Second Paper on Reciprocants. By L. J. ROGERS, B.A.

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§ 1. In my last memoir on Reciprocants, pp. 220–231, I showed the existence of a certain class of mixed homogeneous reciprocants, to which I gave the name of homographic.

I propose in the following pages to show how all such reciprocants can be very conveniently expressed in terms of a certain series of protomorphs slightly differing from the series $M_1, M_2, M_3 \dots$ which I used in my last memoir.

Homographic reciprocants are, as has already been shown, annihilated by the operator

$$t\delta_a + 3a\delta_b + 6b\delta_c + 10c\delta_d + \dots \dots\dots(1),$$

which I call H .

Let G_m denote the mixed generator

$$t\delta_x - \frac{n}{2} a \dots\dots\dots(2),$$

where n is the characteristic of the reciprocant to be operated upon. Then we know that $G_m M_1 = M_2$, $G_m M_2 = M_3$, &c., and generally $G_m M_m = M_{m+1}$. Also the characteristic of M_m is $3m$.

Now, let $N_1, N_2, N_3 \dots$ be a set of protomorphs of characteristic 3, 6, 9 ... respectively, of which the successive formation is given by the general equation

$$N_{m+1} = \lambda_m G_m N_m + \mu_m a^2 N_{m-1} \dots\dots\dots(3),$$

where λ_m, μ_m are numerical and functions of m . It is always possible to suppose this, as the characteristic of N_{m+1} is $3m+3$, and therefore $6 +$ that of N_{m-1} ; and that of a^3 being six also, we see that N_{m+1} and $a^2 N_{m-1}$ are of the same characteristic. The character of the three terms in the equation is likewise seen to be the same.

Let also, if possible, the operation of H on N_{m+1} give $k_m N_m t$ as result, k_m being numerical. By hypothesis, then,

$$HN_{m+1} = k_m N_m t \dots\dots\dots(4).$$

Operating on (3) with H , we therefore get

$$k_m N_m t = \lambda_m H G_m N_m + 2\mu_m a t N_{m-1} + \mu_m t a^2 k_{m-2} N_{m-2} \dots\dots\dots(5).$$

We now have to evaluate $HG_m N_m$.

It is easy to show that

$$H\delta_x - \delta_x H = t\delta_t + 2a\delta_a + 3b\delta_b + \dots$$

We know, too, that

$$(2t\delta_t + 3a\delta_a + 4b\delta_b + \dots) N_m = 3mN_m,$$

and that

$$(t\delta_t + a\delta_a + \dots) N_m = mN_m.$$

Hence

$$(H\delta_x - \delta_x H) N_m = 2mN_m,$$

therefore

$$\begin{aligned} H\delta_x N_m &= \delta_x H N_m + 2mN_m \\ &= k_{m-1} \delta_x (N_{m-1} t) + 2mN_m, \text{ by (4),} \\ &= k_{m-1} a N_{m-1} + k_{m-1} t \delta_x N_{m-1} + 2mN_m \\ &= k_{m-1} a N_{m-1} + k_{m-1} G_m N_{m-1} + k_{m-1} \frac{3m-3}{2} a N_{m-1} + 2mN_m, \text{ by (2),} \\ &= k_{m-1} \left(G_m N_{m-1} + \frac{3m-1}{2} a N_{m-1} \right) + 2mN_m \dots\dots\dots (6). \end{aligned}$$

$$\begin{aligned} \text{Again, } HG_m N_m &= tH\delta_x N_m - \frac{3m}{2} tN_m - \frac{3m}{2} atk_{m-1} N_{m-1} \\ &= tk_{m-1} \left\{ G_m N_{m-1} - \frac{1}{2} a N_{m-1} \right\} + \frac{m}{2} tN_m \dots\dots (7), \end{aligned}$$

as we finally get after substituting for $H\delta_x N_m$ in (6). But

$$G_m N_{m-1} = \frac{N_m - \mu_{m-1} a^2 N_{m-2}}{\lambda_{m-1}}, \text{ by (3),}$$

therefore, substituting in (7), we get

$$\begin{aligned} k_m tN_m &= \lambda \cdot tk_{m-1} \left(\frac{N_m - \mu_{m-1} a^2 N_{m-2}}{\lambda_{m-1}} - \frac{1}{2} a N_{m-1} \right) + \lambda_m \frac{m}{2} tN_m \\ &\quad + 2\mu_m atN_{m-1} + \mu_m a^2 tk_{m-2} N_{m-2} \dots\dots\dots (8), \end{aligned}$$

an equation which ought identically to be true. Hence the coefficient of N_m is zero, since N_m contains linearly a letter which N_{m-1} , N_{m-2} do not contain. Therefore

$$\frac{k_m}{\lambda_m} - \frac{k_{m-1}}{\lambda_{m-1}} = \frac{m}{2} \dots\dots\dots (9).$$

Similarly, equating coefficients of N_{m-1} , N_{m-2} , we get

$$\frac{\lambda_m k_{m-1}}{\mu_m} = 4 \dots \dots \dots (10),$$

and

$$\frac{\lambda_m k_{m-1}}{\mu_m} = \frac{\lambda_{m-1} k_{m-2}}{\mu_{m-1}},$$

which are not independent, since the second follows from the first.

Now, we are free to give λ_m any value we please. Let this value be 2. Then (9) becomes

$$k_m - k_{m-1} = m,$$

and (10)

$$\mu_m = \frac{1}{2} k_{m-1}.$$

It is most convenient to take N_1 , N_2 as the same as M_1 , M_2 , i.e., a and $tb - \frac{3}{2}a^2$, whence

$$HN_1 = t,$$

$$HN_2 = 0;$$

therefore $k_1 = 0$, $k_2 = 2$, $k_3 = 5$, $k_4 = 9$, &c. ...;

and, generally,

$$k_m = \frac{1}{2} (m-1)(m+2),$$

$$\mu_m = \frac{1}{4} (m-2)(m+1).$$

Hence, if

$$N_1 = a,$$

$$N_2 = tb - \frac{3}{2}a^2,$$

$$N_3 = 2G_m N_2,$$

$$N_4 = 2G_m N_3 + a^2 N_2,$$

$$N_5 = 2G_m N_4 + \frac{5}{2}a^2 N_3,$$

$$N_6 = 2G_m N_5 + \frac{9}{2}a^2 N_4,$$

$$\&c. \dots,$$

then $HN_3 = 0$, $HN_4 = 2tN_3$, $HN_5 = 5tN_3$, ... &c.,

which is the same as saying

$$H = t \left(2N_3 \frac{d}{dN_3} + 5N_3 \frac{d}{dN_4} + 9N_4 \frac{d}{dN_5} + \dots \right) \dots \dots \dots (11),$$

when we operate on any function of these N 's. Whence we see that, when homographic reciprocants are expressed in terms of the protomorphs N , they assume a binariant form.

This manner of forming such a series of protomorphs that homo-

graphic reciprocants could be written as a binariant function of them, was suggested to me by a very similar theorem relating to Projective Reciprocants, put forward by Professor Sylvester and Mr. Hammond, —a correction of a theorem of mine I had supposed true, but they had discovered to be erroneous.

Having obtained the above annihilator (11) for homographic reciprocants, we can deduce a similar one for the analogous orthogonals, viz., the circular reciprocants.

$$\text{Let} \quad G_0 = (1+t^2) \delta_x - at,$$

the orthogonal generator. Then, if

$$\begin{aligned} \psi_1 &= a, \\ \psi_2 &= (1+t^2) b - 3a^2t, \\ \psi_3 &= G_0 \psi_2, \\ \psi_4 &= G_0 \psi_3 - a^2 \psi_2, \\ \psi_5 &= G_0 \psi_4 - \frac{5}{2} a^2 \psi_3, \\ &\quad \&c., \quad \&c., \end{aligned}$$

the annihilator for circular reciprocants will be

$$2\psi_3 \frac{d}{d\psi_3} + 5\psi_3 \frac{d}{d\psi_4} + 9\psi_4 \frac{d}{d\psi_5} + \dots \dots \dots (12),$$

when they are expressed as a function of the ψ 's as above defined.

In order to point out more clearly the similarity between the theorems established above and that of Professor Sylvester and Mr. Hammond, I have thought it advisable to introduce here a proof of the latter theorem, without adopting the numerical alterations they make, viz., writing 1.2.*a* for *a*, 1.2.3.*b* for *b*, &c., and consequently $a\delta_b + 2b\delta_c + 3c\delta_d + \dots$ for the projective annihilator Ω .

§ 2. In the unsimplified form

$$\Omega \equiv 3a\delta_b + 8b\delta_c + 15c\delta_d + \dots \dots \dots (1)$$

the differences between each successive coefficient forming the arithmetic series

$$5, 7, 9, \dots$$

Let G_p denote the pure generator

$$a\delta_x - \frac{n}{3} b \dots \dots \dots (2),$$

and, as in my former memoir on *Reciprocants*,

$$R_2 = ac - \frac{5}{3}b^3,$$

$$R_3 = a^2d - 5abc + \frac{40}{9}b^3,$$

and, generally,

$$R = G_p R_{m-1} \dots \dots \dots (3).$$

Here n , the characteristic, is four times the weight.

This series of simplest educts is not, however, convenient for expressing projective reciprocants.

As in § 1, let us take another series of protomorphs $S_2, S_3, S_4, \dots$ of weights 8.12.16... such that

$$\Omega S_{m+1} = k_m a S_m \dots \dots \dots (4),$$

where k_m is numerical, and if possible such that

$$S_{m+1} = \lambda_m G_p S_m + \mu_m R_2 S_{m-1} \dots \dots \dots (5).$$

Operating on each side with Ω and substituting in (4), and remembering that

$$\Omega \delta_x S_m - \delta_x \Omega S_m = (3a\delta_a + 5b\delta_b + \dots) S_m = (2n - 3m) S_m = 5m S_m,$$

we get, proceeding step by step in the same manner as in § 1,

$$\frac{k_m}{\lambda_m} - \frac{k_{m-1}}{\lambda_{m-1}} = m \dots \dots \dots (6),$$

$$\frac{k_{m-2} \lambda_{m-1}}{\mu_{m-1}} = \frac{k_{m-1} \lambda_m}{\mu_m} = -6.$$

Assuming S_3 to be the same as R_3 , we know that

$$\Omega S_3 = 0,$$

i.e.,

$$k_2 = 0.$$

Let $\lambda_m = 1$, so that

$$k_m - k_{m-1} = m,$$

$$k_3 = 3,$$

$$k_4 = 7,$$

$$k_5 = 12, \quad \&c. \dots$$

$$k_m = \frac{1}{2} (m-2)(m+3),$$

and

$$\mu_m = -\frac{1}{6} k_{m-1} = -\frac{1}{12} (m-3)(m+2).$$

Hence, if

$$S_3 = R_3,$$

$$S_4 = G_p S_3,$$

$$S_6 = G_p S_4 - \frac{1}{2} R_2 S_3 = R_6 - \frac{1}{2} R_2 R_3,$$

$$S_8 = G_p S_6 - \frac{1}{6} R_2 S_4,$$

&c.,

$$\text{then} \quad \Omega \equiv a \left(3S_3 \frac{d}{dS_4} + 7S_4 \frac{d}{dS_6} + 12S_6 \frac{d}{dS_8} + \dots \right) \dots \dots \dots (7).$$

And any function of the S 's annihilated by Ω , as given in (7), will be a projective reciprocant.

The simplest projective after S_3 is

$$6S_3 S_6 - 7S_4^2,$$

which is easily seen to be the same as

$$6R_3 R_6 - 7R_4^2 - 3R_2^2 R_3 \dots \dots \dots (8),$$

which is of weight 8, order 8, and characteristic 32.

We can obtain a series of projective protomorphs in two ways, either by forming them on the model of ordinary invariants and altering the numerical coefficients so as to make them admit of annihilation by Ω (7), and so get them expressed in terms of the S 's; or by forming an absolute reciprocant and differentiating. If we differentiate with respect to x , we get a very complicated result, but if with respect to r , where $r = \int a^4 dx$, we have, as shown in my last memoir, $\frac{dR_m}{dr} = R_{m+1}$, so that the production of successive protomorphs is very simple. We thus get them in terms of the R 's.

For instance, from R_3 and $6R_3 R_6 - 7R_4^2 - 3R_2^2 R_3$ can be deduced a projective reciprocant with $R_3^2 R_6$ for its first term, of characteristic 48. From the latter and from R_3 , we deduce another beginning with $R_3^3 R_7$, of characteristic 64, and so on.

§ 3. The method of obtaining a third reciprocant from two given ones admits of easy explanation. For the sake of brevity, let us take the following statement:—

$$S \text{ is rec. } n(q),$$

as meaning, S is a reciprocant of characteristic n and character q ; q being ± 1 .

If, then, R is rec. $m(p)$,
 and S is rec. $n(q)$,
 then will $mR\delta_x S - nS\delta_x R$ be rec. $m+n+1, (pq)$.

For $\frac{S^m}{R^n}$ is rec. 0, $(p^n q^m)$,

whence by taking logarithms and differentiating, we get that

$$\frac{mR\delta_x S - nS\delta_x R}{RS} \text{ is rec. } 1, (+).$$

Hence $mR\delta_x S - nS\delta_x R$ is rec. $m+n+1, (pq)$ (1).

When we are treating of M , ϕ , or R functions, we can put this expression into a similar form. For instance, R and S being two R -functions of weights m and n , we get

$$4mR\delta_x S - 4nS\delta_x R$$

as a reciprocant, by (1), since the characteristic is four times the weight.

That is, $(mR\delta_x S - nS\delta_x R) \frac{dr}{dx}$ is rec.

But $\frac{dr}{dx} = a^{\frac{1}{2}}$ and is a reciprocant. Hence

$$mR\delta_x S - nS\delta_x R \text{ is rec. }(2),$$

of characteristic $4(m+n+1)$, since the weight is $m+n+1$.

On a Class of Projective Reciprocants.

§ 4. Let (ξ, η) be the Boothian coordinates of any tangent to a given curve, and let (x, y) be the Cartesian coordinates of the point of contact.

Then it is well known that three relations exist between ξ, η, x , and

$$y, \text{ viz., } \left. \begin{aligned} x\xi + y\eta &= 1 \\ x d\xi + y d\eta &= 0 \\ \xi dx + \eta dy &= 0 \end{aligned} \right\}(1),$$

Let r be a function of x and y , and ρ the same function of ξ, η . The most convenient function to take for the value r will be determined hereafter.

Let $x_1, x_2, \&c., y_1, y_2, \&c.,$ denote $\frac{dx}{dr}, \frac{d^2x}{dr^2}, \&c.,$ with similar abbreviations for the Greek letters. Then

$$\left. \begin{aligned} x\xi_1 + y\eta_1 &= 0 \\ \xi x_1 + \eta y_1 &= 0 \end{aligned} \right\} \dots\dots\dots(2),$$

$$x\xi_2 + y\eta_2 + (x_1\xi_1 + y_1\eta_1) \frac{dr}{d\rho} = 0,$$

$$\xi x_2 + \eta y_2 + (x_1\xi_1 + y_1\eta_1) \frac{d\rho}{dr} = 0.$$

Eliminating $\frac{dr}{d\rho}$, we get

$$(x\xi_2 + y\eta_2)(\xi x_2 + \eta y_2) = (x_1\xi_1 + y_1\eta_1)^2,$$

or $\frac{\eta y}{x_1\xi_1}(\eta_2\xi_1 - \xi_2\eta_1)(y_2x_1 - x_2y_1) = (x_1\xi_1 + y_1\eta_1)^2$, by (2).

From (2) and (1), we also have

$$x_1\xi_1 + y_1\eta_1 = x_1\xi_1 \left(1 + \frac{x\xi}{y\eta}\right) = \frac{x_1\xi_1}{y\eta}.$$

Hence $(\eta_2\xi_1 - \xi_2\eta_1)(y_2x_1 - x_2y_1) = (x_1\xi_1 + y_1\eta_1)^2 \dots\dots\dots(3).$

Now it is easy to show, by elementary Differential Calculus, that

$$y_2x_1 - x_2y_1 = \frac{d^2y}{dx^2} \left(\frac{dx}{dr}\right)^3.$$

It is, therefore, best to define r as such a function that

$$\frac{d^2y}{dx^2} = \left(\frac{dr}{dx}\right)^3 \dots\dots\dots(4),$$

so that r is the same function as has been used in the preceding sec-

tions. And for ρ that $\frac{d^2\eta}{d\xi^2} = \left(\frac{d\rho}{d\xi}\right)^3.$

Whence (3) becomes $x_1\xi_1 + y_1\eta_1 = 1$

while

$$\left. \begin{aligned} y_2x_1 - x_2y_1 &= 1 \\ \eta_2\xi_1 - \xi_2\eta_1 &= 1 \end{aligned} \right\} \dots\dots\dots(5).$$

In terms of r and x we may very conveniently express the series of pure reciprocant educts $R_3, R_6, R_9, \dots$, though now it is more con-

venient to consider them as the *absolute* pure reciprocants, instead of only the numerators of such. For, by (4),

$$a^{-\frac{1}{3}} = x_1, .$$

$$x_2 = -\frac{1}{3}ba^{-\frac{1}{3}},$$

$$x_3 = -\frac{1}{3}(ca^{-\frac{1}{3}} - \frac{5}{3}b^2a^{-\frac{1}{3}})a^{-\frac{1}{3}} = -\frac{1}{3}R_2x_1.$$

Hence

$$R_2 = -3x_3/x_1,$$

and this, by differentiating this second equation in (5), $= -3y_3/y_1$.

Again,

$$R_3 = \frac{1}{a^{\frac{1}{3}}} \frac{d}{dx} R_2 = \frac{d}{dr} R_2,$$

and generally

$$R_n = \frac{d}{dr} R_{n-1} \dots\dots\dots (6).$$

We shall denote by $P_2, P_3, \dots$ the corresponding absolute reciprocants in (ξ, η) .

Also $r_1, r_2, \dots$ will stand for $\frac{dr}{d\rho} \cdot \frac{d^2r}{d\rho^2} \dots$,

and $\rho_1, \rho_2, \dots$ „ $\frac{d\rho}{dr} \cdot \frac{d^2\rho}{dr^2} \dots$.

By expressing $R_2, R_3, \dots$ in terms of r and ρ , we shall find the connection between the R 's and the P 's.

By (5), we see that, for the value we fix for R , the equations following (2) become

$$x\xi_2 + y\eta_2 = -\frac{dr}{d\rho},$$

which, by (2) and (5), gives

$$\frac{dr}{d\rho} = \frac{x}{\eta_1} = -\frac{y}{\xi_1}, \text{ by (2).}$$

By symmetry $\frac{d\rho}{dr} = \frac{\xi}{y_1} = -\frac{\eta}{x_1},$

therefore $\frac{d\rho}{dr}(y_1x - x_1y) = \xi x + \eta y = \dots$ by (1),

therefore $r_1 = y_1x - x_1y \dots\dots\dots (7).$

Differentiating for ρ , we have

$$\frac{r_2}{r_1} = y_2 x - x_2 y,$$

$$\frac{r_3 r_1 - r_2^2}{r_1^3} = y_3 x - x_3 y + y_2 x_1 - x_2 y_1$$

$$= y_3 x - x_3 y + 1 \text{ by (5)}$$

$$= x_3 \left(\frac{y_1}{x_1} x - y \right) + 1 \text{ because } \frac{y_3}{y_1} = \frac{x_3}{x_1} = -\frac{1}{3} R_2$$

$$= \frac{x_3}{x_1} r_1 + 1 = -\frac{1}{3} R_2 r_1 + 1,$$

therefore
$$-\frac{1}{3} R_2 = \frac{r_3 r_1 - r_2^2}{r_1^4} - \frac{1}{r_1} \dots\dots\dots(8).$$

Differentiating again for ρ , by (6), we have

$$-\frac{1}{3} R_3 = \frac{r_4 r_1 - 5 r_3 r_2}{r_1^5} + \frac{4 r_2^3}{r_1^6} - \frac{r_2}{r_1^3} \dots\dots\dots(9).$$

Now, the right side of (9) is a reciprocant in r and ρ , since the numerator of the first term is the post-Schwarzian in r and ρ , and the other terms are well-known forms.

Since, then, by symmetry,

$$-\frac{1}{3} P_3 = \frac{\rho_4 \rho_1 - 5 \rho_3 \rho_2}{\rho_1^5} + \frac{4 \rho_2^3}{\rho_1^6} - \frac{\rho_2}{\rho_1^3},$$

we have, finally,
$$R_3 = -P_3 \rho_1^3 \dots\dots\dots(10).$$

Thus R_3 is an odd reciprocant in r and ρ , with characteristic 3. Let us write this statement thus, as in § 3,

$$R_3 \text{ is rec. } 3, (-).$$

Now
$$\rho_2 \quad \quad \quad 3, (-).$$

Hence
$$R_4 \rho_2 - R_3 \rho_3 \quad \quad \quad \text{is rec. } 7, (+), \text{ by } \S 3 \dots\dots(11),$$

$$3 R_3 (R_5 \rho_2 - R_4 \rho_3) - 7 R_4 (R_3 \rho_2 - R_3 \rho_3) \quad \quad \quad \text{,, } 11, (-) \dots\dots\dots(12).$$

Again,
$$5 R_4^2 \rho_2^2 - 10 R_4 R_3 \rho_3 \rho_2 + 5 R_3^2 \rho_3^2 \text{ is rec. } 14, (+), \text{ by (11),}$$

and
$$R_3^2 (3 \rho_2 \rho_3 - 5 \rho_3^2) \quad \quad \quad \text{,, } 14, (+).$$

Adding, we get, after dividing by ρ_2 (3-),

$$5R_4^2\rho_2 - 10R_4R_3\rho_3 + 3R_3^2\rho_4 \text{ is rec. 11, } (-) \dots\dots\dots(13).$$

Eliminating the term $R_4R_3\rho_3$ from (12) and (13), we get

$$(30R_3R_5 - 35R_4^2)\rho_1 - 9R_3^2\rho_4 \text{ is rec. 11, } (-) \dots\dots\dots(14).$$

Referring to (8), we easily get

$$-\frac{1}{3}R_3 = -\frac{\rho_3\rho_1 - 2\rho_2^2}{\rho_1^2} - \rho_1,$$

$$\text{therefore} \quad \rho_4 - \frac{1}{3}R_3\rho_2 = \frac{\rho_1\rho_4 - 5\rho_3\rho_2}{\rho_1} + \frac{10}{3}\frac{\rho_2^3}{\rho_1^2} - \frac{5}{3}\rho_1\rho_2,$$

which is recip. 5, (-).

$$\text{Hence} \quad R_3^2(\rho_4 - \frac{1}{3}R_3\rho_2) \text{ is rec. 11, } (-).$$

Eliminating $R_3^2\rho_4$ by means of (14), we get, after dividing by $5\rho_1$,

$$6R_3R_5 - 7R_4^2 - 3R_3^2R_2 \text{ is rec. 8, } (+) \dots\dots\dots(15),$$

$$\text{i.e.,} \quad = (6P_5P_5 - 7P_4^2 - 3P_3^2P_2)\rho_1^8.$$

We have now obtained two R -functions, *i.e.*, two absolute pure reciprocants in x and y , which are also reciprocants for r and ρ . We have therefore one absolute reciprocant for r and ρ , and can by differentiation obtain an infinite number of them.

This series of reciprocants is of high importance geometrically. They lead when integrated, as Mr. Hammond has pointed out, to equations of curves whose polar reciprocals with respect to any point are curves of the same kind.

Thus,

$$R_3 = 0$$

leads to the general equation to a conic. Transforming into Boothians,

$$\text{we have, by (10),} \quad P_3 = 0,$$

i.e., the polar reciprocal is also a conic.

Let us therefore call these reciprocants self-polar. All such reciprocants are projective. For we know that $6R_5R_3 - 7R_4^2 - 3R_3^2R_2$ and R_3 are projective, and that their weights are in the ratio of 3 to 8. So also, treated as reciprocants in r and ρ , are their characteristics in the ratio of 3 to 8. Employing the methods of § 3 for the formation of new reciprocants, it is evident that we shall form a system of self-polar protomorphs identical with the projective protomorphs obtained at the end of § 2.

We see, then, that all self-polar reciprocants are projective, although the converse is not true.