2s-2)!\}^3}{s! (s-1)! (4s-3)!} \right\} a^{2s} \nabla^{2s} f \right].$$

7. Lastly, for the solution of Laplace's equation, which satisfies the conditions $V = f$, ($r = a$); $V = 0$, ($r = b$), we get

$$V = \chi_0 f + \chi_3 \nabla^3 f + \chi_4 \nabla^4 f + \dots \dots\dots (F);$$

where we find

$$\chi_0 = u_1,$$

$$\chi_3 = -\frac{1}{1.2} \frac{r}{a} \{r^2 u_3 - ar \cdot u_1\},$$

$$\chi_4 = \frac{1}{2^3.3.5} \left(\frac{r}{a} \right)^3 \{r^4 u_7 - 6a^2 r^2 u_3 + 5a^3 r u_1\},$$

$$\chi_6 = -\frac{1}{2^4.3^3.5.7} \left(\frac{r}{a} \right)^5 \{r^6 u_{11} - 7a^2 r^4 u_7 + 27a^4 r^2 u_3 - 21a^5 r u_1\};$$

where

$$u_s = \frac{r^{-s} - b^{-s}}{a^{-s} - b^{-s}}.$$

Hence we infer that

$$\chi_{2s} = \sum_{p=0}^{s-1} (-1)^{s+p} \frac{(2s-p-1)! (2s-2p-2)!}{s! (s-p-1)! (4s-2p-1)!} (4s-4p-1) a^{2p} r^{2s-2p} u_{4s-4p} \\ + \frac{1}{(2s)!} a r^{2s-1} u_1.$$

In the same way we may write down the solution of $\nabla^2 V = 0$, which satisfies $V = F$, ($r = b$); $V = 0$, ($r = a$). Adding this to the first solution of this article, we get the expression for the solution of Laplace's equation that satisfies the conditions $V = f$, ($r = a$); $V = F$, ($r = b$). A series for the density of distribution may then be readily derived, but the result is not very elegant.

8. The legitimate employment of the series (E) requires the negligibility of $\phi_n \nabla^{2s+2} f$ when s is infinite, and, secondly, the convergence of the series. If, however, for some finite value of s , $\nabla^{2s} f = 0$, the series becomes a finite one and may be safely employed. On splitting (E) into groups of terms of the same dimensions, there occur groups of order $-2m$, ($m = 1, 2, 3$, etc.), viz.,

$$(-1)^m \frac{(4m-1)(2m-2)!}{(m-1)!} \left(\frac{a^2}{r^2}\right)^m \sum_{s=m}^{\infty} \frac{(-1)^s (s+m-1)!}{s! (2s+2m-1)!} r^{2s} \nabla^{2s} f;$$

and but one group of odd dimensions, viz.,

$$\frac{a}{r} \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s)!} r^{2s} \nabla^{2s} f.$$

If we start with the solution $\phi_0 = 1$, and determine the constants of integration of the other ϕ 's, so that every term of the series shall be of positive dimensions in the variables and, except the first, shall vanish when $r = a$, we get the expression for the internal potential. The result is

$$V = f + \sum_{s=1}^{\infty} \left(\frac{r}{a}\right)^{2s} \phi'_s \nabla^{2s} f \dots \dots \dots (F');$$

where ϕ'_s denotes the result of interchanging a and r in (11), and multiplying by $\frac{(-1)^s 2^{s-2} (2s-2)!}{s! (4s-3)!}$.

9. By writing $2n+d-3$ in place of $2n$, we shall adapt our series to the case of d variables.

E.g., when $r^2 \equiv x_1^2 + x_2^2 + \dots + x_d^2 = 1$,

$$f_n(x_1, x_2, \dots, x_n) = S_n + S_{n-2} + \dots + S_{n-2m} + \text{etc.};$$

where $S_{n-2m} = \left(-\frac{1}{r^2}\right)^m \frac{2(2n-4m+d-2)(n-m+\frac{1}{2}[d-3])!}{m!(2n-2m+d-2)!}$

$$\times \sum_{s=m}^{\infty} \frac{(2n-2s-2m+d-4)!}{(s-m)!(n-s-m+\frac{1}{2}[d-5])!} (-r^2)^s \nabla^{2s} f_n$$

when d is odd, and

$$= \left(-\frac{1}{r^3}\right)^m \frac{1}{m!} \sum_{s=m}^{\infty} \frac{1}{2^{2s+2}} \frac{(n-s-m+\frac{1}{2}d-2)!}{(s-m)!(n-m+\frac{1}{2}d-1)!} (-r^3)^s \nabla^{2s} f_n$$

when d is even; each S satisfying the condition

$$\nabla^2 S \equiv \left(\frac{d^2}{dx_1^2} + \frac{d^2}{dx_2^2} + \dots + \frac{d^2}{dx_d^2}\right) S = 0.$$

[ADDENDUM.—It may be noticed that the series (E) is convergent if the series $\sum_{s=1}^{\infty} \frac{1}{(2s-1)!} r^{2s-2} \nabla^{2s} f$ is so, a series which is the development of the mean value of $\nabla^2 f$ over the surface of a sphere of radius r described round xyz as centre. For, comparison of (11) and (11') shows that each term of the development of ϕ_{2s} is numerically less than $\frac{1}{(2s)!} ar^{2s-1}$, and the sum of the first two terms when s is large is less than $\frac{1}{(2s)!} ar^{2s-1}$. Hence ϕ_{2s} is numerically less than

$$\frac{ar}{2} \frac{1}{(2s-1)!} r^{2s-2}$$

when s is large, and the s^{th} term of the series (E) is numerically less than the s^{th} term of the series

$$\frac{ar}{2} \sum_{s=1}^{\infty} \frac{1}{(2s-1)!} r^{2s-2} \nabla^{2s} f,$$

when s is large.

Moreover, since from (E),

$$\nabla^2 V = Lt_{s=\infty} \phi_{2s} \nabla^{2s+2} f,$$

and therefore

$$< \frac{a}{2r} Lt_{s=\infty} \frac{r^{2s}}{(2s-1)!} \nabla^{2s+2} f,$$

we see that $\nabla^2 V = 0$ will be satisfied, if the series

$$\sum_{s=1}^{\infty} \frac{1}{(2s-1)!} r^{2s-2} \nabla^{2s} f$$

is as convergent as a series $\sum_{s=1}^{\infty} \frac{1}{s^\alpha}$, where α is some positive quantity greater than 2. Therefore, when this is the case, the series (E) may always be legitimately employed.]