On some Forms of Cubic Determinants. By Mr. R. F. SCOTT.

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1. If we form the strata of a cubic determinant O from the square arrays

$$\begin{array}{ccc} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{array} \quad \begin{array}{ccc} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} \end{array}$$

by repeating the first array m times, and the second $n-m$ times, then [it being remembered that the general cubic determinant of order n $|a_{ijk}|$ can be represented by the product of n lineo-linear functions of alternate units, such as

$$\begin{aligned} & a_{111} e_1 e_1 + a_{112} e_1 e_2 + \dots + a_{11n} e_1 e_n \\ & + a_{121} e_2 e_1 + a_{122} e_2 e_2 + \dots + a_{12n} e_2 e_n \\ & + \dots \dots \dots \dots \dots \dots \\ & + a_{in1} e_n e_1 + a_{in2} e_n e_2 + \dots + a_{inn} e_n e_n \end{aligned}$$

where $e_1 e_2 \dots e_n = e_1 e_2 \dots e_n = 1$], if we write

$$\begin{aligned} A_i &= a_{i11} e_1 + a_{i12} e_2 + \dots + a_{in} e_n, \\ B_i &= b_{i11} e_1 + b_{i12} e_2 + \dots + b_{in} e_n, \end{aligned}$$

it is clear that the value of our cubic determinant is

$$O = (A_1 e_1 + A_2 e_2 + \dots + A_n e_n)^m (B_1 e_1 + B_2 e_2 + \dots + B_n e_n)^{n-m}.$$

Now products of pairs of alternate numbers such as $A_i e_i$ or $B_i e_i$ are what Mr. Spottiswoode* has called concurrent numbers; namely, we have

$$\begin{aligned} (A_i e_i)^2 &= 0 = (B_i e_i)^2, \\ (A_i e_i) (A_k e_k) &= (A_k e_k) (A_i e_i). \end{aligned}$$

Hence it is clear that the first expression on the right for O reduces to the sum of such terms as

$$m! A_p e_p A_q e_q \dots A_r e_r,$$

where $p, q \dots r$ is any m -ad selected from $1, 2 \dots n$.

Similarly, the second expression on the right for O reduces to the sum of such terms as

$$(n-m)! B_u e_u B_v e_v \dots B_w e_w,$$

where $u, v \dots w$ is any $(n-m)$ -ad selected from $1, 2 \dots n$. The product of these two sums will give O . On multiplying the two sums

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together, we shall get a series of terms, such as

$$m! (n-m)! A_p e_p A_q e_q \dots A_r e_r B_u e_u B_v e_v \dots B_w e_w,$$

and in this series all terms vanish where the combinations

$$p, q \dots r \text{ and } u, v \dots w$$

contain any suffix in common. They must therefore be complementary combinations selected from $1, 2 \dots n$. Bringing the order of the suffixes both of the units e and of the alternate numbers A and B to the order $1, 2 \dots n$, we leave the sign of the term unaltered, and hence

$$O = m! (n-m)! S;$$

where S is the sum of the $\frac{n!}{m! (n-m)!}$ ordinary determinants ob-

tained by substituting for each combination m at a time of the rows of the determinant $B = |b_{ik}|$ the corresponding m rows of the determinant $A = |a_{ik}|$.

If the determinants A and B become identical, each term of the sum S becomes equal to A , and hence

$$O = n! A,$$

a result which is well known.

2. In the theorem of Art. 1, let us suppose that the system b_{ik} is the system A_{ik} of first minors of the determinant $|a_{ik}|$. Also let $m = 1$. Then the sum S is the sum of n such determinants as

$$\begin{vmatrix} A_{11} & \dots & A_{1n} \\ A_{21} & \dots & A_{2n} \\ \dots & \dots & \dots \\ a_{i1} & \dots & a_{in} \\ \dots & \dots & \dots \\ A_{n1} & \dots & A_{nn} \end{vmatrix}$$

Expanding this according to the elements of the i^{th} row, we get, by a well-known theorem regarding the first minors of the reciprocal system,

$$A^{n-2} (a_{i1}^2 + a_{i2}^2 + \dots + a_{in}^2).$$

Hence in this case the determinant O is given by the equation

$$O = (n-1)! A^{n-2} \Sigma a_{ik}^2;$$

the sum being the squares of the elements of the determinant $A = |a_{ik}|$.

3. If in the theorem of Art. 1 we repeat the stratum $|a_{ik}|$ $n-1$ times and the stratum $|A_{ik}|$ only once, the resulting cubic determinant is in like manner

$$O = (n-1)! \Sigma A_{ik}^2;$$

the sum being the sum of the squares of first minors of the determinant A .

4. If in the theorem of Art. 1 we replace the stratum b_{ik} by the reciprocal system A_{ik} of a_{ik} and expand each element of the sum S by Laplace's theorem keeping the m rows from the determinant A together, we have, by known theorems regarding minors of reciprocal systems,

$$S = A^{n-m-1} \sum p_{ik}^2,$$

where p_{ik} is any minor of $A = |a_{ik}|$ of order m . Hence, if we form the strata of a cubic determinant by repeating the determinant $A = |a_{ik}|$ m times, and the reciprocal system $|A_{ik}|$ $n-m$ times; this cubic determinant is given by

$$O = m! (n-m)! A^{n-m-1} \sum p_{ik}^2.$$

5. We can, however, obtain another cubic determinant which also, contains as a factor the sum of the squares of all minors of $A = |a_{ik}|$

of order m . Namely, if $\frac{n!}{m!(n-m)!} = \mu = n_m$, and if

$$\begin{array}{ccccc} p_{11} & \dots & p_{1r} & q_{11} & \dots & q_{1r} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{r1} & \dots & p_{rr} & q_{r1} & \dots & q_{rr} \end{array}$$

are the systems of minors of the determinant A , of orders m and $n-m$ respectively. Then, if we form a cubic determinant of order μ by taking $\mu-1$ strata, all identical with q_{ik} , and one stratum p_{ik} ; then, by the theorems of Arts. 1 and 2 and by known theorems in the theory of compound determinants,

$$\begin{aligned} O &= (\mu-1)! A^{n-1-(n-1)m-1} \sum p_{ik}^2 \\ &= (\mu-1)! A^{(n-1)m-1} \sum p_{ik}^2. \end{aligned}$$

The ratio of this cubic determinant to that in Art. 4 is (up to a numerical factor) a power of A .

If we form the strata of a cubic determinant of order μ by repeating the stratum p_{ik} r times, and the stratum q_{ik} $\mu-r$ times, the resulting cubic determinant is in like manner given by

$$O = r! (\mu-r)! A^{n-r-(n-1)m-1} \sum \pi_{ik}^2,$$

where π_{ik} is the system of minors of order r of the determinant $|p_{ik}|$.

6. It is interesting to consider a cubic determinant in which all elements vanish except those situated in the three diagonal planes passing through pairs of opposite edges of the cube in which the elements are arranged.

First, consider the diagonal plane which cuts the strata orthogonally;

then all the elements of the determinant vanish except those lying in the principal diagonals of the several strata. For convenience of notation, let the several rows of the determinant $|a_{ik}|$ be taken to form these diagonals, so that the first row is taken to form the principal diagonal of the first stratum, and so on. Then the cubic determinant reduces to the product

$$C = \Pi (a_{11} e_1 e_1 + a_{22} e_2 e_2 + \dots + a_{nn} e_n e_n)$$

But such products as $e_i e_i$ are concurrent numbers, hence

$$C = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Where the function on the right is formed like a determinant, only all the terms have the positive sign. If we have a cubic determinant in which all the elements vanish except those lying in one of the two diagonal planes of the cube which cut the strata at an angle of 45° , it is clear that in the one case all the elements of each stratum vanish except those of one row, and in the other all vanish except those of one column. For convenience, let us again take the several rows or columns to be those of the determinant $|a_{ik}|$. Then, with the notation of Art. 1, it is clear that in the first case the cubic determinant reduces to

$$C = \Pi (A_i e_i) \\ = |a_{ik}|,$$

and in the second case to

$$C = \Pi (a_{1i} e_i + a_{2i} e_i + \dots + a_{ni} e_i) e_i \\ = |a_{ik}|.$$

Thus, in each of these two cases, the cubic determinant reduces to the ordinary determinant which can be formed from the non-vanishing elements of the cubic determinant.

7. A considerable number of cubic determinants can be obtained by forming their strata from an ordinary determinant. As an example, if we form a cubic determinant from the ordinary determinant $|a_{ik}|$ by multiplying the i^{th} row by $1+h$ ($i = 1, 2 \dots n$), and using this as the i^{th} stratum of the cubic determinant, then the cubic determinant is given

$$\text{by} \quad C = \Pi (A_1 e_1 + A_2 e_2 + \dots + A_n e_n + h A_i e_i).$$

Or, if we write for brevity

$$X = A_1 e_1 + A_2 e_2 + \dots + A_n e_n, \\ c_i = A_i e_i,$$

then

$$\begin{aligned} O &= (X + hc_1)(X + hc_2) \dots (X + hc_n) \\ &= X^n + X^{n-1}h(c_1 + c_2 + \dots + c_n) + \dots \\ &\quad + X^{n-r}h^r p_r + \dots + h^n c_1 c_2 \dots c_n; \end{aligned}$$

where p_r is the sum of the products of the quantities $c_1, c_2, \dots, c_n$ taken r at a time. But, clearly,

$$\begin{aligned} X^{n-r} p_r &= (n-r)! |a_{ik}| \frac{n!}{r! (n-r)!} \\ &= \frac{n!}{r!} A. \end{aligned}$$

Hence
$$O = A n! \left\{ 1 + \frac{h}{1!} + \frac{h^2}{2!} + \dots + \frac{h^n}{n!} \right\}.$$

If we had formed the strata of the cubic determinant by simply repeating the system a_{ik} n times, the value of the resulting cubic determinant would have been

$$O' = A n!$$

Hence we see that, whatever the system $|a_{ik}|$ may be,

$$\frac{O}{O'} = 1 + \frac{h}{1!} + \frac{h^2}{2!} + \dots + \frac{h^n}{n!}.$$

$$\left\{ \text{If we increase } n \text{ indefinitely } \frac{O}{O'} = e^h. \right\}$$

8. If we form the i^{th} stratum of a cubic determinant by adding the quantities $c_1, c_2, \dots, c_n$ to the i^{th} row of the determinant $|a_{ik}|$, the resulting cubic determinant is given by

$$O = \Pi (A_1 e_1 + A_2 e_2 + \dots + A_n e_n + c e_i),$$

where

$$c = c_1 e_1 + c_2 e_2 + \dots + c_n e_n.$$

By effecting the multiplication we get

$$O = n! A + (n-1)! \Sigma \begin{vmatrix} a_{11} & \dots & a_{n1} \\ \dots & \dots & \dots \\ c_1 & \dots & c_n \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}^p$$

where the suffix p means that we have substituted the row $c_1, c_2, \dots, c_n$ for the p^{th} row of the determinant A . The summation extending over all the values of p from 1 to n , it is clear that the coefficient of c in

the sum is

$$A_{1r} + A_{2r} + \dots + A_{nr};$$

also that the sum can be written in the form

$$- \begin{vmatrix} a_{11} & \dots & a_{1n} & 1 \\ \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} & 1 \\ c_1 & \dots & c_n & \dots \end{vmatrix}$$

Hence the cubic determinant of order n reduces to an ordinary determinant of order $(n+1)$; viz., we have

$$C = -(n-1)! \begin{vmatrix} a_{11} & \dots & a_{1n} & 1 \\ \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} & 1 \\ c_1 & \dots & c_n & -n \end{vmatrix}$$

9. If for each element a_{ik} of an ordinary determinant we write $a_{ik} + c$, we get n^2 new determinants a_{ik} . These can be taken to form another determinant $|a_{ik}|$, whose value we shall now proceed to find.

We have clearly $a_{ik} = A + cA_{ik}$.

$$\begin{aligned} \text{Hence } |a_{ik}| &= \begin{vmatrix} A + cA_{11} & \dots & A + cA_{1n} \\ \dots & \dots & \dots \\ A + cA_{n1} & \dots & A + cA_{nn} \end{vmatrix} \\ &= \begin{vmatrix} 1, & 0 & \dots & 0 \\ 1, & A + cA_{11} & \dots & A + cA_{1n} \\ \dots & \dots & \dots & \dots \\ 1, & A + cA_{n1} & \dots & A + cA_{nn} \end{vmatrix} \end{aligned}$$

Or, multiplying the first column by A and subtracting it from each of the other columns,

$$|a_{ik}| = \begin{vmatrix} 1, & -A & \dots & -A \\ 1, & cA_{11} & \dots & cA_{1n} \\ \dots & \dots & \dots & \dots \\ 1, & cA_{n1} & \dots & cA_{nn} \end{vmatrix}$$

Expanding this determinant, by Cauchy's theorem, according to products of elements in the first row and column, we have

$$|a_{ik}| = c^n A^{n-1} + c^{n-1} A^{n-1} S,$$

where S is the sum of all the elements a_{ik} .

This theorem has been introduced as the first of a series, of which two more will be proved in the following paragraphs by the use of alternate numbers. It will be useful therefore to give the proof of this theorem by aid of alternate numbers. We have then

$$|a_{ik}| = \Pi \{c(A_{i1}e_1 + A_{i2}e_2 + \dots + A_{in}e_n) + AE\},$$

where

$$E = e_1 + e_2 + \dots + e_n.$$

But this product already

$$= c^n A^{n-1} + A c^{n-1} \Sigma \begin{vmatrix} A_{11} & \dots & A_{1n} \\ \dots & \dots & \dots \\ 1 & \dots & 1 \\ \dots & \dots & \dots \\ A_{n1} & \dots & A_{nn} \end{vmatrix} p$$

the suffix p , as before, meaning that in the determinant $|A_{ik}|$ we have replaced the p^{th} row by the row 1, 1 ... 1. The summation extends to all values of p from 1 to n .

Expanding each of the determinants in this sum according to the elements of the p^{th} row, this expression reduces to the same result as before.

10. If now we have the n quantities $c_1, c_2 \dots c_n$, and if we substitute $c_p + a_{pk}$ for a_{pk} in the determinant A , we get a series of n^3 elements,

$$c_{pik} = A + c_p A_{ik},$$

which we can take to form a cubic determinant C . Representing this determinant by a product of alternate numbers in the usual way, we have

$$C = \Pi \{ AEE' + c_p (A_{11}e_1e_1 + A_{12}e_1e_2 + \dots + A_{1n}e_1e_n \\ + A_{21}e_2e_1 + A_{22}e_2e_2 + \dots + A_{2n}e_2e_n \\ + \dots \\ + A_{n1}e_ne_1 + A_{n2}e_ne_2 + \dots + A_{nn}e_ne_n) \};$$

where

$$E = e_1 + e_2 + \dots + e_n,$$

$$E' = e_1 + e_2 + \dots + e_n.$$

If we write

$$a_i = A_{i1}e_1 + A_{i2}e_2 + \dots + A_{in}e_n,$$

then it is clear that we have

$$C = \Pi \{ AEE' + c_p (u_1e_1 + a_2e_2 + \dots + a_ne_n) \} \\ = n! c_1c_2 \dots c_n A^{n-1} + (n-1)! c_1c_2 \dots c_n A \Sigma \frac{1}{c_p} \begin{vmatrix} A_{11} & \dots & A_{1n} \\ \dots & \dots & \dots \\ 1 & \dots & 1 \\ \dots & \dots & \dots \\ A_{n1} & \dots & A_{nn} \end{vmatrix}$$

the suffix p having the same meaning as before.

Reducing the determinant, we have

$$C = (n-1)! A^{n-1} c_1c_2 \dots c_n \left\{ n + \Sigma \frac{a_{p1} + a_{p2} + \dots + a_{pn}}{c_p} \right\};$$

the summation extending to all values of p .

11. Let us now take the n^2 quantities

$$\begin{array}{cccc} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} \end{array}$$

and let the determinant formed with them be denoted by B . Then, if for the element a_{ik} of the determinant A we substitute $a_{ik} + b_{pk}$ we get

$$n^4 \text{ elements} \quad c_{pqik} = A + b_{pk} A_{ik},$$

which we can take to form the elements of a determinant C of the fourth class. Representing this as a product of alternate numbers, we have

$$C = \Pi \{ A E_1 E_2 E_3 + B_p X \};$$

where

$$E_1 = e_1 + e_2 + \dots + e_n,$$

$$E_2 = e_1 + e_2 + \dots + e_n,$$

$$E_3 = \eta_1 + \eta_2 + \dots + \eta_n,$$

$$B_p = b_{p1} e_1 + b_{p2} e_2 + \dots + b_{pn} e_n,$$

$$X = a_1 \eta_1 + a_2 \eta_2 + \dots + a_n \eta_n.$$

Multiplying out this product, we obtain

$$C = n! A^{n-1} B + \Sigma X^{n-1} A E_1 E_2 E_3 \begin{vmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ 1 & \dots & 1 \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} \end{vmatrix}$$

$$= n! A^{n-1} B + (n-1)! A^{n-1} S_1 S_2;$$

where

$$\begin{aligned} S_1 = & a_{11} + a_{12} + \dots + a_{1n} \\ & + a_{21} + a_{22} + \dots + a_{2n} \\ & + \dots \dots \dots \\ & + a_{n1} + a_{n2} + \dots + a_{nn}. \end{aligned}$$

$$\begin{aligned} S_2 = & B_{11} + \dots + B_{1n} \\ & + \dots \dots \dots \\ & + B_{n1} + \dots + B_{nn}. \end{aligned}$$

Or, since

$$S_2 = - \begin{vmatrix} b_{11} & \dots & b_{1n} & 1 \\ \dots & \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} & 1 \\ 1 & \dots & 1 & \dots \end{vmatrix}$$

$$A^{n-2} S_1 = - \begin{vmatrix} A_{11} & \dots & A_{1n} & 1 \\ \dots & \dots & \dots & \dots \\ A_{n1} & \dots & A_{nn} & 1 \\ 1 & \dots & 1 & \dots \end{vmatrix}$$

We may, if we please, write O in the form

$$O = n! A^{n-1} B + (n-1)! A \begin{vmatrix} A_{11} & \dots & A_{1n} & 1 \\ \dots & \dots & \dots & \dots \\ A_{n1} & \dots & A_{nn} & 1 \\ 1 & \dots & 1 & \dots \end{vmatrix} \begin{vmatrix} b_{11} & \dots & b_{1n} & 1 \\ \dots & \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} & 1 \\ 1 & \dots & 1 & \dots \end{vmatrix}$$

12. The following theorem is clearly involved in Arts. 10 and 11.

If $X = A_1 e_1 + A_2 e_2 + \dots + A_n e_n$,

with the same notation as before, and if

$$B_i = b_{i1} e_1 + b_{i2} e_2 + \dots + b_{in} e_n,$$

$$O_k = c_{k1} e_1 + c_{k2} e_2 + \dots + c_{nk} e_n;$$

then $X^{n-1} B_i O_k = -(n-1)! \begin{vmatrix} a_{11} & \dots & a_{1n} & c_{1k} \\ a_{21} & \dots & a_{2n} & c_{2k} \\ \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} & c_{nk} \\ b_{i1} & \dots & b_{in} & \dots \end{vmatrix}$

For one term of X^{n-1} is

$$(n-1)! A_1 e_1 \dots A_{r-1} e_{r-1} A_{r+1} e_{r+1} \dots A_n e_n.$$

Multiplying this into $B_i O_k$, we have

$$\begin{aligned} & (n-1)! A_1 e_1 \dots A_{r-1} e_{r-1} A_{r+1} e_{r+1} \dots A_n e_n B_i O_k \\ &= (-1)^{n-r} (n-1)! \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{r-11} & \dots & a_{r-1n} \\ a_{r+11} & \dots & a_{r+1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \\ b_{i1} & \dots & b_{in} \end{vmatrix} c_{rk}. \end{aligned}$$

Summing this for all values of r , we get the product $X^{n-1} B_i O_k$ by Cauchy's theorem. This gives us at once the theorem in question.

This remarkable theorem can be generalised as follows:—If from the three determinants of order n ,

$$A = |a_{ik}|, \quad B = |b_{ik}|, \quad O = |c_{ik}|,$$

we form a determinant of order $n+m$ by bordering the determinant A with m rows of B and m columns of O , then, if

$$A_i = a_{i1} e_1 + \dots + a_{in} e_n,$$

$$B_k = b_{k1} e_1 + \dots + b_{kn} e_n,$$

$$O_r = c_{r1} e_1 + \dots + c_{rn} e_n,$$

$$X = A_1 e_1 + \dots + A_n e_n,$$

the determinant obtained by bordering A with the $u^{\text{th}}, v^{\text{th}} \dots$ rows of B and the $p^{\text{th}}, q^{\text{th}} \dots$ columns of C , is equal to

$$\frac{(-1)^m}{(n-m)!} X^{n-m} B_u B_v \dots C_p C_q \dots$$

13. From the expressions in Art. 12, can be deduced with the the greatest facility theorems regarding compound determinants whose elements are bordered determinants such as the above, these determinants being in fact reduced to determinants of alternate numbers. The following theorem is an example of the class of theorems referred to. Taking the cubic determinant of order n

$$A = | a_{ijk} | ,$$

if we border the i^{th} stratum by a row and column from the ordinary determinants $B = | b_{ik} |$, $C = | c_{ik} |$, we can form a new cubic determinant whose elements are of the form

$$P_{irk} = \begin{vmatrix} a_{i11} & \dots & a_{i1n} & c_{1k} \\ \dots & \dots & \dots & \dots \\ a_{i1n} & \dots & a_{i1n} & c_{nk} \\ b_{r1} & \dots & b_{rn} & \dots \end{vmatrix}$$

Then it can be proved that this cubic determinant is equal to

$$(-1)^n BC | A_{ijk} | ;$$

where A_{ijk} is the minor of a_{ijk} in the ordinary determinant formed from the elements of the i^{th} stratum of the given cubic determinant; namely, by Cauchy's theorem,

$$\begin{aligned} P_{irk} &= -\Sigma b_{r1} c_{1k} A_{i11} \\ \text{or } -P_{irk} &= c_{1k} (b_{r1} A_{i11} + b_{r2} A_{i12} + \dots) \\ &\quad + c_{2k} (b_{r1} A_{i21} + b_{r2} A_{i22} + \dots) \\ &\quad + \dots \\ &= c_{1k} a_{ir1} + c_{2k} a_{ir2} + \dots \text{ say.} \end{aligned}$$

$$\begin{aligned} \text{Thus } | P_{irk} | &= (-1)^n C | a_{irk} | \\ &= (-1)^n BC | A_{irk} | \end{aligned}$$

by the rule for multiplying a cubic by an ordinary determinant.