negative, it would appear that there must be a *superior* limit to the order n of a curve which could satisfy the N_n conditions.

If we write π for the deficiency of C_n , we have $\delta_n - \epsilon_n = \nu^2 - \Sigma i^2 =$ *constant* number of ordinary points in which two curves C_n intersect, and $\rho_n + \sigma_n + \pi - 1 = n\nu - \Sigma i =$ number of ordinary points in which C_n and C_n intersect. Reasoning from analogy we should expect 0 to be the least possible value of ϵ_n , corresponding to the breaking-point (p. 679), and $\nu^2 - \Sigma i^2$ to be the least possible value of δ_n .

The Conformal Representation of a Pentagon on a Half Plane.

By Miss M. E. BARWELL. Read June 9th, 1898. Received, in revised form, September 15th, 1898.

1. The conformal representation of a rectilinear polygon on a half plane was first attempted by Schwarz and Christoffel, who arrived independently at the same result. They have shown that the area of the w -plane included by a polygon, whose sides do not cross, can be conformally represented by the northern half of the z -plane, the boundary of the polygon corresponding to the axis of real quantities on the z -plane.

The necessary transformation is

$$w = M \int (z-a)^{\alpha-1} (z-b)^{\beta-1} \dots (z-l)^{\lambda-1} dz + M',$$

where $a, b, \dots, l$ are the points on the real axis of z corresponding to the angular points of the polygon taken in order, and all lying in the finite part of the z -plane.

$\alpha\pi, \beta\pi, \dots, \lambda\pi$ are the internal angles of the polygon at the respective points. The constant M' is determined by fixing the origin in the w -plane. Any three of the real quantities $a, \dots, l$ may be chosen arbitrarily, and the remainder must be determined in terms of these three, and the constants of the polygon $\alpha, \beta, \dots, \lambda$. The case of the quadrilateral is given in Forsyth's *Theory of Functions*, p. 546. There is one unknown quantity besides M to be determined, and the solution involves Gauss' hypergeometric functions.

With the pentagon, there are two unknown quantities, in addition to M , and the necessary equations contain functions analogous to those of Gauss, but satisfying differential equations of the third order (Schäfli, *Crelle*, Vol. LXXVIII.). They may be called "hypergeometric functions of the third order," and they form a special case of the functions investigated by Pochhammer, in his paper on "Hypergeometrische Functionen n^{ter} Ordnung" (*Crelle*, Vol. LXXI.).

Let $\alpha\pi$, $\beta\pi$, $\gamma\pi$, $\delta\pi$, $\kappa\pi$ be the internal angles of the pentagon at A , B , C , D , K respectively. Then

$$\alpha + \beta + \gamma + \delta + \kappa = 3.$$

Also let 0, 1, and ∞ be the homologues in the z -plane of the points A , B , K .

The required transformation is

$$\frac{dw}{dz} = Mz^{\alpha-1}(1-z)^{\beta-1}(1-sz)^{\gamma-1}(1-tz)^{\delta-1},$$

where $\frac{1}{s}$ and $\frac{1}{t}$ correspond to the points C and D , and where

$$1 > s > t > 0.$$

Suppose $w = 0$ at the point A , and that AB is taken as the real axis of w .

$$\text{Then } w = M \int_0^z z^{\alpha-1} (1-z)^{\beta-1} (1-sz)^{\gamma-1} (1-tz)^{\delta-1} dz,$$

and M must be real.

Integrating along the real axis of z , we obtain the following equations :—

$$AB = M \int_0^1 X dx, \quad (1)$$

$$BC \cdot e^{i\pi(1-\beta)} = M \int_1^{1/s} X dx, \quad (2)$$

$$CD \cdot e^{i\pi(2-\beta-\gamma)} = M \int_{1/s}^{1/t} X dx, \quad (3)$$

$$DK \cdot e^{i\pi(3-\beta-\gamma-\delta)} = M \int_{1/t}^{\infty} X dx, \quad (4)$$

$$KA \cdot e^{i\pi(4-\beta-\gamma-\delta-\alpha)} = M \int_{-\infty}^0 X dx, \quad (5)$$

where $X = x^{\alpha-1} (1-x)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1}$.

There are thus five equations to determine M , s , and t ; and they must all be consistent by virtue of the relations between the sides and angles of the pentagon. Any three may be selected for the determination of the three unknown quantities.

It is first necessary to reduce equations (1) to (5) to a real form.

$$\begin{aligned} \text{Now} \quad \int_1^{1/s} X dx &= \int_1^{1/s} x^{\alpha-1} (1-x)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx \\ &= e^{\pm i\pi(\beta-1)} \int_1^{1/s} x^{\alpha-1} (x-1)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx. \end{aligned}$$

The lower sign must be taken, for the argument of $1-x$ is $-\pi$, when

$$\frac{1}{s} > x > 1.$$

$$\text{Hence} \quad \int_1^{1/s} X dx = e^{i\pi(1-\beta)} \int_1^{1/s} x^{\alpha-1} (x-1)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx.$$

Similarly,

$$\begin{aligned} \int_{1/s}^{1/t} X dx &= e^{i\pi(2-\beta-\gamma)} \int_{1/s}^{1/t} x^{\alpha-1} (x-1)^{\beta-1} (sx-1)^{\gamma-1} (1-tx)^{\delta-1} dx, \\ \int_{1/t}^{\infty} X dx &= e^{i\pi(3-\beta-\gamma-\delta)} \int_{1/t}^{\infty} x^{\alpha-1} (x-1)^{\beta-1} (sx-1)^{\gamma-1} (1-tx)^{\delta-1} dx, \end{aligned}$$

$$\begin{aligned}\text{and } \int_{-\infty}^0 X dx &= e^{i\pi(a-1)} \int_{-\infty}^0 (-x)^{a-1} (1-x)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx \\ &= e^{i\pi(a-1)} \int_{-\infty}^0 (-x)^{a-1} (1-x)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx\end{aligned}$$

(since the argument of x is $+\pi$, for $0 > x > -\infty$)

$$= e^{i\pi(2-\beta-\gamma-\delta-\kappa)} \int_{-\infty}^0 (-x)^{a-1} (1-x)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx,$$

since

$$\alpha + \beta + \gamma + \delta + \kappa = 3.$$

Hence the five equations become

$$AB = M \int_0^1 x^{a-1} (1-x)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx, \quad (\text{i.})$$

$$BC = M \int_1^{1/t} x^{a-1} (x-1)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx, \quad (\text{ii.})$$

$$CD = M \int_{1/t}^{1/s} x^{a-1} (x-1)^{\beta-1} (sx-1)^{\gamma-1} (1-tx)^{\delta-1} dx, \quad (\text{iii.})$$

$$DK = M \int_{1/t}^{\infty} x^{a-1} (x-1)^{\beta-1} (sx-1)^{\gamma-1} (tx-1)^{\delta-1} dx, \quad (\text{iv.})$$

$$KA = M \int_{-\infty}^0 (-x)^{a-1} (1-x)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx. \quad (\text{v.})$$

From these equations we can obtain two which will determine s and t , and for this purpose we are at liberty to select any two pairs of equations. For instance, from (ii.) and (iv.),

$$\left. \begin{aligned} DK \int_1^{1/t} x^{a-1} (x-1)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx \\ = BC \int_{1/t}^{\infty} x^{a-1} (x-1)^{\beta-1} (sx-1)^{\gamma-1} (tx-1)^{\delta-1} dx, \end{aligned} \right\} \quad (\text{A})$$

and, from (i.) and (v.),

$$\left. \begin{aligned} AB \int_0^1 (-x)^{a-1} (1-x)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx \\ = KA \int_0^1 x^{a-1} (1-x)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx. \end{aligned} \right\}$$

2. It is to be observed that the definite integrals in equations (1) to (5) on p. 697 are hypergeometric functions as defined by Pochhammer

in his paper already quoted (*Crelle*, Vol. LXXI.). He has established as hypergeometric functions of the n^{th} order the integrals typified by

$$\int_g^h (\xi - a_1)^{b_1-1} (\xi - a_2)^{b_2-1} \dots (\xi - a_n)^{b_n-1} (\xi - s)^{\lambda-1} d\xi,$$

where g and h may be any two of the quantities $a_1, a_2, \dots, a_n, s, \infty$.

Now, in the typical function of the third order, let

$$\xi = \frac{1}{x}, \quad a_1 = 0, \quad a_2 = t, \quad a_3 = 1,$$

$$\lambda = \gamma, \quad b_1 = \kappa, \quad b_2 = \delta, \quad b_3 = \beta;$$

and therefore $3 - \lambda - b_1 - b_2 - b_3 = 3 - \gamma - \kappa - \delta - \beta =$

We then obtain the definite integrals given on p. 697.

3. Expansions of the Integrals on p. 698.

$$\text{i.} \quad \phi_1 = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx$$

(1 > s > t > 0)

is finite, uniform, and continuous, if $\alpha > 0$, $\beta > 0$, and if s and t have no real positive value between 1 and ∞ . Hence the integral can, in the interior of a circle of radius mod $(1-t)$ and centre $s=t$, be developed in a converging series of ascending powers of $s-t$.

$$\begin{aligned} & \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx \\ &= \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} (1-tx)^{\delta-1} \{1-tx-(s-t)x\}^{\gamma-1} dx \\ &= \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} (1-tx)^{\delta+\gamma-2} \left\{1-(\gamma-1) \frac{s-t}{1-tx} x + \dots\right\} dx \\ &= \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} (1-tx)^{\delta+\gamma-2} dx \\ &\quad - (\gamma-1)(s-t) \int_0^1 x^{\alpha} (1-x)^{\beta-1} (1-tx)^{\delta+\gamma-3} dx \\ &\quad + \dots \\ &\quad + (-1)^k \frac{(\gamma-1)(\gamma-2)\dots(\gamma-k)}{k!} (s-t)^k \\ &\quad \times \int_0^1 x^{\alpha+k-1} (1-x)^{\beta-1} (1-tx)^{\delta+\gamma-k-2} dx \\ &\quad + \dots \end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} F(2 - \delta - \gamma, \alpha, \alpha + \beta, t) \\
&\quad - (\gamma - 1) \frac{\Gamma(\alpha + 1) \Gamma(\beta)}{\Gamma(\alpha + \beta + 1)} F(3 - \delta - \gamma, \alpha + 1, \alpha + \beta + 1, t)(s - t) \\
&\quad + \dots \\
&\quad + (-1)^k \frac{\Gamma(\alpha + k) \Gamma(\beta)}{\Gamma(\alpha + \beta + k)} \frac{(\gamma - 1)(\gamma - 2) \dots (\gamma - k)}{k!} \\
&\quad \quad \quad \times F(k + 2 - \delta - \gamma, \alpha + k, \alpha + \beta + k, t)(s - t)^k \\
&\quad + \dots,
\end{aligned}$$

where $F(\alpha, \beta, \gamma, x)$ is the hypergeometric series

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \dots$$

Since

$$2 - \delta - \gamma = \alpha + \beta + \kappa - 1,$$

the series may be denoted by $F_{3,\kappa}(\alpha + \beta + \kappa - 1, \alpha, \alpha + \beta, 1 - \gamma, t, s - t)$, where the first suffix means that the expansion is that of a hypergeometric function of the *third* order, the second that α is the element which increases by unity from term to term.

$$\text{ii.} \quad \phi_2 = \int_1^{1/t} x^{\alpha-1} (x-1)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx,$$

after division by $(1-s)^{\beta+\gamma-1}$, is finite, uniform, and continuous, if $\beta > 0$, $\gamma > 0$, and if t has no real positive value between s and 1. The integral can therefore be expanded in the neighbourhood of $t = 0$ in a converging series of powers of t , and also, in the vicinity of $s = 1$, in a converging series of powers of $1 - s$.

$$\text{Let} \quad x = \frac{1}{1 - (1-s)\xi};$$

$$\text{then} \quad dx = \frac{(1-s) d\xi}{\{1 - (1-s)\xi\}^2},$$

$$1 - sx = \frac{(1-s)(1-\xi)}{1 - (1-s)\xi}, \quad x - 1 = \frac{(1-s)\xi}{1 - (1-s)\xi},$$

$$1 - tx = 1 - \frac{t}{1 - (1-s)\xi}.$$

$$\begin{aligned}
 \text{Then } \phi_s &= (1-s)^{\beta+\gamma-1} \int_0^1 \xi^{\beta-1} (1-\xi)^{\gamma-1} (1-\overline{1-s}\xi)^{1-\beta-\gamma} \\
 &\quad \times \left\{ 1 - \frac{t}{1-(1-s)\xi} \right\}^{\delta-1} dx \\
 &= (1-s)^{\beta+\gamma-1} \int_0^1 \xi^{\beta-1} (1-\xi)^{\gamma-1} (1-\overline{1-s}\xi)^{1-\beta-\gamma} \\
 &\quad \times \left\{ 1 - (\delta-1) \frac{t}{1-\overline{1-s}\xi} + \dots \right\} d\xi \\
 &= (1-s)^{\beta+\gamma-1} F_{\beta, \beta-\kappa-\delta} (2-\kappa-\delta, \beta, \beta+\gamma, 1-\delta, 1-s, t),
 \end{aligned}$$

in accordance with our notation.

$$\text{iii. } \phi_s = \int_{1/s}^{1/t} x^{s-1} (x-1)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx,$$

after division, by $(s-t)^{\gamma+\delta-1}$, is uniform, finite, and continuous, if $\gamma > 0$, $\delta > 0$, and if s and t have only values lying between 0 and 1. Thus ϕ_s can, in the vicinity of $t=s$, be developed in a converging series of powers of $s-t$, the coefficients of which will involve s .

$$\text{Let } \frac{sx-1}{x(s-t)} = \xi;$$

$$\begin{aligned}
 \text{then } x &= \frac{1}{s-(s-t)\xi}, \quad dx \frac{s-t}{\{s-(s-t)\xi\}^2} d\xi, \\
 x-1 &= \frac{1-s+(s-t)\xi}{s-(s-t)\xi}, \quad sx-1 = \frac{(s-t)\xi}{s-(s-t)\xi}, \\
 1-tx &= \frac{(s-t)(1-\xi)}{s-(s-t)\xi}.
 \end{aligned}$$

$$\begin{aligned}
 \phi_s &= (s-t)^{\gamma+\delta-1} \int_0^1 \xi^{\beta-1} (1-\xi)^{\gamma-1} \{s-(s-t)\xi\}^{2-\beta-\gamma-\delta} \\
 &\quad \times \{1-s+(s-t)\xi\}^{\beta-1} d\xi \\
 &= (s-t)^{\gamma+\delta-1} s^{\beta-1} (1-s)^{\beta-1} \int_0^1 \xi^{\beta-1} (1-\xi)^{\gamma-1} \left(1 - \frac{s-t}{s}\xi\right)^{\kappa-1} \\
 &\quad \times \left(1 - \frac{s-t}{1-s}\xi\right)^{\beta-1} d\xi
 \end{aligned}$$

$$= \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} s^{\kappa-1} (1-s)^{\beta-1} (s-t)^{\gamma+\delta-1}$$

$$\times \left[1 - \frac{\gamma}{\gamma+\delta} \left(\frac{\kappa-1}{s} + \frac{\beta-1}{1-s} \right) (s-t) + \frac{\gamma(\gamma+1)}{(\gamma+\delta)(\gamma+\delta+1)} \right.$$

$$\left. \left\{ \frac{(\kappa-1)(\kappa-2)}{2! s^2} + \frac{(\kappa-1)(\beta-1)}{s(s-1)} + \frac{(\beta-1)(\beta-2)}{2! (1-s)^2} \right\} (s-t)^2 + \dots \right].$$

$$\text{iv.} \quad \phi_4 = \int_{1/t}^{\infty} x^{\alpha-1} (x-1)^{\beta-1} (sx-1)^{\gamma-1} (tx-1)^{\delta-1} dx.$$

When $t = 0$ the integral becomes

$$\text{Lt}_{t \rightarrow 0} \int_{1/t}^{\infty} x^{\alpha-1} (x-1)^{\beta-1} (sx-1)^{\gamma-1} dx,$$

and is therefore infinite and of the order $\alpha + \beta + \gamma - 2$, *i.e.*, $1 - \kappa - \delta$. Hence, after division, by $t^{\kappa+\delta-1}$, ϕ_4 will be finite, uniform, and continuous, if $\delta > 0$, $3 - \alpha - \beta - \gamma - \delta > 0$, *i.e.*, if $\delta > 0$ and $\kappa > 0$, and if, moreover, s has no real positive value between 0 and t . Hence $\frac{\phi_4}{t^{\kappa+\delta-1}}$ can be developed in a converging series of powers of t , and also of $1-s$.

$$\text{Let} \quad x = \frac{1}{t\xi};$$

$$\text{then} \quad dx = -\frac{1}{t\xi^2} d\xi,$$

$$x-1 = \frac{1-t\xi}{t\xi}, \quad sx-1 = \frac{s-t\xi}{t\xi}, \quad tx-1 = \frac{1-\xi}{\xi},$$

$$\phi_4 = t^{2-\alpha-\beta-\gamma} \int_0^1 \xi^{2-\alpha-\beta-\gamma-\delta} (1-\xi)^{\delta-1} (1-t\xi)^{\beta-1} (1-t\xi-\overline{1-s})^{\gamma-1} d\xi$$

$$= t^{\kappa+\delta-1} \int_0^1 \xi^{\kappa-1} (1-\xi)^{\delta-1} (1-t\xi)^{\beta+\gamma-2} \left\{ 1 - (\gamma-1) \frac{1-s}{1-t\xi} + \dots \right\} d\xi$$

$$= t^{\kappa+\delta-1} F_{3, 2-\beta-\gamma} (2-\beta-\gamma, \kappa, \kappa+\delta, 1-\gamma, t, 1-s).$$

This result might have been obtained from the expansion of ϕ_4 on p. 701, by interchanging β and κ , γ and δ , t and $1-s$.

$$\text{v.} \quad \phi_5 = \int_{-\infty}^0 (-x)^{\alpha-1} (1-x)^{\beta-1} (1-sx)^{\gamma-1} (1-tx)^{\delta-1} dx$$

is uniform, finite, and continuous, if $\alpha > 0$, and $3 - \alpha - \beta - \gamma - \delta > 0$ (*i.e.*, if $\alpha > 0$ and $\kappa > 0$), and if, moreover, s and t have no real negative

value. Hence, in the neighbourhood of $s = 1$, ϕ_s can be developed in a converging series of powers of $1-s$, and again, when considered as a function of t , it can be expanded in a similar series of powers of $s-t$.

$$\text{Let } x = \frac{\xi}{\xi-1} = 1 + \frac{1}{\xi-1}, \quad dx = \frac{-d\xi}{(1-\xi)^2},$$

$$1-x = \frac{1}{1-\xi}, \quad 1-sx = \frac{1-(1-s)\xi}{1-\xi}, \quad 1-tx = \frac{1-(1-t)\xi}{1-\xi},$$

$$\begin{aligned} \phi_s &= \int_0^1 \xi^{a-1} (1-\xi)^{2-a-\beta-\gamma-t} (1-\overline{1-s}\xi)^{\gamma-1} (1-\overline{1-t}\xi)^{s-1} d\xi \\ &= \int_0^1 \xi^{a-1} (1-\xi)^{a-1} (1-\overline{1-s}\xi)^{\gamma+t-2} \left\{ 1-(\delta-1) \frac{s-t}{1-\overline{1-s}\xi} \xi + \dots \right\} d\xi \\ &= F_{3,a}(a+\beta+\kappa-1, a, a+\kappa, 1-\delta, 1-s, s-t). \end{aligned}$$

The two equations on p. 698 for the determination of s and t can now be written in the form

$$\left. \begin{aligned} (1) \quad & (1-s)^{s+\gamma-1} F_{3,2-\kappa-t}(2-\kappa-\delta, \beta, \beta+\gamma, 1-\delta, 1-s, t) \\ &= \frac{BC}{DK} t^{t+\delta-1} F_{3,2-\beta-\gamma}(2-\beta-\gamma, \kappa, \kappa+\delta; 1-\gamma, t, 1-s), \\ (2) \quad & F_{3,a}(a+\beta+\kappa-1, a, a+\beta, x-\gamma, t, s-t) \\ &= \frac{AB}{KA} F_{3,a}(a+\beta+\kappa-1, a, a+\kappa, 1-\delta, 1-s, s-t). \end{aligned} \right\} \quad (B)$$

4. The Numerical Determination of s and t .

The questions naturally arise: Are the equations (B) at the end of the last paragraph suitable for the determination of the numerical values of s and t , and why should one pair of the equations (i.) to (v.) on p. 698 be selected rather than another?

With regard to the latter question, let us consider the expansions of the functions ϕ , that is, of the five definite integrals.

- ϕ_1 can be expanded in a double series of powers of t and $s-t$;
- ϕ_2 , after division by $(1-s)^{s+\gamma-1}$, in powers of $1-s$ and t ;
- ϕ_3 , after division by $(s-t)^{s+\delta-1}$, in powers of $s-t$, the coefficients of which contain s ;
- ϕ_4 , after division by $t^{t+\kappa-1}$, in powers of $1-s$ and t ;
- ϕ_5 in a double series of powers of $s-t$ and $1-s$.

It is in general advantageous to have an equation connecting ϕ_2 and ϕ_4 , since they can both be expanded in powers of $1-s$ and t , and for this reason equations (ii.) and (iv.) on p. 698 would be selected. The choice of the other pair would probably depend on the numerical data of the pentagon in question.

Thus an equation connecting t and $1-s$ is

$$(1-s)^{\beta+\gamma-1} \cdot F_{3,2-\kappa-\delta}(2-\kappa-\delta, \beta, \beta+\gamma, 1-\delta, 1-s, t) \\ = \frac{BC}{DK} \cdot t^{\kappa+\delta-1} F_{3,2-\beta-\gamma}(2-\beta-\gamma, \kappa, \kappa+\delta, 1-\gamma, t, 1-s).$$

Suppose $\beta+\gamma-1$ and $\kappa+\delta-1$ can each be expressed as the ratio of two integers.

$$\text{Let} \quad \beta+\gamma-1 = \frac{p'}{p}, \quad \kappa+\delta-1 = \frac{q'}{q},$$

where p, p', q, q' are integers, which will be in general positive, and let

$$\sigma^p = 1-s, \quad \tau^q = t;$$

σ and τ may be supposed small.

We have

$$\sigma^{p'} \cdot F_{3,2-\kappa-\delta}(2-\kappa-\delta, \beta, \beta+\gamma, 1-\delta, \sigma^p, \tau^q) \\ = \frac{BC}{DK} \cdot \tau^{q'} \cdot F_{3,2-\beta-\gamma}(2-\beta-\gamma, \kappa, \kappa+\delta, 1-\gamma, \tau^q, \sigma^p).$$

This may be written in the form

$$f(\sigma, \tau) \equiv (p' \cdot 0) \sigma^{p'} + (0 \cdot q') \tau^{q'} + \dots = 0,$$

where (m, n) denotes the coefficient of $\sigma^m \tau^n$, the series being absolutely convergent for small values of σ and τ .

First, suppose that p' and q' are both unity, *i.e.*, that $\beta+\gamma-1$ and $\kappa+\delta-1$ are aliquot parts of unity. Then, if $f(\sigma, \tau) = 0$ be considered as a curve, the origin is an ordinary point, and σ can be expanded in a series of the form

$$c_0 \tau + c_1 \tau^2 + \dots,$$

convergent for sufficiently small values of σ and τ . Similarly τ can be expanded in a like series of powers of σ .

Next, let one of the quantities p' , q' be unity, p' say. Then σ can be expanded in a converging series of integral powers of τ , the exponent of the first term being q' . But τ cannot be expanded in a similar power-series of σ . Thus, in both these cases, we can obtain an expansion for $1-s$ in a series of fractional or integral powers of $1-t$.

Another equation connecting s and t is equation (2) of (B) viz.,

$$F_{3,a}(a+\beta+\kappa-1, a, a+\beta, 1-\gamma, t, s-t) \\ = \frac{AB}{KA} \cdot F_{3,a}(a+\beta+\kappa-1, a, a+\kappa, 1-\delta, 1-s, s-t).$$

Substituting in this equation the expansion for $1-s$ in terms of t , we have an equation for t in the form of an infinite series.

Now suppose that neither p' nor q' is unity. Then the origin is a singular point of the curve $f(\sigma, \tau) = 0$, and σ and τ have respectively p' -ple and q' -ple values. There are, however, p' different convergent expansions for σ of the form

$$\sigma = \Sigma C_r \tau^r,$$

where the exponents r form a series of increasing positive fractional numbers (for the theorem proved in Chrystal's *Algebra*, Vol. II., ch. xxx., (d), § 22, for an algebraic function, can be extended to a sufficiently converging infinite series).

Hence for a singular point, by a process such as the use of Newton's parallelogram, corresponding to that employed for an algebraic equation, we can obtain expansions of σ in fractional powers of τ , and consequently expansions of $1-s$ in powers of t . These can be substituted in equation (2) of (B), or in any other of equations (B) which may be convenient.

In the special case of a pentagon which is symmetrical about a line joining A to the middle point of CD , we have

$$\beta = \kappa, \quad \gamma = \delta, \quad AB = AK, \quad BC = DK.$$

The equation (1) of (B) gives $1-s = t$,

and B (2) will be identically satisfied. Combining equations (i.) and (ii.) on p. 698, we have

$$F_{3,a}(a+\beta+\kappa-1, a, a+\beta, 1-\gamma, t, s-t) \\ = \frac{AB}{BC} (1-s)^{a+\gamma-1} \cdot F_{3,1-\delta}(2-\kappa-\delta, \beta, \beta+\gamma, 1-\delta, 1-s, t);$$

i.e., $F_{3,2}(a+2\beta-1, a, a+\beta, 1-\gamma, t, 1-2t)$

$$= \frac{AB}{BC} t^{\beta+\gamma+1} F_{3,2-\beta-\gamma}(2-\beta-\gamma, \beta, \beta+\gamma, 1-\gamma, t, t)$$

$$\begin{aligned} \text{or } \frac{\Gamma(a)}{\Gamma(a+\beta)} & \left\{ F(2-2\gamma, a, a+\beta, t) \right. \\ & + (1-\gamma) \frac{a}{a+\beta} F(3-2\gamma, a+1, a+\beta+1, t)(1-2t) \\ & + \frac{(1-\gamma)(2-\gamma)}{2!} \frac{a(a+1)}{(a+\beta)(a+\beta+1)} \\ & \quad \times F(4-2\gamma, a+2, a+\beta+2, t)(1-2t)^2 + \dots \left. \right\} \\ & = \frac{AB}{BC} t^{\beta+\gamma+1} \frac{\Gamma(\gamma)}{\Gamma(\beta+\gamma)} \\ & \quad \times \left\{ F(2-\beta-\gamma, \beta, \beta+\gamma, t) + (1-\gamma) F(3-\beta-\gamma, \beta, \beta+\gamma, t) t \right. \\ & \quad \left. + \frac{(1-\gamma)(2-\gamma)}{2!} F(4-\beta-\gamma, \beta, \beta+\gamma, t) t^2 + \dots \right\}, \end{aligned}$$

an equation for t in the form of a power-series.

If $\beta + \gamma = 1$,

that is, if AB and CD are parallel, the powers of t are all integral.

If the pentagon is regular, the equation becomes

$$\begin{aligned} F\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, t\right) + \frac{2}{3} \cdot \frac{2}{3} \cdot F\left(\frac{4}{3}, \frac{2}{3}, \frac{1}{3}, t\right)(1-2t) + \dots \\ = t^{\frac{1}{3}} \left\{ F\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, t\right) + \frac{2}{3} \cdot F\left(\frac{4}{3}, \frac{2}{3}, \frac{2}{3}, t\right) t + \dots \right\}. \end{aligned}$$

On certain Regular Polygons in Modular Network. By L. J. ROGERS. Received June 7th, 1898; in revised form, August 29th, 1898. Read June 9th, 1898.

1. In connexion with the theory of modular functions, it is proved that a plane may be entirely covered with a network of circles whose interstices are curvilinear triangles equiangular to one another whose angles are all sub-multiples of two right angles. These networks are of three kinds, according as the sum of the angles in each triangle is greater, equal, or less than two right angles. In the