On the Rectification of Certain Curves. By R. A. ROBERTS, M.A.

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1. I purpose in this paper to give some results concerning the rectification of algebraic curves, both plane and twisted. Amongst these is the rectification of a series of twisted unicursal curves, whose arcs in particular cases are shown to afford a very general representation of elliptic and hyper-elliptic integrals. I arrive at several theorems also involving geometrical relations between the arcs of curves in cases in which these arcs are expressible by elliptic integrals.

2. Before proceeding, however, to the main subject of this paper, it may be of interest to give a brief account of what has been done in the subject of the rectification of curves, and to mention those points which have been of the most interest to geometers. J. Bernoulli and Euler appear to have been the first to consider the question of finding algebraic curves whose arcs should represent given integrals. The latter discovered certain curves whose arcs, measured from a fixed point, are equal to a single circular function (see "New Memoirs of St. Petersburg," t. xi., and Serret's "Cours de Calcul Différentiel et Intégral," t. ii., p. 252). These curves derived their interest from the fact that their arcs could be multiplied and divided algebraically, exactly in the same manner as those of the circle.

3. Legendre then, after having noticed that the arc of the lemniscate of Bernoulli was expressible by an elliptic integral of the first kind with a modulus equal to $\sqrt{\frac{1}{2}}$ (a result first given by Fagnani), appears to have attempted to find a curve whose arc could be expressed by a similar integral with an arbitrary modulus, as by the theory of these functions the arc of such a curve could be multiplied and divided algebraically. In this he was unsuccessful, his nearest approach to a solution being the curve enveloped by a perpendicular to the central radius vector of an ellipse at its extremity. This curve is of the fourth class and sixth degree. The actual determination of its Cartesian equation baffled Legendre, and was accomplished afterwards by Tortolini (*Crelle*, t. 32, p. 90); but the result can be easily obtained by taking the discriminant of a binary quartic. The solu-

tion is, however, evidently imperfect, as the arc of the curve is an elliptic integral together with an algebraic quantity, and therefore cannot be divided or multiplied algebraically.

Not long afterwards Abel gave some interesting investigations, in which he showed how to sub-divide algebraically the arc of the lemniscate, that is, an integral of the kind mentioned above. Amongst his results was included the resolution of the semi-perimeter into seventeen parts (*Crelle*, t. iv., p. 131; see also Libri, *Crelle*, t. x., p. 167).

4. At the time, then, that Serret took up the problem, the lemniscate was the only curve known whose arc exactly represented an elliptic integral of the first kind. In his first paper on the subject (*Liouville's Journal*, t. viii., p. 145), Serret showed that the arc of the ellipse of Cassini was expressible by the sum or difference of two elliptic integrals of the first kind with complementary moduli, and hence that an arc of the curve with variable extremities represented exactly an elliptic integral. This is the simplest representation we have in a plane, but, since the extremities of the arc which is equal to the single integral are variable, we cannot divide or multiply algebraically any assigned arc of the curve, and therefore the solution is incomplete. Shortly afterwards my father, the Rev. W. Roberts, gave a spherical curve of the fourth order whose arc, measured from a fixed point, is equal to an elliptic integral of the first kind with an arbitrary modulus (*Liouville's Journal*, t. viii., p. 263). Since then, it may be observed that several other twisted curves have been found whose arcs can be expressed in the same manner (see Hermite's *Cours d'Analyse de l'Ecole Polytechnique*, p. 421). The plane curves, however, remained still to be found, and it was to this problem that Serret especially directed his attention (see Cayley's "Elliptic Functions," Chap. 15). He discovered a series of curves which are algebraic provided a certain number n is commensurable. The arc, then, is an elliptic integral of the first kind whose modulus is a function of n . Thus, for each curve corresponding to an assigned value of n , the arc represents an elliptic integral with a numerical modulus. This solution of Serret's is in all probability the best to be obtained, but still it evidently falls short of what is desirable.

5. Since then, it may be noticed, my father gave two very simple plane curves whose arcs are expressible by elliptic integrals of the first kind with certain numerical moduli (see *Journal de Mathématiques*, t. xii., p. 445). We see thus that geometers directed their special attention to finding curves whose arcs can be multiplied or

divided algebraically, that is, are expressible by a circular function or an elliptic integral of the first kind. The case of the logarithm seems to have been overlooked; but there does not appear to be much difficulty in finding curves whose arcs are expressible as a single logarithmic function.

6. With reference to curves whose arcs represent other integrals, we may notice that, as the arc of the plane ellipse represents the elliptic integral of the second kind, so the arc of the spherical conic represents the elliptic integral of the third kind with a circular parameter (Gudermann, *Crelle*, t. xiv., p. 169). One of the most remarkable results in the rectification of curves was the discovery by an Italian geometer, Genocchi, that the arc of the Cartesian oval was equal to the sum of the arcs of three ellipses (see Tortolini's *Annali*, 1864, p. 97). Since then, Dr. Casey has shown by geometrical considerations that the arc of the general bicircular quartic can be resolved into the sum of four quantities, each of which can be expressed by elliptic integrals. (See his Memoir in the *Philosophical Transactions of the Royal Society*, Vol. 167, Pt. 2, and Professor Cayley's addition thereto.) This reduction, however, in a certain case, is not effected by real transformations; so that the arc then depends upon elliptic integrals with imaginary moduli, and has to be expressed in a real manner by hyper-elliptic integrals of the first kind; a result which, I believe, has not been noticed before, and which I proceed to demonstrate here.

7. Let
$$F \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

be the equation of a focal conic, and

$$J \equiv (x - \alpha)^2 + (y - \beta)^2 - k^2 = 0$$

that of the corresponding Jacobian circle (see Casey's Memoir on Bicircular Quartics); then, if r, ϑ are the polar coordinates of a point on the quartic referred to the centre of J , we have, from the theory of

the curve,
$$r = p + \sqrt{(p^2 - k^2)}, \quad \vartheta = \omega,$$

where p is the perpendicular from α, β on a tangent to F , and ω is the angular coordinate of that tangent. Hence

$$ds = \sqrt{(dr^2 + r^2 d\vartheta^2)} = \left\{ 1 + \frac{p}{\sqrt{(p^2 - k^2)}} \right\} \left\{ p^2 + \left(\frac{dp}{d\omega} \right)^2 - k^2 \right\}^{\frac{1}{2}} d\omega;$$

but, if $a \cos \vartheta, b \sin \vartheta$ are the coordinates of the point on the conic,

we have

$$p = \frac{ba \cos \vartheta + a\beta \sin \vartheta - ab}{\sqrt{(a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta)}},$$

$$d\omega = \frac{abd\vartheta}{a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta},$$

and

$$p^2 + \left(\frac{dp}{d\omega}\right)^2 = (a + a \cos \vartheta)^2 + (\beta + b \sin \vartheta)^2.$$

We thus obtain for ds an expression of the form

$$\frac{P d\vartheta}{\sqrt{U}} + \frac{Q d\vartheta}{\sqrt{UV}},$$

where P, Q are rational functions of $\cos \vartheta, \sin \vartheta$, and

$$U \equiv (a + a \cos \vartheta)^2 + (\beta + b \sin \vartheta)^2 - k^2,$$

$$V \equiv (ba \cos \vartheta + a\beta \sin \vartheta - ab)^2 - k^2 (a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta).$$

Now, the first part of this value of s , namely,

$$\int \frac{P d\vartheta}{\sqrt{U}},$$

evidently depends upon elliptic integrals; we therefore confine our attention to the latter part. Putting $\tan \frac{1}{2}\vartheta = \mu$, this integral becomes

$$I = \int \frac{R d\mu}{\sqrt{UV}},$$

where R is a rational function of μ , and U, V are two binary quartics in μ . Now, it is easy to see the geometrical meaning of U and V ; U , in fact, determines the parameters of the rays joining a point P on F to the points of intersection of F with J , and V determines the rays joining P to the points on H at which the tangents touch J . But, referring F and J to their common self-conjugate triangle, that is, writing

$$F = x^2 + y^2 - z^2, \quad J = ax^2 + by^2 - cz^2,$$

and putting then $x = \lambda^2 - \mu^2$, $y = 2\lambda\mu$, $z = \lambda^2 + \mu^2$ for a point on F , we see that U is of the form

$$\lambda^4 + \mu^4 + 6m\lambda^2\mu^2,$$

and that V is of the same form, that is, can be written $U + \lambda H$, where H is the Hessian of U ; for it arises from the intersection of F with the conic

$$bcx^2 + cay^2 + abz^2 = 0.$$

Now, if the four intersections of F and J are all real or all imaginary, we may write

$$U \equiv \mu^4 + 6m\mu^2 + 1, \quad V \equiv \mu^4 + 6n\mu^2 + 1.$$

Hence, putting $R = A + \mu B$,

where A, B are functions of μ^2 , we have

$$I = \int \frac{A + B\sqrt{y}}{\sqrt{(y^2 + 6my + 1)(y^2 + 6ny + 1)}},$$

where $\mu^2 = y$. Now, part of this integral depends upon elliptic integrals, and the remaining part upon

$$\int \frac{yBdy}{\sqrt{\{y(y^2 + 6my + 1)(y^2 + 6ny + 1)\}}}.$$

But, putting $y + \frac{1}{y} = z$,

this latter expression is found to depend upon two elliptic integrals with different moduli. That is, if the four points of intersection of F and J are either all real or all imaginary, or, which is the same thing, if the four real foci of the curve are concyclic, the arc of the quartic can be broken up into the sum of four expressions, each of which can be expressed by elliptic integrals. This is Dr. Casey's result. If, however, the real foci of the curve are not concyclic, two points of intersection of F and J will be real and two imaginary, and U, V may be reduced to the forms

$$U \equiv \mu^4 + 6m\mu^2 - 1, \quad V \equiv \mu^4 + 6n\mu^2 - 1,$$

respectively. Hence, in this case, we have

$$I = \int \frac{A + B\sqrt{y}}{\sqrt{\{(y^2 + 6my - 1)(y^2 + 6ny - 1)\}}},$$

where $\mu^2 = y$, and, as before, the first part depends upon elliptic integrals; but the second part, namely,

$$\int \frac{yBdy}{\sqrt{\{y(y^2 + 6my - 1)(y^2 + 6ny - 1)\}}},$$

cannot be reduced in a real manner any further. This result, we see depends upon the fact that the vanishing of the skew invariant E of a sextic X is not a sufficient condition that

$$\int \frac{E dx}{\sqrt{X}}$$

should break up into elliptic integrals with real moduli. It is also necessary that the double points of the involution determined by the three quadratic factors of X should be real.

8. As I have remarked already, geometers do not appear to have noticed those curves whose arcs represent a pure logarithmic function. I shall proceed here to obtain a few of such curves. First, I shall show that there are some simple curves of the fourth order whose arcs can be expressed in this manner.

Let us consider the inverse of the parabola from an arbitrary point, then the arc of this curve is equal to

$$\int \frac{ds}{r^3} \dots\dots\dots(1),$$

where ds is the element of an arc of the parabola, and r is the distance of the element from the fixed point. Now, r^2 can be expressed rationally in terms of a parameter; hence we see that (1) depends upon logarithms and circular functions. Let us consider now the case in which the centre of inversion lies on the axis of the parabola, and see whether (1) can be expressed as a pure logarithm. Taking, then, $m\mu^2$, $2m\mu$ as the coordinates of a point on the parabola, and α , 0 as those of the centre of inversion, we have

$$s = \int \frac{2m\sqrt{(1+\mu^2)} d\mu}{(\alpha + m\mu^2)^2 + 4m^2\mu^2}.$$

Hence, if α is greater than $3m$, we obtain

$$s = \frac{1}{2\sqrt{(m^2 + m\alpha)}} \left\{ p \log \left(\frac{1+p \sin \omega}{1-p \sin \omega} \right) - q \log \left(\frac{1+q \sin \omega}{1-q \sin \omega} \right) \right\} \dots(2),$$

from $\mu = 0$ to $\mu = \tan \omega$, where

$$p^2 = \frac{\alpha + m + 2\sqrt{(m^2 + m\alpha)}}{\alpha + 2m + 2\sqrt{(m^2 + m\alpha)}},$$

$$q^2 = \frac{\alpha + m - 2\sqrt{(m^2 + m\alpha)}}{\alpha + 2m - 2\sqrt{(m^2 + m\alpha)}}.$$

In order, then, that the arc of this curve should represent the logarithm of an algebraic function of the variable, we must take the ratio of p to q commensurable, that is, if $\alpha + m = m\lambda^2$, we must have

$$\frac{(\lambda + 2)}{(\lambda - 2)} \frac{(1 - \lambda)^2}{(1 + \lambda)^2} = \frac{s^2}{t^2},$$

where s and t are numbers. This gives a cubic equation to determine λ or α , when s , t are assigned, and the arc of the curve from (2) is then proportional to

$$\log \left\{ \left(\frac{1+p \sin \omega}{1-p \sin \omega} \right)^s \left(\frac{1-q \sin \omega}{1+q \sin \omega} \right)^t \right\}.$$

The simplest case is obtained by taking $t = 0$, when we get $\alpha = 3m$

and
$$s = \frac{1}{3m\sqrt{2}} \log \left\{ \frac{3+2\sqrt{2} \sin \omega}{3-2\sqrt{2} \sin \omega} \right\}.$$

As an example, then, of the algebraic subdivision of the arcs of this curve, we may notice that the semiperimeter is bisected at the point corresponding to $\sin \omega = \frac{2}{3}$.

9. I now mention a system of curves whose arcs in each case represent either a logarithm or a circular function. Consider the hypocycloid whose equation in p , ω coordinates is $p = b \cos n\omega$ (see Salmon's "Curves," Art. 311). Then, for the arc s of the inverse of the given curve with respect to the origin, we have

$$s = \int \frac{\left(p + \frac{d^2 p}{d\omega^2} \right) d\omega}{p^3 + \left(\frac{dp}{d\omega} \right)^2} = \int \frac{(1-n^2) \cos n\omega d\omega}{b (\cos^2 n\omega + n^2 \sin^2 n\omega)} \dots\dots\dots (3),$$

in the case we are considering. Hence, putting $\sin n\omega = z$, we get

$$s = \frac{1-n^2}{nb} \int \frac{dz}{1-(1-n^2)z^2} \dots\dots\dots (4).$$

Thus, if n be less than unity, we have a series of curves whose arcs represent a pure logarithm. For instance, for the envelope of the system of circles

$$b(x \cos \omega + y \sin \omega) - \cos \frac{1}{2}\omega (x^2 + y^2) = 0,$$

we get

$$s = \frac{1}{2}b\sqrt{3} \log \left(\frac{1+\frac{1}{2}\sqrt{3} \sin \frac{1}{2}\omega}{1-\frac{1}{2}\sqrt{3} \sin \frac{1}{2}\omega} \right).$$

Again, taking $n = \frac{1}{3}$, we have

$$s = 2b\sqrt{2} \log \left(\frac{1+\frac{2}{3}\sqrt{2} \sin \frac{1}{3}\omega}{1-\frac{2}{3}\sqrt{2} \sin \frac{1}{3}\omega} \right).$$

In the latter case, the curve is an inverse of the cardioid, and therefore, it is easy to see, the inverse of a parabola; in fact, it is the

curve already considered above. By taking n greater than unity, we get curves whose arcs represent a circular function. Thus, for example, putting $n = 3$, the arc of the curve

$$b^4 + 8b^3r \cos 3\theta + 18b^2r^2 = 27r^4$$

(Salmon's "Curves," Art. 310, Ex. 5) is a circular function.

10. I now mention some methods of generating curves whose arcs can be expressed linearly in terms of arcs of a given curve, or of several given curves. This is the idea of John Bernoulli (see on "Curva Reptoria"). Consider several points $x_1, y_1, x_2, y_2, \&c.$, on a curve, or on different curves at which the tangents are parallel; then, if x_1, y_1 are the coordinates of the centre of mean position of these points for the system of multiples $l, m, \&c.$, we have

$$x = lx_1 + mx_2 + \&c., \quad y = ly_1 + my_2 + \&c.,$$

where

$$\Sigma l = 1.$$

Hence, since

$$\frac{dx_1}{ds_1} = \frac{dx_2}{ds_2} = \&c. = \cos \phi, \text{ say,}$$

$$\frac{dy_1}{ds_1} = \frac{dy_2}{ds_2} = \&c. = \sin \phi,$$

we have

$$dx = (l ds_1 + m ds_2 + \&c.) \cos \phi,$$

$$dy = (l ds_1 + m ds_2 + \&c.) \sin \phi;$$

from which we see that the tangent to the locus of xy is parallel to the other tangents, and that its arc is equal $\Sigma l s$. Hence, by taking $l, m, \&c.$ as numbers, we see that, if the arc of a curve be equal to a logarithm, a circular function, or an elliptic integral of the first kind, we can generate from this curve an infinite number of other curves whose arcs can be expressed in the same manner; for, by the properties of these functions, the several expressions for the different arcs can be compounded into a single expression which is a function of the same kind as is involved in each of the arcs.

As an example of this result, we may consider the locus of the middle point of the chord of contact of two parallel tangents to the lemniscate of Bernoulli. Putting $x + iy = u$, $x - iy = v$, in the equation

$$(u^2 + v^2)^2 = 2(u^2 - v^2),$$

we get

$$u^2 v^2 = u^2 + v^2.$$

Now, taking

$$u^2 = 1 + t^2, \quad v^2 = 1 + t^{-2} \dots\dots\dots (5),$$

we find that t^3 has the same value for parallel tangents; hence, for the two points on the curve, we may take $\mathfrak{J}\mu$, $\mathfrak{J}^2\mu$ as the parameters, where $\mathfrak{J}$ is a cube root of unity. Thus for the locus we have, from (5),

$$2u = (1 + \mathfrak{J}^3\mu^3)^{\frac{1}{3}} + (1 + \mathfrak{J}\mu^3)^{\frac{1}{3}},$$

$$2\mu v = (\mu^3 + \mathfrak{J})^{\frac{1}{3}} + (\mu^3 + \mathfrak{J}^2)^{\frac{1}{3}}.$$

Hence, eliminating μ , we get

$$\{16u^3v^3 - 8(u^3 + v^3) + 3\}^3 = 16(u^4 + v^4 - u^3v^3) - 12(u^3 + v^3) + 9 \dots (6),$$

which represents a curve of the eighth order. It may be observed that the locus of the points dividing in a given ratio $\frac{l}{m}$ the points of contact of the parallel tangents, is also of the eighth order. From what we have shown, then, we see that, if l and m are numbers, the arc of such a curve can be expressed as a single elliptic integral of the first kind whose modulus is equal to $\sqrt{(\frac{1}{2})}$.

11. I mention here a class of curves from each of which an infinite number of other curves can be generated whose arcs are exactly equal to corresponding arcs of the particular curve considered. Putting $x + iy = u$, $x - iy = v$, where x , y are rectangular coordinates, let us consider the curve whose equation can be expressed in the form

$$\phi(u) \phi(v) = 1 \dots \dots \dots (7);$$

then, if we have $u' = \int \{\phi(u)\}^n du$,

$$v' = \int \{\phi(v)\}^n dv,$$

where u , v are the coordinates of a point lying on the curve (7), it is evident that the point u' , v' will describe a curve whose arc is equal to the corresponding arc of (7); for we have

$$du' = \{\phi(u)\}^n du, \quad dv' = \{\phi(v)\}^n dv,$$

and therefore, from (7),

$$du' dv' = du dv, \quad \text{or} \quad ds'^2 = ds^2.$$

For a considerable number of curves included in (7) the generated curves will be algebraic; as, for instance, the curves whose polar

equations are $r^m = a^m \cos m\mathfrak{J}$,

being capable of being written in the form

$$(1-u^m)(1-v^m) = 1,$$

the generated curves will be always algebraic, if we suppose n to be a positive integer. For the lemniscate $m = 2$, and, putting

$$u = \frac{2z}{z^2+1}, \quad v = \frac{2iz}{z^2-1}$$

for a point on the curve, we have for the generated curve

$$u' = 2 \int \frac{(1-z^2)^{2n+1} dz}{(1+z^2)^{2n+2}}, \quad v' = -2i \int \frac{(1+z^2)^{2n+1} dz}{(1-z^2)^{2n}} \dots\dots\dots (8),$$

which give the coordinates of a point on the curve in terms of a parameter. I shall show now that this curve belongs to Serret's system; for his general curves are given by the equation

$$x+iy = Ce^{i\alpha} \int \frac{(z-a)^m (z+a)^n}{(z-a)^{m+1} (z+a)^{n+1}} dz, \quad x-iy = \&c.$$

(see *Liouville*, t. x., p. 267); but, putting $m = n$ in these equations we find $a^3 + a^3 = 0$ as the necessary condition that the curve should be algebraic. Hence we see that the curves (8) belong to Serret's system, and correspond to the case of $m = n$.

In the same way we can generate, from the curves whose polar equations are

$$r^3 = a^3 \cos 3\theta, \quad r^4 = a^4 \cos \left(\frac{3\theta}{2} \right),$$

an infinite number of curves whose arcs can be expressed in the same manner, that is, are expressible by elliptic integrals of the first kind with certain numerical moduli. These curves, it may be observed, do not appear to belong to Serret's system. As a further example, we may notice the curves derived from the ellipse of Cassini. The equation of this curve can be written in the form

$$(1-u^2)(1-v^2) - k^4 = 0,$$

where

$$u = x+iy, \quad v = x-iy;$$

hence, for a point on the derived curve, we may take

$$k^{2n} u' = \int (1-u^2)^n du, \quad k^{2n} v' = \int (1-v^2)^n dv.$$

Thus, from the known property of the ellipse of Cassini, the arcs of

these curves are expressible as the sum of two elliptic integrals of the first kind with complementary moduli.

12. I now proceed to consider the arc of the general twisted unicursal curve. If for every point of the curve we have

$$x : y : z : 1 = f_1 : f_2 : f_3 : f,$$

where f_1 , &c. are binary quantics of the n^{th} degree in a parameter $\mathcal{J}$, and the arcs are rectangular, we obtain

$$ds = \frac{\sqrt{\phi}}{f^2} d\mathcal{J} \dots \dots \dots (9),$$

in which

$$\phi \equiv J_1^2 + J^2 + J_2^2,$$

where J_1 , J , J_2 are the respective Jacobians of the pairs of quantics f_1, f ; f_2, f ; f_3, f . Now the curves which I intend to consider more particularly are those of the $(2m)^{\text{th}}$ and $(2m+1)^{\text{th}}$ degrees which intersect the imaginary circle at infinity Ω in $2m$ points. In these cases the expression (9) will admit of considerable simplification; for the roots of ϕ evidently determine the points on the curve at which the tangents intersect Ω , and if $2m$ of these points lie on Ω , ϕ must be divisible by f^2 for $n = 2m$, and by all the squares of the linear factors of f but one for $n = 2m+1$. This may be readily inferred from the fact that the section of the circumscribed developable by a plane has the points where the plane meets the curve for cusps. For the circular curve, then, of the $(2m)^{\text{th}}$ degree, we have

$$ds = \sqrt{\psi} \frac{d\mathcal{J}}{f} \dots \dots \dots (10),$$

where ψ is a binary quantic in $\mathcal{J}$ of the degree $4(m-1)$. Now, ψ and f together have $4m-4+2m-3 = 6m-7$ independent absolute invariants, namely, the number of independent invariants less by unity; for the number of the independent invariants of two quantics of the

degrees p, q is

$$p+q-2.$$

Now, the general unicursal curve of the n^{th} degree has $4n-15$ absolute invariants; for, if the curve be referred to four stationary planes, which are fundamental planes of the curve, each coordinate will be expressible in the form

$$(\mathcal{J}-\alpha)^4 f_{n-4},$$

so that we thus have a biquadratic and four quantics of the degree $n-4$, and these involve $4n-15$ absolute invariants. Also a plane,

namely, in the case we are considering, that at infinity, contains three constants, and $2m-5$ conditions are satisfied by $2m$ points lying on a conic in a given plane, namely Ω . Thus there are

$$8m-15+3-(2m-5) = 6m-7$$

absolute invariants, which, as we have seen, is the same number as that involved in (10). Hence we see that the integral

$$\int \frac{\sqrt{\psi} d\mathcal{P}}{f},$$

where ψ and f are general binary quantics in $\mathcal{P}$ of the degrees $4m-4$ and $2m$, respectively, can be represented by the arc of a twisted curve, with this limitation, however, that all the linear factors of ψ and f must be imaginary. For the curve of the $(2m+1)^{\text{th}}$ degree meeting Ω in $2m$ points, we may put $f = \chi_{2m} \chi_1$, where χ_{2m} gives the $2m$ points on Ω , and χ_1 gives the single real point of the curve at infinity. Hence, since ϕ is divisible by χ_{2m}^2 , we have in this case

$$ds = \frac{\sqrt{\psi} d\mathcal{P}}{\chi_{2m} \chi_1^2} \dots\dots\dots(11),$$

where ψ is of the degree $4m$. Now, this curve, subject to the condition of meeting Ω in $2m$ points, has $6m-3$ absolute invariants; and ψ, χ_{2m}, χ_1 , if they were the most general expressions of their kind, would have $6m-2$ absolute invariants. We see thus that ψ, χ_{2m}, χ_1 are connected by a single invariant relation, and, therefore, that the integral (11) cannot be represented by the arc of a curve unless this relation is satisfied.

13. We may now proceed to consider each curve of given degree more particularly. For the circular twisted cubic, we have

$$ds = \frac{\sqrt{\psi} d\mathcal{P}}{\chi_2 \chi_1^2},$$

where ψ is of the fourth degree in $\mathcal{P}$. Now, in order that we may find the relation connecting ψ, χ_2, χ_1 in this case, we refer the cubic to the imaginary tetrahedron formed by an arbitrary point on the curve and the three points at infinity, when we can write

$$x : y : z : u = (t-\alpha)^{-1} : (t-\beta)^{-1} : (t-\gamma)^{-1} : 1.$$

We then easily find

$$x : y : z = (t-\alpha)^{-2} : (t-\beta)^{-2} : (t-\gamma)^{-2} \dots\dots\dots(12),$$

for the point where a tangent meets the plane u ; but, since Ω passes through the points xu, yu , there is

$$\Omega \equiv cz^2 + fyz + gzx + hxy.$$

Hence, substituting for x, y, z from (12) in Ω , and dividing by $(t-\alpha)^2(t-\beta)^2$, that is χ_2^2 , we get

$$\psi = (t-\gamma)^2 \{f(t-\alpha)^2 + g(t-\beta)^2 + h(t-\gamma)^2\} + c(t-\alpha)^2(t-\beta)^2;$$

from which it follows that $t-\gamma$, that is χ_1 , is a factor of the Jacobian of ψ and χ_2 . If we suppose then γ infinite, and put $\chi_2 = 1+t^2$, which is evidently permissible, we get

$$ds = \left(t^2 + \frac{at^2 + bt + c}{1+t^2} \right) \frac{dt}{\sqrt{\psi}},$$

where

$$\psi = t^4 + (a+1)t^2 + bt + c.$$

The first part of this integral for s , namely,

$$\int \frac{t^2 dt}{\sqrt{\psi}},$$

evidently depends upon algebraic, circular, and elliptic functions of the three different kinds. The other part of the integral

$$\int \frac{at^2 + bt + c}{1+t^2} \frac{dt}{\sqrt{\psi}}$$

will be expressed by two elliptic integrals of the third kind with complementary imaginary parameters, which, as has been shown by Legendre in his *Traité des Fonctions Elliptiques*, t. 1, depend upon two similar integrals, one with a circular and the other with a logarithmic parameter.

If in the above we write

$$\psi \equiv \{(t-\alpha)^2 + \beta^2\} \{(t-\alpha')^2 + \beta'^2\} \dots \dots \dots (13),$$

we have $\alpha' = -\alpha$; and then, if the single condition $\alpha = 0$ is satisfied, there is

$$s = \int \left\{ t^2 + \frac{\beta^2 \beta'^2 + (\beta^2 + \beta'^2 - 1) t^2}{1+t^2} \right\} \frac{dt}{\sqrt{\{(t^2 + \beta^2)(t^2 + \beta'^2)\}}} \dots (14),$$

in which, putting $t = \beta \tan \phi$, we get the arc, minus an algebraic part, equal to

$$lE(\phi) + m\Pi(n, \phi).$$

If, in (13), $\beta' = \beta$, and then we put

$$t = \sqrt{(\alpha^2 + \beta^2)} \tan \frac{1}{2}\phi,$$

we see that the arc in this case also involves a single elliptic integral of the third kind.

14. Proceeding now to the case of the circular twisted quartic, we have

$$s = \int \frac{\sqrt{\psi} d\mathfrak{J}}{f} = \int \frac{\psi}{f} \frac{d\mathfrak{J}}{\sqrt{\psi}} \dots\dots\dots(15),$$

where ψ and f are general binary quantics with imaginary roots. If, then, $\mathfrak{J}^2 + p\mathfrak{J} + q$, $\mathfrak{J}^2 + p'\mathfrak{J} + q'$ are the real factors of f , we may write

$$s = \int \left\{ \frac{a\mathfrak{J}^2 + b\mathfrak{J} + c}{\mathfrak{J}^2 + p\mathfrak{J} + q} + \frac{a' + b'\mathfrak{J} + c'}{\mathfrak{J}^2 + p'\mathfrak{J} + q'} \right\} \frac{d\mathfrak{J}}{\sqrt{\psi}}.$$

Hence from the result of Legendre, referred to above, we see that in the expression for the arc there are involved four elliptic integrals of the third kind with different parameters, two of which are circular and two logarithmic. I shall now transform this integral (15) into another in which the consideration of particular cases becomes easier. Let P , Q be the two real quadratic factors of ψ ; then, if J is the Jacobian of P , Q , it is evident that a , b , h , l , m can be determined so that

$$aP^2 + bQ^2 + 2hPQ + J(lP + mQ)$$

should represent any given binary quartic; and this is the form which I shall assume for f . We have then

$$s = \int \frac{PQ}{aP^2 + bQ^2 + 2hPQ + J(lP + mQ)} \frac{d\mathfrak{J}}{\sqrt{PQ}} \dots\dots\dots(16);$$

hence, putting $P = Qz$, and observing that, by a known theorem, we may assume

$$J^2 = -(P - \lambda_1 Q)(P - \lambda_2 Q),$$

in which λ_1 , λ_2 are real quantities, we obtain

$$s = \int \frac{z}{\{az^2 + 2hz + b + (lz + m)\Delta\}} \frac{dz}{\Delta\sqrt{z}},$$

where

$$\Delta^2 = (\lambda_1 - z)(z - \lambda_2).$$

In the general case, then, there is

$$s = \int \frac{(az^2 + 2hz + b)}{\{(az^2 + 2hz + b)^2 - (lz + m)^2 \Delta^2\}} \frac{z dz}{\Delta \sqrt{z}} \\ - \int \frac{(lz + m) \sqrt{z} dz}{\{(az^2 + 2hz + b)^2 - (lz + m)^2 \Delta^2\}} \dots\dots\dots (17),$$

the latter integral evidently depending on circular and logarithmic functions. Putting then

$$\phi(z) = \sqrt{z(\lambda_1 - z)(z - \lambda_2)},$$

we may express

$$\frac{z(az^2 + 2hz + b)}{(az^2 + 2hz + b)^2 - (lz + m)^2 \Delta^2}$$

in the form $\frac{m_1 \sqrt{\phi(\alpha)}}{z - \alpha} + \frac{m_2 \sqrt{\phi(\beta)}}{z - \beta} + \frac{m_3 \sqrt{\phi(\gamma)}}{z - \gamma} + \frac{m_4 \sqrt{\phi(\delta)}}{z - \delta},$

where $\alpha, \beta; \gamma, \delta; m_1, m_2; m_3, m_4$ are supposed to be conjugate imaginaries. Thus the first part of the expression (17) for the arc becomes

$$\frac{1}{2} (m_1 + m_2) \int \left\{ \frac{\sqrt{\phi(\alpha)}}{z - \alpha} + \frac{\sqrt{\phi(\beta)}}{z - \beta} \right\} \frac{dz}{\sqrt{\phi(z)}} \\ + \frac{1}{2} (m_1 - m_2) \int \left\{ \frac{\sqrt{\phi(\alpha)}}{z - \alpha} - \frac{\sqrt{\phi(\beta)}}{z - \beta} \right\} \frac{dz}{\sqrt{\phi(z)}} \\ + \frac{1}{2} (m_3 + m_4) \int \left\{ \frac{\sqrt{\phi(\gamma)}}{z - \gamma} + \frac{\sqrt{\phi(\delta)}}{z - \delta} \right\} \frac{dz}{\sqrt{\phi(z)}} \\ + \frac{1}{2} (m_3 - m_4) \int \left\{ \frac{\sqrt{\phi(\gamma)}}{z - \gamma} - \frac{\sqrt{\phi(\delta)}}{z - \delta} \right\} \frac{dz}{\sqrt{\phi(z)}} \dots\dots\dots (18).$$

Now, by the theory of the elliptic integral of the third kind [that is, by means of the result

$$\Pi(\alpha, u) = \Pi(u, \alpha) + uZ(\alpha) - \alpha Z(u),$$

and by means of the addition formula for these integrals], each of the four integrals in (18) can be expressed as a real Π function. We may put, in fact,

$$\int \left\{ \frac{\sqrt{\phi(\alpha)}}{z - \alpha} + \frac{\sqrt{\phi(\beta)}}{z - \beta} \right\} \frac{dz}{\sqrt{\phi(z)}} = \int \frac{\sqrt{\phi(p)}}{z - p} \frac{dz}{\sqrt{\phi(z)}} + \lambda \mathfrak{J} + \mu \int \frac{dz}{\sqrt{\phi(z)}},$$

$$\int \left\{ \frac{\sqrt{\phi(\alpha)}}{z-\alpha} - \frac{\sqrt{\phi(\beta)}}{z-\beta} \right\} \frac{dz}{\sqrt{\phi(z)}} \\ = i \int \frac{\sqrt{-\phi(q)}}{z-q} \frac{dz}{\sqrt{\phi(z)}} + i\lambda'\phi + i\mu' \int \frac{dz}{\sqrt{\phi(z)}} \dots (19),$$

where $\mathfrak{J}$ is a logarithm and ϕ a circular function, and $i^2 = -1$.

15. We can now easily find the conditions that the arc should involve a lesser number of different integrals of the third kind. Since

$$\int \left\{ \frac{m\sqrt{\phi(p)}}{z-p} + \frac{n\sqrt{\phi(p')}}{z-p'} \right\} \frac{dz}{\sqrt{\phi(z)}},$$

where m, n are numbers, and p, p' are real quantities, depends upon a single integral, we see that the arc can be made to involve but three distinct integrals, if one of the quantities

$$\frac{m_1 \pm m_2}{m_3 \pm m_4}$$

is commensurable, and that it will involve two integrals only if both these quantities are commensurable. A simple example is obtained by taking

$$m_1 \pm m_2 = 0, \quad \text{or} \quad m_3 \pm m_4 = 0,$$

in which case, it may be easily shown, the relation between f and ψ is

$$f'(\alpha)\sqrt{\psi(\beta)} - f'(\beta)\sqrt{\psi(\alpha)} = 0,$$

where α, β are two conjugate imaginary roots of f .

16. There is another case in which the number of the integrals is reduced, and which does not come under the preceding. Suppose

$$\phi(p) = 0,$$

or, which comes to the same thing, putting

$$z = \lambda_1 \cos^2 \phi + \lambda_2 \sin^2 \phi = \lambda_1 \Delta^2 am u, \text{ say,}$$

and $a = \lambda_1 \text{cn}^2 a_1 + \lambda_2 \text{sn}^2 a_1, \quad \beta = \lambda_1 \text{cn}^2 a_2 + \lambda_2 \text{sn}^2 a_2,$

if we have

$$a_1 + a_2 = K,$$

then, since $\Pi(K-a, u)$ can be expressed in terms of $\Pi(a, u)$, we see that only three distinct integrals are involved. In this case, a real quadratic factor of f is harmonically connected with the Jacobian of

the two real quadratic factors of ψ , that is, in (16), $lP+mQ$ is a factor of aP^2+bQ^2+2hPQ . It is easy to see then, at once, that the expression for s (17) only involves three different integrals.

17. Again, if f have two factors of the form $\lambda P+\mu Q$, l, m , both vanish in (16), and we get

$$s = \int \frac{z}{az^2+2hz+b} \frac{dz}{\sqrt{\{z(\lambda_1-z)(z-\lambda_2)\}}} \dots\dots\dots(20).$$

Hence, in the latter case, the arc is expressed by two elliptic integrals of the third kind. Furthermore, if f is of the form $\psi+\lambda H$, where H is the Hessian of ψ , we can show that the arc depends upon a single elliptic integral of the third kind. We have

$$s = \int \frac{\psi}{\psi+\lambda H} \frac{d\psi}{\sqrt{\psi}} = \int \frac{1}{1+\lambda z} \frac{d\psi}{\sqrt{\psi}},$$

putting $H = \psi z$, according to Weierstrass's transformation; but

$$\psi^2 dz = \psi dH - H d\psi = J d\psi,$$

where J is the Jacobian of ψ and H . Now, by the theory of the binary quartic, J^2 is proportional to

$$-(4H^3 - SH\psi^2 + T\psi^3) = -\psi^3(4z^3 - Sz + T),$$

where S and T are the invariants of ψ . Hence we have

$$\frac{d\psi}{\sqrt{\psi}} = \frac{dz}{\sqrt{-(4z^3 - Sz + T)}},$$

and

$$s = \int \frac{1}{1+\lambda z} \frac{dz}{\sqrt{-(4z^3 - Sz + T)}} \dots\dots\dots(21),$$

which is of the form mentioned above. I have already considered the rectification of the curve in this case (see *Proceedings of the London Mathematical Society*, Vol. xvii., p. 39), and have shown that the arc is exactly equal to the arc of a sphero-conic. This sphero-conic is, however, not necessarily real, in which case the integral has a logarithmic parameter, so that the arc of the curve we are considering represents the elliptic integral of the third kind in the most general manner, which the arc of the sphero-conic does for integrals with circular parameters only. The particular case noticed *loc. cit.*, in which the arc of the curve is a pure elliptic integral of the first kind, evidently arises from making f and ψ coincide. It is easy to see, then, that Ω touches the section by the plane at infinity of the circumscribed developable at its four cusps, namely, the four points where

the plane at infinity meets the curve, or, which is the same thing, the osculating planes to the curve at the four points on Ω touch Ω .

18. We may notice, also, two simple cases in which the expression (17) is made to depend upon three integrals. Suppose f and ψ are such that $f \equiv \psi + R^2$, identically, where R is a quadratic; then, by a particular case of Abel's theorem, there is $\alpha_1 \pm \alpha_2 + \alpha_3 \pm \alpha_4 = \omega$, where α_1 , &c. are the arguments of the parameters, from which the result stated follows at once. In fact, there is then either $p = p'$ or $q = q'$, where p, q refer to α, β , and p', q' to γ, δ , in (19). Again, if P, Q are the real quadratic factors of ψ , and f is of the form $Px^2 + Qy^2$, where x, y are linear expressions in the variable, then, by another particular case of Abel's theorem (see Legendre's *Traité des Fonctions Elliptiques*, t. III., p. 192), there is $\alpha_1 \pm \alpha_2 + \alpha_3 \pm \alpha_4 = \frac{1}{2}\omega$, from which the dependence on three integrals becomes manifest. These results would also follow at once from the respective identities,

$$\int \frac{J(R, \psi) d\vartheta}{(\psi + R^2)\sqrt{\psi}} = \tan^{-1} \left(\frac{\sqrt{\psi}}{R} \right),$$

$$\int \frac{\{PQ + xyJ(P, Q)\}}{Px^2 + Qy^2} \frac{d\vartheta}{\sqrt{PQ}} = \tan^{-1} \left\{ \frac{x}{y} \sqrt{\left(\frac{P}{Q} \right)} \right\} \dots\dots (22);$$

for by means of them we should evidently obtain a linear relation connecting the four integrals involved in each case.

19. Let us now consider the case in which Ω is touched by the osculating planes at two of the points where it meets the curve. It is evident then that f and ψ have a common quadratic factor P , say. Hence, if $f = PR$, we have

$$s = \int \frac{Q}{R} \frac{d\vartheta}{\sqrt{PQ}} \dots\dots\dots (23),$$

which is expressible by means of two elliptic integrals of the third kind, one of which has a circular and the other a logarithmic parameter. Furthermore, if we suppose R to be of the form $lP + mQ$, and put $P = Qz$, we get

$$s = \int \frac{1}{lz + m} \frac{dz}{\sqrt{\{z(\lambda_1 - z)(z - \lambda_2)\}}} \dots\dots\dots (24),$$

which is an elliptic integral of the third kind with a parameter either circular or logarithmic. The arc, then, of this curve also gives a very general representation of such integrals. It may be observed that if,

in this case, we take $m = 0$, or suppose $f = H$ in (21), we get two curves whose arcs are exactly equal to those of ellipses.

20. The integral
$$\int \frac{\sqrt{PQ}}{f} d\mathfrak{J}$$

would admit of considerable simplification in the case also in which f contained the Jacobian J of P, Q as a factor; but this can never obtain for the arcs of the curves we are considering, for, since P, Q have imaginary roots, their Jacobian J has real roots, and therefore can never be a factor of f , all of whose roots are imaginary.

21. For the curve which has double contact with Ω , we put f equal to the square of a quadratic R^2 , say. Now, it is evident that any quadratic R can be expressed in the form

$$lP + mQ + nJ;$$

hence we have
$$s = \int \frac{PQ}{(lP + mQ + nJ)^2} \frac{d\mathfrak{J}}{\sqrt{PQ}}$$

$$= \int \frac{z}{(lz + m + n\Delta)^2} \frac{dz}{\Delta\sqrt{z}} \dots\dots\dots(25),$$

where $P = Qz$ and $\Delta^2 = (\lambda_1 - z)(z - \lambda_2)$.

In this case, then, the arc involves an algebraic part and an elliptic integral of the second kind, besides two elliptic integrals of the third kind, one with a circular and the other with a logarithmic parameter. If, however, the single condition $n = 0$ is satisfied, there is but one integral of the third kind, whose parameter may be either circular or logarithmic.

22. We may now consider another particular form of the curve, namely, such that the section of the circumscribed developable by the plane at infinity has double contact with Ω at points other than those on the curve. In this case ψ is evidently a perfect square, and there is

$$s = \int \frac{Rd\mathfrak{J}}{PQ},$$

where P, Q are the two real factors of f ; hence, if we write R in the form

$$lP + mQ + nJ,$$

we find
$$s = l \int \frac{d\mathfrak{J}}{Q} + m \int \frac{d\mathfrak{J}}{P} + n \log \left(\frac{P}{Q} \right) \dots\dots\dots(26),$$

$\begin{matrix} 1 & 2 \end{matrix}$

that is, the arc is expressible by two circular arcs and a logarithmic function. If we put $n = 0$, and suppose $l : m$ commensurable, the arc of the curve, we see, will represent a circular function.

23. Passing by the case of the quintic, which does not seem to be of much interest, we may proceed at once to consider the circular sextic. For this curve, we have

$$s = \int \frac{\psi}{f} \frac{d\mathcal{J}}{\sqrt{\psi}} \dots\dots\dots (27),$$

where ψ and f are general expressions in $\mathcal{J}$ of the eighth and sixth degrees, respectively. Let us suppose, now, that the osculating planes of the curve at the six points on Ω are touched by Ω , then ψ will evidently contain f as a factor. Hence, if $\psi_1, \psi_2, \psi_3, \psi_4$ are the four real quadratic factors of ψ , we may take

$$f = \psi_1 \psi_2 \psi_3,$$

and we have

$$s = \int \frac{\psi_4 d\mathcal{J}}{\sqrt{(\psi_1 \psi_2 \psi_3 \psi_4)}} \dots\dots\dots (28),$$

which evidently gives a very general representation of the first kind of the second class of hyper-elliptic integrals. Further, if we suppose $\psi_1, \psi_2, \psi_3, \psi_4$ to belong to a system in involution, these quadratics can be transformed linearly so as to involve the square of $\mathcal{J}$ only, that is,

$$\text{we may take} \quad \psi_1 = \mathcal{J}^2 + \alpha^2, \text{ \&c. ;}$$

$$\text{hence, putting} \quad \mathcal{J}^2 = z, \quad \alpha^2 = a, \text{ \&c.,}$$

$$\text{we have} \quad s = \int \frac{(z-d) dz}{\sqrt{\{z(z+a)(z+b)(z+c)(z+d)\}}} \dots\dots\dots (29),$$

which gives a very general representation of the first kind of the first class of hyper-elliptic integrals. Hence, by the theory of Abelian integrals, if we take three variable points A, B, C on the latter curve, the condition

$$\text{arc } OA \pm \text{arc } OB \pm \text{arc } OC = \text{a constant,}$$

where O is a fixed point, is consistent with two algebraic relations. Also, for the former curve, if we take a fourth point, a similar relation connecting the arcs will be consistent with three algebraic relations.

24. It may be observed that, if we have $ab = cd$ in (29), and if we

put, then, $z + \left(\frac{ab}{z}\right) = 2y,$

we find that the arc consists of the sum of two elliptic integrals of the first kind with different moduli. We may notice, also, a general case in which the arc of the sextic depends upon elliptic integrals. Suppose the section of the circumscribed developable by the plane at infinity to have double contact with Ω , then ψ assumes the form $\phi^2\chi$, where ϕ is a quadratic, and χ a quartic; we have then

$$s = \int \frac{\phi\chi}{f} \frac{d\theta}{\sqrt{\chi}} \dots\dots\dots(30),$$

which evidently depends, in general, on six elliptic integrals of the third kind, with different parameters and the same modulus. By considering special cases, several of these different parameters may coincide until the arc is expressed by a single elliptic integral of the third kind.

25. From the fact that the arcs of curves are in general expressed by Abelian integrals, we should be led, from the properties of those functions, to expect certain geometrical theorems depending on the comparison of different arcs. That is, in the case of elliptic and hyper-elliptic integrals of the first kind, we should expect simple geometrical relations connecting the extremities of arcs whose algebraic sums were given; and, in the case of the integrals of the second and third kind, we should expect the sums of arcs connected with algebraic or circular quantities, respectively. The only important result of this nature seems to be the theorem which is called Dr. Graves's, and its extension to the lines of curvature of a quadric. This theorem was originally obtained by Dr. Graves for the case of the sphero-conic (see his "Translation of Chasles' two Memoirs on Cones of the Second Degree and Spherical Conics," Dublin, 1841, p. 77, where the result is given as the reciprocal of another theorem, namely, if two spherical conics have the same cyclic arcs, any arc touching the inner curve will cut off from the outer one a segment of constant area). An extension of Dr. Graves's theorem was discovered by MacCullagh, and both results were shortly afterwards given independently by Chasles (*Comptes Rendus*, t. xvii., 1843, pp. 838—844), who added some properties also connecting circles with elliptic and hyperbolic arcs whose difference is rectifiable. Several properties of equal arcs of the lemniscate of Bernoulli were thence derived by Chasles (*Comptes Rendus*, t. xxv., 1845, pp. 190—201).

26. It has been remarked that the theorems alluded to above were not anticipated by those who had applied the integral calculus to the rectification of conic sections; but this, indeed, could hardly have been expected to have been the case, as the results flow from a simple geometrical property of the conic sections. In fact, the geometrical method would, in general, infer theorems from some particular property, and then consider whether there was any known curve that possessed this property. The general theorem, of which Dr. Graves's result is the particular application to confocal conics, was originally given by Leibnitz in a letter dated 3rd Jan., 1704, addressed to John Bernoulli, as has been pointed out by Mr. Samuel Roberts (see *Proceedings of the London Mathematical Society*, Vol. XII., p. 121).

27. I purpose now to give a few simple properties of arcs of certain curves included in the form

$$r^m = a^m \cos m\vartheta,$$

where r, ϑ are polar coordinates. Considering the cubic corresponding to the case $m = -3$, we have the arc proportional to

$$\int \frac{z \, dz}{\sqrt{(z^3 - 1)}},$$

where

$$r^2 = a^2 z.$$

Now, eliminating ϑ between the equation of the curve and the line

$$r \cos (\vartheta - \omega) - p = 0,$$

we get $(4p^3 - a^3 \cos 3\omega - 3pr^2)^3 - \sin^3 3\omega (r^6 - a^6) = 0 \dots\dots\dots(31),$

which may be written

$$(a + \beta z)^3 - (z^3 - 1) \equiv \phi(z), \text{ say } = 0 \dots\dots\dots(32).$$

But, differentiating this relation, we have

$$\phi'(z) \, dz + 2(a + \beta z)(d\alpha + z \, d\beta) = 0,$$

or, from (32), for a root of ϕ ,

$$\frac{dz}{\sqrt{(z^3 - 1)}} + \frac{2(d\alpha + z \, d\beta)}{\phi'(z)} = 0,$$

whence we get

$$\Sigma \frac{dz}{\sqrt{(z^3 - 1)}} = 0, \text{ identically,}$$

and

$$\Sigma \frac{z \, dz}{\sqrt{(z^3 - 1)}} = 0,$$

if
$$d\beta = 0,$$

that is, if β is constant. Hence, from (31), we see that tangents to the tricuspidal hypocycloid

$$p = b \sin 3\omega$$

meet the curve in three points, the algebraic sum of whose distances, measured along the curve from fixed points, is constant. We say fixed points, and not a fixed point, because collinear points are situated on different branches of the curve.

In the same way, if we seek the intersection of a circle with the curve, we get an equation of the form

$$z^3 - 1 - (\alpha + \beta z + \gamma z^2 + \delta z^3)^2 = 0 \dots\dots\dots(33),$$

whence, by Abel's theorem, we get for the six roots

$$\sum \frac{dz}{\sqrt{(z^3-1)}} = 0, \quad \sum \frac{z dz}{\sqrt{(z^3-1)}} = 0,$$

from which we infer that, if a circle meet the curve in six real points, the algebraic sum of the arcs, measured from these six points to fixed points on the curve, is constant. The same, it may be observed, will also be true for a conic having the origin for centre, or, which is the same thing, if the tangents at the six points are touched by a circle.

28. Let us consider now the inverse of the curve mentioned in the preceding article, that is, the curve whose polar equation is

$$r^3 = a^3 \cos 3\theta \dots\dots\dots(34).$$

In this case the arc is proportional to

$$\int \frac{dz}{\sqrt{\{z(z^3-1)\}}}.$$

But, putting z^{-1} for z in this expression, we get the integral which has been called the argument of a point on the inverse cubic curve. Hence, by inverting the known properties of points on a cubic curve, we can arrive at certain results involving arcs of the curve (34). Thus, if a circle be described through the origin to meet the curve in three points, the sum of the arcual distances of these points from the origin is equal to the perimeter of one of the ovals.

Again, we find that two real circles can be described through the origin and a point on the curve to touch the curve elsewhere, and that the arc joining the points of contact is equal to half the perimeter of an oval. Also we may deduce that, if a circle intercept three

arcs on the curve, three properly selected points of bisection of these arcs will lie on a circle passing through the origin.

29. I may notice here a certain locus connected with the lemniscate, namely, that of the centre of gravity of an arc of given length. Let the curve be written

$$r^2 = a^2 \cos 2\theta,$$

then

$$ds = \frac{a d\theta}{\sqrt{\cos 2\theta}},$$

and, if x, y are the coordinates of the centre of gravity of an arc of length l , we find

$$lx = a^2 (\sin \theta_1 - \sin \theta_2), \quad ly = a^2 (\cos \theta_1 - \cos \theta_2),$$

whence $l^2 (x^2 + y^2) = 4a^4 \sin^2 \frac{1}{2} (\theta_1 - \theta_2), \quad \tan \frac{1}{2} (\theta_1 + \theta_2) = y/x \dots (35).$

Now, since the arc is of given length, there is a relation of the form

$$m \cos \theta_1 \cos \theta_2 + n \sin \theta_1 \sin \theta_2 = 1,$$

where

$$m^2 + n^2 = 2.$$

We thus get, for the equation of the locus, from (35),

$$l^2 (m+n)(x^2 + y^2)^2 - 4a^4 \{ (m-1)x^2 + (n-1)y^2 \} = 0 \dots (36),$$

which represents the inverse of a concentric ellipse.

30. I shall now consider some more general relations connecting the arcs of curves; and, for this purpose, we must have a general method of comparing transcendents. I commence, therefore, by demonstrating a general theorem concerning the points of intersection of a conic with a given curve.

Let

$$U \equiv f(x, y) = 0$$

be the equation of a curve of the n^{th} degree in Cartesian coordinates, and let us take

$$x : y : 1 = f_1 : f_2 : f \dots (37)$$

for a point on a conic, where f_1, f_2, f are quadratic expressions in a parameter θ . Substituting, then, the coordinates of a point on the conic in $U = 0$, we get, to determine the $2n$ points of intersection,

$$f^n U \equiv \phi(\theta) = 0 \dots (38),$$

where $\phi(\theta)$ is a rational integral expression in θ of the $(2n)^{\text{th}}$ degree.

Considering now the roots of

$$\phi(\mathfrak{P}) = 0$$

as functions of the constants which occur in the equation of the conic, that is, in f_1, f_2, f , we get, by differentiation,

$$\phi'(\mathfrak{P}) d\mathfrak{P} + f^n \delta U = 0,$$

$$\text{or} \quad \phi'(\mathfrak{P}) d\mathfrak{P} + f^n \left\{ \frac{dU}{dx} \delta \left(\frac{f_1}{f} \right) + \frac{dU}{dy} \delta \left(\frac{f_2}{f} \right) \right\} = 0 \dots\dots\dots (39),$$

where δ represents differentiation with regard to the constants just referred to. But, calling u the argument of a point on U , we have

$$\frac{dx}{dU} = - \frac{dy}{dU} = du.$$

Hence, from (39), we may write

$$\phi'(\mathfrak{P}) du + f^n \left\{ \left(\frac{dx}{d\mathfrak{P}} \right) \delta \left(\frac{f_2}{f} \right) - \left(\frac{dy}{d\mathfrak{P}} \right) \delta \left(\frac{f_1}{f} \right) \right\} = 0 \dots\dots\dots (40).$$

$$\text{Now} \quad \frac{dx}{d\mathfrak{P}} = \frac{d}{d\mathfrak{P}} \left(\frac{f_1}{f} \right) + \frac{\delta}{d\mathfrak{P}} \left(\frac{f_1}{f} \right), \quad \frac{dy}{d\mathfrak{P}} = \frac{d}{d\mathfrak{P}} \left(\frac{f_2}{f} \right) + \frac{\delta}{d\mathfrak{P}} \left(\frac{f_2}{f} \right);$$

$$\begin{aligned} \text{and} \quad & \frac{d}{d\mathfrak{P}} \left(\frac{f_1}{f} \right) \delta \left(\frac{f_2}{f} \right) - \frac{d}{d\mathfrak{P}} \left(\frac{f_2}{f} \right) \delta \left(\frac{f_1}{f} \right) \\ &= \frac{J_2}{f^4} (f \delta f_2 - f_2 \delta f) + \frac{J_1}{f^4} (f \delta f_1 - f_1 \delta f) \\ &= \frac{1}{f^3} (J_1 \delta f_1 + J_2 \delta f_2) - \frac{\delta f}{f^4} (J_1 f_1 + J_2 f_2) \dots\dots\dots (41), \end{aligned}$$

where J_1, J_2, J_3 are the Jacobians of $f_2, f; f, f_1; f_1, f_2$, respectively.

But we have $J_1 f_1 + J_2 f_2 + J_3 f_3 = 0$,

identically; hence (41) becomes

$$\frac{1}{f^3} (J_1 \delta f_1 + J_2 \delta f_2 + J_3 \delta f_3).$$

Thus, finally, we have, from (40),

$$du + \frac{f^{n-3} G}{\phi'(\mathfrak{P})} = 0 \dots\dots\dots (42),$$

where

$$G \equiv J_1 \delta f_1 + J_2 \delta f_2 + J_3 \delta f_3,$$

or

$$\begin{vmatrix} \delta f_1, & \delta f_2, & \delta f \\ \frac{df_1}{d\lambda}, & \frac{df_2}{d\lambda}, & \frac{df}{d\lambda} \\ \frac{df_1}{d\mu}, & \frac{df_2}{d\mu}, & \frac{df}{d\mu} \end{vmatrix},$$

if f_1, f_2, f are written as homogeneous functions of λ, μ . I now mention the geometrical meaning of $G = 0$. Suppose the conic to vary subject to four conditions, then it will evidently, in general, touch an envelope in four points. These points are found from $\delta V = 0$, where V is the equation of the conic; hence, from (37), we easily find

$$\frac{d}{d\vartheta} \left(\frac{f_1}{f} \right) \delta \left(\frac{f_2}{f} \right) - \frac{d}{d\vartheta} \left(\frac{f_2}{f} \right) \delta \left(\frac{f_1}{f} \right) = 0.$$

We thus see that $G = 0$ gives the four points where the conic is intersected by a consecutive.

31. I shall now notice the principal applications of this result. Let U be a conic, then we have

$$du + \frac{G}{f\phi'(\vartheta)} = 0 \dots\dots\dots(43).$$

Now, if the variable conic have double contact with a fixed conic S , we may put

$$G = G_1 f \sqrt{S},$$

where G_1 is a quadratic factor of G ; for if G_2 be the other factor of G corresponding to the points of contact with S , we have

$$G_2 = fP,$$

where P is the chord of contact, and $P^2 = S$ is the equation of the variable conic. We obtain thus

$$\frac{du}{\sqrt{S}} + \frac{G_1}{\phi(\vartheta)} = 0 \dots\dots\dots(44),$$

whence, for the four points of intersection, we get

$$\sum_4 \frac{du}{\sqrt{S}} = 0 \dots\dots\dots(45).$$

Again, suppose the variable conic to touch a quartic curve W in four points, then, in the same way, we may put

$$G = f^2 \sqrt{W},$$

whence we get $\frac{du}{\sqrt{W}} + \frac{f}{\phi'(\mathfrak{P})} = 0;$

we have then $\Sigma \frac{du}{\sqrt{W}} = \Sigma \frac{x du}{\sqrt{W}} = \Sigma \frac{y du}{\sqrt{W}} = 0 \dots\dots\dots(46).$

Applying the result (42) to the case of the cubic, we find that, if a conic having double contact with a fixed conic S meet the curve, then, for the six points of intersection, we have

$$\Sigma \frac{L du}{\sqrt{S}} = 0 \dots\dots\dots(47),$$

where L is an arbitrary line, besides the known relation

$$\Sigma du = 0.$$

If the conic have quartic contact with a quartic curve W , we find

$$\Sigma \frac{V du}{\sqrt{W}} = 0 \dots\dots\dots(48),$$

where V is an arbitrary conic.

For the curves of the fourth order, we find that, if a conic having double contact with a given conic S meet the curve, then

$$\Sigma \frac{V du}{\sqrt{S}} = 0 \dots\dots\dots(49),$$

where V is an arbitrary conic. Also, if the variable conic have quartic contact with a quartic W , then

$$\Sigma \frac{V' du}{\sqrt{W}} = 0 \dots\dots\dots(50),$$

where V' is an arbitrary cubic.

32. There is a modification of the preceding results if the variable conic pass through two fixed points on the curve. In the case of the cubic, we have

$$du + \frac{G}{\phi'(\mathfrak{P})} = 0 \dots\dots\dots(51),$$

where G gives the two points of contact with the envelope, and $\phi(\mathfrak{P})$ involves the four variable points of intersection. Applying this result to the circular cubic and circle, we find that du is proportional to

$$\frac{AP \cdot AQ}{AB \cdot AC \cdot AD} \dots\dots\dots(52),$$

where A, B, C, D are the four points of intersection, and P, Q the two points of contact of the circle with its envelope. The expression (52), it may be remarked, can be replaced by $\Delta_1 p_1$, where Δ_1 is the area of the triangle BOD , and p_1 is the perpendicular from A on the chord of contact. By inversion, then, we find that the preceding result holds for the intersection of a circle with a bicircular quartic. Hence, if a circle have double contact with a bicircular quartic W , we have, for the four points of intersection with a circular cubic or bicircular quartic,

$$\Sigma \frac{L du}{\sqrt{W}} = 0 \dots \dots \dots (53),$$

where L is an arbitrary line.

33. I now proceed to deduce some theorems by means of the foregoing results. I first consider particularly the bicircular quartic with a centre. In this case there are two focal conics, which may be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad \frac{x^2}{a^2 - k^2} + \frac{y^2}{b^2 - k^2} - 1 = 0,$$

the corresponding Jacobian circles being

$$x^2 + y^2 = \pm k^2,$$

respectively. I may remark here then that, if $a^2 + b^2 > k^2$, the arc of the curve is equal to the sum or difference of an arc of each of the mutually reciprocal sphero-conics

$$\frac{b^2 x^2}{a^2 - k^2} + \frac{a^2 y^2}{b^2 - k^2} - z^2 = 0,$$

$$(a^2 - k^2) \frac{x^2}{b^2} + (b^2 - k^2) \frac{y^2}{a^2} - z^2 = 0,$$

where

$$x^2 + y^2 + z^2 = a^2 + b^2 - k^2.$$

These curves are the sections of the sphere by the cylinders standing on the focal conics. This suggests the existence of some theorem connecting the arcs of the curve with those of a circle; and, in fact, I proceed to prove that, supposing the curve to be written in the form

$$SS' = d^4,$$

where S, S' are two equal circles, if a variable circle Σ meet the curve and the two circles S, S' , so that the sum of the arcs intercepted on the curve, minus the sum of the arcs intercepted on S, S' , is constant, then Σ has double contact with a concentric bicircular quartic. I

demonstrate this result in the converse form, which evidently comes to the same thing. Putting

$$S \equiv (u-c)(v-c)-r^2,$$

$$S' \equiv (u+c)(v+c)-r^2,$$

where

$$x+iy = u, \quad x-iy = v,$$

if

$$U \equiv SS' - d^4,$$

$$\begin{aligned} \text{we have } \frac{dU}{du} \frac{dU}{dv} &= S^2 (S' + r^2) + S'^2 (S + r^2) + 2SS' (uv - c^2) \\ &= 4d^4 uv + 4r^2 c^2 (u+v)^2 \\ &= 4 \{ d^4 (x^2 + y^2) + 4r^2 c^2 x^2 \} \\ &= 16 \{ a^2 (a^2 - k^2) x^2 + b^2 (b^2 - k^2) y^2 \}; \end{aligned}$$

$$\text{for } d^4 = 4b^3 (b^2 - k^2), \quad r^2 = a^2 + b^2 - k^2, \quad c^2 = a^2 - b^2.$$

Now, if du is the argument of a point on the curve

$$\begin{aligned} ds &= \sqrt{\left\{ \left(\frac{dU}{dx} \right)^2 + \left(\frac{dU}{dy} \right)^2 \right\}} du \\ &= \sqrt{\{ a^2 (a^2 - k^2) x^2 + b^2 (b^2 - k^2) y^2 \}} du = P du \text{ say... (54).} \end{aligned}$$

Hence, if a circle have double contact with the bicircular quartic

$$a^2 (a^2 - k^2) x^2 + b^2 (b^2 - k^2) y^2 + \lambda U \dots\dots\dots (55),$$

$$\text{we have, from (51), } \frac{du}{P} \propto \frac{f}{\phi'(g)};$$

therefore, from (54),

$$ds = P du \propto f \{ d^4 uv + r^2 c^2 (u+v)^2 \} / \phi'(g).$$

Now, for any point on a circle, we may take

$$u : v : 1 = \alpha + \rho t : \beta + \rho t^{-1} : 1 \dots\dots\dots (56).$$

Hence, for the four points of intersection with the curve, we get

$$\begin{aligned} \Sigma ds &\propto \Sigma \frac{1}{t \phi'(t)} \{ d^4 t (\alpha + \rho t) (\beta t + \rho) + r^2 c^2 (\alpha t + \beta t + \rho + \rho t^3)^2 \} \\ &= r^2 c^2 \rho^3 \left(1 - \frac{1}{t_1 t_2 t_3 t_4} \right) \dots\dots\dots (57). \end{aligned}$$

But, substituting the values (56) of u, v in the equation of the curve,

$$\text{we get } (\beta^2 - c^2) t^4 + \dots + \alpha^2 - c^2 = 0,$$

from which we see that

$$t_1 t_2 t_3 t_4 = \frac{a^2 - c^2}{\beta^2 - c^2} \dots\dots\dots (58).$$

We see thus, from (57) and (58), that the expression for Σds is independent of d^4 , and therefore retains the same value when d vanishes, in which case the curve breaks up into the two circles S, S' . Hence the theorem which I have stated above follows at once.

34. If the circle passing through two pairs of inverse points on the curve has double contact with the bicircular quartic

$$(x^2 + y^2 + k^2)^2 - 4(a^2 x^2 + b^2 y^2) + \frac{4}{\lambda^2} \{a^2(a^2 - k^2)x^2 + b^2(b^2 - k^2)y^2\} = 0,$$

it can be easily shown that the generating circles of the curve at the points intersect on a confocal bicircular quartic; that is, in the case which we are considering, the generating circles at the extremities of arcs, connected in the manner mentioned above, intersect on a confocal bicircular quartic.

As a particular case of this result, let $r = 0$, then the curve is an ellipse of Cassini, and we have

$$\Sigma ds = 0.$$

Hence we see that, if generating circles of the ovals of Cassini intercept arcs whose sum or difference is given, then their points of intersection will lie on a confocal bicircular quartic.

Again, suppose the envelope (55) to coincide with the curve, that is, let $\lambda = 0$, and we see that the generating circles of the curve intercept constant arcs on the circles S, S' . In fact, we can easily verify that the circle

$$x^2 + y^2 - 2ax \cos \phi - 2by \sin \phi + k^2 = 0 \dots\dots\dots (59)$$

cuts off constant arcs from the circles

$$y^2 + (x \pm c)^2 - (a^2 + b^2 - k^2) = 0 \dots\dots\dots (60);$$

for the chords of intersection of (59) with (60) touch the circles

$$x^2 + y^2 \pm 2cx - b^2 = 0.$$

The theorem given above connects arcs of a certain quartic with those of circles, and is a geometrical consequence of the fact that the arc of the curve depends upon elliptic integrals of the third kind with circular parameters. But, in the case of the integrals having logarithmic parameters, the arcs of the curve cannot be connected

with arcs of circles, from which we may infer that the arcs of the general bicircular quartic cannot be connected with circular arcs; for the expression found at p. 100 involves imaginary parameters, and therefore, by transformation, is expressed by integrals with circular and logarithmic parameters. We may, however, notice the following result for the general curve. If two generating circles of the same system intersect on a confocal bicircular quartic, the sum or difference of the arcs intercepted by the points of contact is rectifiable by logarithmic and circular functions.

35. We may also arrive at some results concerning arcs intercepted on the curve by conics which satisfy certain conditions. Let U be a general bicircular quartic, then we can easily show that

$$\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2,$$

where x, y are rectangular coordinates, can, by combination with $U=0$, be expressed in a form which, being equated to zero, represents a bicircular quartic. In fact, if the quartic, being referred to its four Jacobian circles J_1, J_2, J_3, J_4 , be expressed in the form

$$\sum_1^4 \frac{a_i^2 J_i^2}{k_i^2} = 0,$$

where k_i is the radius of J_i , a_i is the semi-axis major of the corresponding focal conic, and

$$\sum_1^4 \frac{J_i^2}{k_i^2} = 0 \text{ identically}$$

(see Casey's *Memoir*), then we can easily find that

$$\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2$$

is proportional to $\sum_1^4 \frac{a_i^4 J_i^2}{k_i^2}$.

Calling this expression V , let us seek the relations connecting the points where a conic having quartic contact with the curve $V+\lambda U$

meets U . We have $ds = \sqrt{V} du$,

and $\sum ds = \sum \sqrt{V} du$;

hence, from (42), we find

$$\Sigma ds = \Sigma_8 \frac{Q}{t\phi'(t)},$$

where $x : y : 1 = (at^2 + bt + c) : (a't^2 + b't + c') : t \dots \dots \dots (61),$

and Q is the expression of the eighth degree in t , which gives the intersection of the conic with V . We thus have, if

$$Q = a_0 t^8 + \dots + a_8,$$

Σds proportional to $a_0 - \frac{a_8}{\omega},$

where $\omega = t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8.$

Now, substituting the values (61) in U and V , respectively, we find that ω has the same value in each case, and, therefore, that Σds vanishes; that is, a conic having quartic contact with $V + \lambda U$ cuts off from the curve four arcs whose sum is constant. In this result we suppose four arcs taken positively, and the other four negatively.

Again, suppose the quartic

$$V + \lambda U - k^2 = 0$$

to intersect U , then we have

$$\Sigma ds = \Sigma \sqrt{V} du = k \Sigma du = 0,$$

from which we infer a result similar to the preceding concerning the intercepted arcs.

Considering the central bicircular quartic

$$U \equiv (x^2 + y^2 + k^2)^2 - 4(a^2 x^2 + b^2 y^2) = 0,$$

we have seen that $\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2$

for any point on U is proportional to

$$a^2 (a^2 - k^2) x^2 + b^2 (b^2 - k^2) y^2 \equiv V.$$

Hence, if the conic $V - L^2 = 0,$

where L is a line, meet the curve, we have, for the eight points of intersection,

$$\Sigma ds = \Sigma \sqrt{V} du = \Sigma L du = 0.$$

That is, a conic touching the pair of lines represented by $V = 0$ cuts

off from the curve four arcs whose sum is constant. In the case of the ovals of Cassini, $k^2 = a^2 + b^2$, and V is proportional to $x^2 + y^2$, from which we see that for this curve the conic has the origin for one of its foci.

36. With respect to the circular twisted quartic for whose arc I have found an expression above (21), I may notice the envelope of a plane cutting off from the curve two arcs whose sum or difference is rectifiable by circular functions and logarithms. With the notation I have used above, it is easy to see, by a particular case of Abel's theorem, that we must have $\phi + \lambda\psi =$ a perfect square in this case,

where
$$\phi = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4,$$

and $\alpha, \beta, \gamma, \delta$ are the coordinates of the plane. Substituting, then, the roots of ψ successively in this identity, and eliminating the constants in the quadratic expression, we get a result which may be

written
$$\lambda^4 + \mu^4 + \nu^4 + \rho^4 = 0 \dots\dots\dots(62),$$

where λ, μ, ν, ρ are the tangential equations of the four points on the curve corresponding to the roots of $\psi = 0$. Now, this equation represents the surface whose equation in tetrahedral coordinates is

$$x^{-1} + y^{-1} + z^{-1} + u^{-1} = 0 \dots\dots\dots(63),$$

that is, a quadrinodal cubic. Hence we see that the envelope of the plane is a quadrinodal cubic.

In the case of the particular curve whose arc is expressible by a pure elliptic integral of the first kind, we easily find that the plane cutting off two equal arcs from the curve touches a cone of the fourth class whose equation is of the form

$$l\beta^2\gamma^2 + m\gamma^2\alpha^2 + n\alpha^2\beta^2 = 0.$$

37. For curves of higher orders, it may be noticed that, if the plane curve whose polar equation is

$$r^6 - 2a^2r^3 \cos 3\theta + k^6 = 0$$

be met by a circle, we have, for the six points of intersection,

$$\sum ds = 0, \text{ or } \sum s = \text{a constant.}$$

In this case, it may be observed, we cannot take the arcs with different signs.