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LONDON MATHEMATICAL SOCIETY.

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TWENTY-NINTH SESSION, 1892-93.

November 10th, 1892.

ANNUAL GENERAL MEETING, held at 22 Albemarle Street, W.

Prof. GREENHILL, F.R.S., President, in the Chair.

Miss P. G. Fawcett was admitted into the Society.

The Treasurer (Mr. A. B. Kempe) read his Report. Its reception was moved by Mr. S. Roberts, seconded by Lt.-Col. Cunningham, and carried unanimously.

At the request of the Chairman, the meeting elected Mr. Hoppel to act as Auditor, if he should be able to undertake the office.

From the Report of the Secretaries, it appeared that the number of the members during the Session had been 212, now reduced to 207, by the deaths of Drs. Hirst and Wolstenholme, and by the retirement of three members.

The number of compounders is 88.

The Society had to regret the loss, by death, of Prof. Enrico Betti, who was elected a foreign member December 14th, 1871; of Dr. Leopold Kronecker, who was elected a foreign member January 14th, 1875; of Dr. Thomas Archer Hirst, F.R.S., who was an original member of the Society; and of Prof. Joseph Wolstenholme, Sc.D., who was elected a member February 9th, 1871.*

* A detailed memoir of Prof. Kronecker, by M. Hermite, appeared in the *Comptes Rendus* for 4th January, 1892, and an interesting sketch is given by Prof. H. B. Fine in "Kronecker and his Arithmetical Theory of the Algebraic Equation," see *Bulletin of the New York Mathematical Society*, Vol. i., No. 8 (May, 1892); cf. also Vol. ii., No. 3, p. 50. Notices of Dr. Hirst will be found in the *Biograph*, Vol. vi., and in *Nature*, Vol. xiv., p. 399. A brief sketch of Dr. Wolstenholme's career appeared in the "Obituary" of the *Times*.

The following communications had been made or received :—

On Selective and Metallic Reflection : Mr. A. B. Basset.

On the Classification of Binodal Quartic Curves : Mr. H. M. Jeffery.

On a Class of Automorphic Functions : Mr. W. Burnside.

Note on the Motion of a Fluid Ellipsoid under its own Attraction : Prof. M. J. M. Hill.

Note on Finding the G -points of a given Circle with respect to a given Triangle of Reference : Mr. J. Griffiths.

On Clifford's paper "On Syzygetic Relations among the Powers of Linear Quantics" : Prof. Cayley.

Note on the Identity $4(x^p-1)/(x-1) = Y^2 \pm pZ^2$: Prof. G. B. Mathews.

On the Theory of Elastic Wires : Mr. A. B. Basset.

The Equations of Propagation of Disturbances in Gyrostatically Loaded Media, and of the Circular Polarization of Light : Mr. J. Larmor.

On the Contacts of Systems of Circles : Mr. A. Larmor.

Note on Dirichlet's Formula for the Number of Classes of Binary Quadratic Forms for a Complex Determinant : Prof. G. B. Mathews.

Some Theorems relating to a System of Conxal Circles : Mr. R. Lachlan.

On the Logical Foundations of Applied Mathematical Sciences : Mr. E. T. Dixon.

Note on the Inadmissibility of the usual reasoning by which it appears that the Limiting Value of the Ratio of two Infinite Functions is the same as the Ratio of their First Derived, with instances in which the result obtained by it is erroneous : Mr. E. P. Culverwell.

On Saint-Venant's Theory of the Torsion of Prisms : Mr. Basset.

The Simplest Specification of a given Optical Path, and the Observations required to determine it : Mr. J. Larmor.

On the Form of Hyper-elliptic Integrals of the First Order, which are Expressible as the Sum of Two Elliptic Integrals : Mr. W. Burnside.

On the Analytical Theory of the Congruency : Prof. Cayley.

Notes on Dualistic Differential Transformations : Mr. E. B. Elliott.

On certain Quartic Curves of the Fourth Class, and the Porism of the Inscribed and Circumscribed Polygon : Mr. R. A. Roberts.

Second Note on a Quaternary Group of 51840 Linear Substitutions : Dr. G. G. Morrice.

Note on the Skew Surfaces applicable upon a given Skew Surface : Prof. Cayley.

A Newtonian Fragment relating to Centripetal Forces : Mr. W. W. Rouse Ball.
Applications of a Theory of Permutations in Circular Progression to the Theory of Numbers : Major MacMahon.

The Harmonic Functions for the Elliptic Cone : Mr. E. W. Hobson.

Researches in the Calculus of Variations : Mr. E. P. Culverwell.

On an Operator which produces all the Covariants and Invariants of any System of Quantics : Mr. W. E. Story.

Note on the Algebraic Theory of Elliptic Transformation : Mr. J. Griffiths.

Further Note on Automorphic Functions : Mr. W. Burnside.

Note on Approximate Evolution : Mr. H. W. Lloyd Tanner.

A Proof of the Exactness of Cayley's Number of Seminvariants of a given Type: Mr. E. B. Elliott.

The Second Discriminant of the Ternary Quantic $x'U + y'V + z'W$: Mr. J. E. Campbell.

On the Reflection and Refraction of Light from a Magnetized Transparent Medium: Mr. Basset.

The same journals had been subscribed for as in the preceding session. An additional exchange of *Proceedings* had been made, with the Mathematical Society of Edinburgh.

The Library had been enriched by a valuable bequest of books from the late Dr. Hirst.

The meeting next proceeded to the election of the new Council.

The Scrutators (Prof. Hudson and Mr. Ralph Holmes), having examined the balloting lists, declared the following gentlemen duly elected:—Mr. A. B. Kempe, F.R.S., President; Mr. A. B. Basset, F.R.S., Mr. E. B. Elliott, F.R.S., Prof. Greenhill, F.R.S., Vice-Presidents; Dr. J. Larmor, F.R.S., Treasurer; Messrs. M. Jenkins and R. Tucker, Secretaries. Other Members of the Council:—Mr. H. F. Baker, Dr. A. R. Forsyth, F.R.S., Dr. J. W. L. Glaisher, F.R.S., Mr. J. Hammond, Prof. M. J. M. Hill, Dr. E. W. Hobson, Mr. A. E. H. Love, Major MacMahon, R.A., F.R.S., and Mr. J. J. Walker, F.R.S.

The new President, having taken the chair, at once called upon Prof. Greenhill to read his Valedictory Address, of which the title was "Collaboration in Mathematics." On the motion of Major MacMahon, seconded by Prof. M. J. M. Hill, Prof. Greenhill was requested to allow his address to be printed in the *Proceedings*.

The following further communications were made:—

Some Properties of Homogeneous Isobaric Functions: Mr. E. B. Elliott.

On certain General Limitations affecting Hypermagic Squares: Mr. S. Roberts.

Note on Secondary Tucker-Circles: Mr. J. Griffiths.

Note on the Equation $y^2 = x^5 - x$: Prof. W. Burnside.

On a Group of Triangles inscribed in a given Triangle ABC , whose Sides are Parallel to Connectors of any Point P with A, B, C : Mr. R. Tucker.

A Note on Triangular Numbers: Mr. R. W. D. Christie.

The following presents were received :—

- "Proceedings of the Edinburgh Mathematical Society," Vol. x.; 1892.
- "Proceedings of the Royal Society of Edinburgh," Vol. xviii.; Session 1890-91.
- "Jahrbuch über die Fortschritte der Mathematik," Band xxi., Heft 3; 1892.
- "Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 4^e Serie, Tome ii.; 1891.
- "Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. v., No. 2.
- "Proceedings of the Royal Society," Vol. li., No. 314; Vol. lii., No. 315.
- "Proceedings of the Physical Society of London," Vol. xi., Part 4; October, 1892.
- "Jornal de Sciencias Mathematicas e Astronomicas," Vol. xi., No. 1; Coimbra, 1892.
- "Bulletin of the New York Mathematical Society," Vol. ii., No. 1.
- "Royal Dublin Society—Scientific Proceedings," Vol. vii., Parts 3, 4.
- "Royal Dublin Society—Scientific Transactions," Vol. iv., Parts 9-13.
- "Nyt Tidsskrift for Mathematik," A. Tredje Aargang, Nos. 4, 5, 6; Copenhagen, 1892.
- "Nyt Tidsskrift for Mathematik," B. Tredje Aargang, No. 3; Copenhagen, 1892.
- "Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," 1892, xxvi.-xxvii.
- "Annali di Matematica," Tomo xx., Fasc. 2, 3.
- "Educational Times," November, 1892.
- "Journal für die reine und angewandte Mathematik," Band cx., Heft 3.
- "Annals of Mathematics," Vol. vi., No. 7; June, 1892.
- "Journal de l'Ecole Polytechnique," Cahier 61, 1891, and 62; 1892, Paris.
- "Cambridge Philosophical Society—Transactions," Vol. xv., Pt. 3.
- "Cambridge Philosophical Society—Proceedings," Vol. vii., Pt. 6.
- "Annales de l'Ecole Polytechnique de Delft," Tome vii., 1891, Livr. 4; Leide, 1892.
- Ruyet (Mons. G.).—"Observations Pluviométriques et Thermométriques, faites dans le Département de la Gironde, de Juin 1890 à Mai 1891"; 8vo, Bordeaux, 1891.
- Aiyar (S. Radhakrishna).—"Leaves from a Manual of Arithmetic"; 8vo, Madras, 1892.
- "Invention," Vol. xiv., 703, 704.
- "Atti della Reale Accademia dei Lincei," Vol. i., Fasc. 6, 7; Roma, 1892.
- "Bollettino della Pubblicazione Italiana ricevute per Diritto di Stampa," 1892, Nos. 163, 164.
- "Traité Élémentaire des Quaternions," par P. G. Tait, translated by G. Plarr. (2 vols.; Paris, 1884.)
- "Wiskundige Opgaven met de Oplossingen. Door de Leden van het Wiskundig Genootschap," vij.^{de} Deel, 2^{de} Stuk-5^{de} Stuk; Amsterdam, 1891-92.
- "Nieuw Archief voor Wiskunde," Deel xix.; Amsterdam, 1892.

Collaboration in Mathematics. (Valedictory Address.) By Prof.

A. G. GREENHILL. Read November 10th, 1892.

On the occasion of being admitted to this Society, it was my privilege to be present at the address delivered by Professor Stephen Smith, when he retired from the office of President.

Professor Smith at the outset asked permission to confine himself entirely to the history of Pure Mathematics, on the plan that his immediate successor, Lord Rayleigh, in looking round in his turn for a subject, would be attracted by the domain of Applied Mathematics, of which, as Professor Smith remarked, he is so well fitted to take a comprehensive view.

My predecessor, Mr. J. J. Walker, selected as his subject "The Influence of Applied on the Progress of Pure Mathematics," and he showed us how many of the most abstruse theorems of pure analysis owe their origin to ideas, which arose in connexion with concrete and even practical requirements.

The developments of Pure Mathematics, and of Applied Mathematics, and the influence of Pure and Applied Mathematics on each other, having thus received such eloquent and exhaustive treatment from my predecessors, I was compelled to turn elsewhere for a subject suitable for a short valedictory address.

A conference of mathematicians was to be held last summer at Nuremberg; and there, I thought, would be the opportunity of meeting, in conclave and in society, the leading mathematicians of the day. It would interest you, I hoped, to hear a short account of their methods and ideas, and perhaps more, to learn what they thought of us in return.

But, unfortunately for these plans, the state of quarantine in Germany caused the Nuremberg meeting to be postponed till next year; so I am obliged to come before you this evening in the condition of one who has to make shift with a few general remarks on the state of our subject, with some special reference to Collaboration in Mathematics.

But in the first place, we must congratulate those members who have worked so efficiently to fill up the gaps pointed out by Professor Smith, in the subjects requiring advanced text-books.

At the time Professor Smith made this observation he could only instance Cayley's *Elliptic Functions*; but at the present day we possess (or shall shortly possess), as works of our members, advanced treatises on the theory of numbers, of equations, of functions and differential equations, on advanced trigonometry, analytical statics, hydrodynamics, optics, and elasticity. The authors' names are familiar to you; and we hope that other treatises are in course of preparation, to gratify acknowledged needs.

A large part of Professor Smith's address was devoted, even if indirectly, to the subject of Elliptic Functions, in consequence of its important bearing on all the domain of Pure Analysis; and it is interesting at this date to reflect on the vicissitudes of its cultivation, as a branch of academic study, by the light of the remarks on this point in Dr. Glaisher's address.

In my own case I have been attracted to this subject, from the point of view of Mr. Walker's address, by looking at the theorems as they arise in the solution of definite mechanical problems.

Even in this way some of its most abstruse analytical theorems receive new interest; to mention an instance, the treatment by Abel of the *pseudo-elliptic integrals*, as derived from the expression of the square root of a quartic in a periodic continued fraction.

I am not aware whether it has been pointed out, that in Abel's method we assign an arbitrary value to the elliptic function of an aliquot part of a period, and thence determine the corresponding value of the modulus; in this way the degree of the equation requiring solution is considerably reduced. Thus, for trisection, the degree is reduced from four to unity; for quinquisection, from six to two; for division by seven, from eight to three; and so on.

I should like, with your permission, to indicate briefly these results on the board.

Abel's integrals, in the form (*Œuvres Complètes*, t. II., p. 162)

$$I = (n+2) \int \frac{x+k}{\sqrt{X}} dx,$$

where X is a quartic, can be reduced, when we know a factor $x-a$ of X , to the Weierstrassian form

$$I = \int \frac{Pz + (n+2)\delta}{z\sqrt{Z}} dz,$$

by the substitution $x-a = \frac{M}{z}$;

and now the cubic Z can be made to assume the form

$$Z = 4z (az - \beta)^2 - (\gamma z - \delta)^2,$$

so that the roots of $Z = 0$ are the squares of the roots of a simpler cubic

$$2at^3 - \gamma t^2 - 2\beta t + \delta = 0;$$

also $z = 0$ corresponds to the parameter of this pseudo-elliptic integral of the third kind.

When n is odd, the result can be written in the form

$$\begin{aligned} I &= 2 \cos^{-1} \frac{z^{\frac{1}{2}(n-3)} + Bz^{\frac{1}{2}(n-5)} + \dots}{2az^{\frac{1}{2}n}} \sqrt{Z} \\ &= 2 \sin^{-1} \frac{Pz^{\frac{1}{2}(n-1)} + Qz^{\frac{1}{2}(n-3)} + \dots}{2az^{\frac{1}{2}n}}. \end{aligned}$$

It will be noticed that the knowledge of a factor of the quartic X has the effect of reducing the degree in the results to one half that given by Abel.

Now, in the case of trisection,

$$n + 2 = 3;$$

we find $\beta = 0$, and we can put $a = 1$, $\gamma = 1$; and then Klein's parameter r is connected with δ by the linear equation

$$r = -\frac{4-27\delta}{27\delta};$$

$$\text{also} \quad J : J-1 : 1 = (r-1)(9r-1)^3 : (27r^3-18r-1)^2 : -64r$$

(Klein, "On the Transformation of the Elliptic Functions," *Proc. Lond. Math. Soc.*, Vol. ix.).

We may now write Abel's pseudo-elliptic integral, for this case of trisection, in the form

$$\begin{aligned} I &= \int \frac{z-3\delta}{z\sqrt{\{z^3-(z-\delta)^3\}}} dz \\ &= 2 \sin^{-1} \frac{\sqrt{\{z^3-(z-\delta)^3\}}}{z^{\frac{3}{2}}} = 2 \cos^{-1} \frac{z-\delta}{z^{\frac{1}{2}}}. \end{aligned}$$

Quinquisition : $n+2=5$;

here we find, from Abel's results, we can put

$$\alpha = 1, \quad \beta = c, \quad \gamma = c-1, \quad \delta = -c;$$

and now Klein's $r = c+11 - \frac{1}{c}$,

a quadratic for c when r is given, corresponding to the parameters

$$\frac{2}{3}\omega \quad \text{and} \quad \frac{4}{3}\omega,$$

ω denoting a period (the imaginary period, according to our result, where the integral is circular).

Here $J : J-1 : 1$

$$= (r^2-10r+5)^3 : (r^2-22r+125)(r^2-4r-1)^3 : -1728r.$$

On comparison with Halphen's equations (*Fonctions Elliptiques*, t. III., p. 3), we find

$$12g_2 = 1-12c+14c^2+12c^3+c^4,$$

$$216g_3 = (1+c^3)(1-18c+74c^2+18c^3+c^4),$$

$$\Delta = -c^3 \left(c+11 - \frac{1}{c} \right);$$

and

$$t = c^2$$

is a root of Halphen's equation (9), p. 5,

$$5t^6 - 12g_2t^5 + 10\Delta t^3 + \Delta^2 = 0.$$

Also

$$x = -\frac{1}{6}(1+c^2)$$

is a root of Halphen's equation (4), p. 3,

$$x^6 - 5g_2x^4 - 40g_3x^3 - 5g_2^2x^2 - 8g_2g_3x - 5g_3^2 = 0.$$

We can now write Abel's pseudo-elliptic integral for this case in the form

$$\begin{aligned} I &= \int \frac{(c+3)z-5c}{z\sqrt{[4z(z-c)^2 - \{(c-1)z+c\}^2]}} dz \\ &= 2 \cos^{-1} \frac{(z-1)\sqrt{Z}}{2z^{\frac{1}{2}}} \\ &= 2 \sin^{-1} \frac{(c+3)z^2 - (2c+1)z+c}{2z^{\frac{3}{2}}}. \end{aligned}$$

7-section : $n+2 = 7.$

We find, from Abel's results,

$$\alpha = 1, \quad \beta = -c^2 - c^3, \quad \gamma = 1 + 3c + c^2, \quad \delta = -c^3 - c^4;$$

and here
$$r = -\frac{1+8c+5c^2-c^3}{c(1+c)},$$

a cubic equation for c , corresponding to the parameters

$$\frac{3}{4}\omega, \quad \frac{4}{4}\omega, \quad \frac{5}{4}\omega;$$

and $J : J-1 : 1 = (r^3+13r+49)(r^3+5r+1)^3$
 $: (r^4+14r^3+63r^2+70r-7)^3 : 1728r.$

Here
$$t = -c^4(1+c)^4$$

is a root of Halphen's equation (59), p. 75, *F. E.*, t. III.,

$$7t^3 - 2^3 \cdot 3^3 g_3 t^7 - 2 \cdot 5 \cdot 7 \Delta t^6 - 3^3 \cdot 7 \Delta^2 t^4 - 2 \cdot 7 \Delta^3 t^2 - \Delta^4 = 0;$$

while
$$x = -\frac{1}{4}(1+c+c^2)^3$$

is a root of equation (15), p. 51,

$$x^3 - 21g_3x^6 - 2 \cdot 3^3 \cdot 7g_3x^5 \dots = 0.$$

Also
$$12g_3 = (1+c+c^2)(1+11c+30c^2+15c^3-10c^4-5c^5+c^6),$$

$$\Delta = -c^7(1+c)^7(1+8c+5c^2-c^3);$$

and Abel's corresponding pseudo-elliptic integral may be written

$$\begin{aligned} I &= \int \frac{(1+3c-3c^2)z+7(1+c)}{z\sqrt{[4cz(cz+1+c)^2-\{(1+3c+c^2)z+1+c\}^2]}} dz \\ &= 2 \cos^{-1} \frac{z^2+(2-c)z+1}{2c^{\frac{1}{4}}z^{\frac{1}{4}}} \sqrt{Z} \\ &= 2 \sin^{-1} \frac{(1+3c-3c^2)z^3+(3+4c-3c^2+c^3)z^2+(3+2c-2c^3)z+1+c}{2c^{\frac{1}{4}}z^{\frac{1}{4}}}. \end{aligned}$$

9-section : $n+2 = 9.$

We find we can put

$$\alpha = (1-c)^2, \quad \beta = -c+c^2-c^3, \quad \gamma = 1-2c+c^2+c^3, \quad \delta = \beta;$$

and now Kiepert's parameter ξ (*Math. Ann.*, xxxii., p. 66) is connected with c by the relation

$$\xi = -\frac{1+3c-6c^2+c^3}{c(1-c)},$$

a cubic equation for c , corresponding to the parameters

$$\frac{3}{5}\omega, \quad \frac{4}{5}\omega, \quad \frac{5}{9}\omega;$$

$$\begin{aligned} \text{also } J : J-1 : 1 &= (\xi+3)^3 (\xi^3+9\xi^2+27\xi+3)^3 \\ &: (\xi^9+18\xi^8+135\xi^4+504\xi^3+891\xi^2+486\xi-27)^2 \\ &: 1728\xi (\xi^2+9\xi+27). \end{aligned}$$

Abel's corresponding pseudo-elliptic integral is

$$\begin{aligned} I &= \int \frac{Pz+Q\delta}{z\sqrt{Z}} dz \\ &= 2 \cos^{-1} \frac{z^3+Bz^2+Cz+D}{2(1-c)^{\frac{1}{2}} z^{\frac{1}{2}}} \sqrt{Z} \\ &= 2 \sin^{-1} \frac{Pz^4+Qz^3+Rz^2+Sz+T}{2(1-c)^2 z^{\frac{1}{2}}}, \end{aligned}$$

where

$$\begin{aligned} B &= -3(2-2c+c^2), \\ C &= (1-c+c^2)(5-3c+c^2), \\ D &= -(1-c+c^2)^2, \\ P &= 7-18c+15c^2-5c^3, \\ Q &= -14+53c-76c^2+61c^3-25c^4+5c^5, \\ R &= (1-c+c^2)(7-33c+38c^2-23c^3+6c^4-c^5), \\ S &= -(1-c+c^2)^2(1-9c+6c^2-2c^3), \\ T &= -c(1-c+c^2)^3. \end{aligned}$$

11-section : $n+2=11.$

Here I have found that, in Abel's notation (*Oeuvres Complètes*, t. II., p. 162), with $q_0=0$, or $q_4=q_6$, then, if

$$\begin{aligned} q(1+q) &= c(1+c)^3, \\ q_4=q_6 &= 2c(1+c+q); \end{aligned}$$

and we may put $\alpha=1, \quad \beta=(c^2+c^3)q,$

$$\gamma=1+2c-c^3+(2+c)q, \quad \delta=-(c^2+c^3)^2q.$$

A quintic relation is now to be expected, corresponding to the parameters

$$\frac{2}{11}\omega, \frac{4}{11}\omega, \frac{6}{11}\omega, \frac{8}{11}\omega, \frac{10}{11}\omega,$$

connecting c with the parameter ξ employed by Kiepert (*Math. Ann.*, xxxii., p. 93), or with the parameter τ of Klein (Klein und Fricke, *Vorlesungen über die Theorie der Elliptischen Modulfunctionen*, II., p. 437, &c.); but I have not yet succeeded in determining this relation, and I am relying upon the assistance of others.

Similar simple results hold for the even values of n ,

$$2, 4, 6, 8, \dots;$$

but now Abel's quartic X is found to break up into two quadratic factors X_1 and X_2 , and the Weierstrassian form is unsuitable.

We can, however, express the result, if logarithmic or hyperbolic, in the form

$$\begin{aligned} I &= (n+2) \int \frac{x+k}{\sqrt{(X_1 X_2)}} dx \\ &= 2 \cosh^{-1} (Ax^{1/2} + Bx^{1/2(n-2)} + \dots) \sqrt{X_1} \\ &= 2 \sinh^{-1} (Px^{1/2} + Qx^{1/2(n-2)} + \dots) \sqrt{X_2}. \end{aligned}$$

Thus, for 4-section; $n+2=4$;

we find

$$X_1 = x^2 + (1+2c)x + c^2,$$

$$X_2 = x^2 + (1-2c)x + c^2;$$

$$I = 2 \sinh^{-1} \frac{x+1-c}{2c^{1/2}} \sqrt{X_1} = 2 \cosh^{-1} \frac{x+1+c}{2c^{1/2}} \sqrt{X_2}.$$

6-section :

$$n+2=6;$$

$$X_1 = x^3 + (1+c)^2 x + c^3 (1+c)^2,$$

$$X_2 = x^3 + (1-c)^2 x + c^3 (1-c)^2;$$

$$\begin{aligned} I &= 2 \sinh^{-1} \frac{x^3 + (2-c+c^3)x + (1-c)(1-c^3)}{2c^{1/2}(1-c^3)} \sqrt{X_1} \\ &= 2 \cosh^{-1} \frac{x^3 + (2+c+c^3)x + (1+c)(1+c^3)}{2c^{1/2}(1-c^3)} \sqrt{X_2}; \end{aligned}$$

and so on; the results being of half the degree given by Abel, in consequence of the knowledge of the factors X_1 and X_2 .

From Abel's results we are able to construct a series of solvable cases of the complicated motion assumed by a body under no forces,

by a top or gyrost, or by a rotating projectile in a surrounding fluid medium; as indicated in Halphen's *Fonctions elliptiques*.

Mr. Burnside tells me he has discovered an application of the pseudo-elliptic integrals to the flow of electricity on an *anchor-ring*, when the differences of the longitude and quasi-latitude of the electrodes are aliquot parts of the circumference; and then the current itself divides into parts in a simple ratio.

In Abel's method it is not necessary (as in Jacobi's treatment, *Orelle*, Bd. VII., *Gesammelte Werke*, Bd. I., p. 329) to consider the resolution of the quartic X under the radical; and now the parameter of the corresponding elliptic integral of the third kind is an aliquot part of one of the periods.

But, in the applications to dynamical problems, the parameter has an added quarter or half period, implying that the resolution of the quartic has been effected; and this requirement introduces a fresh difficulty, which may in competent hands lead to important developments of Abel's method.

To illustrate these curious analytical results which may thus arise, requiring the resources of pure analysis, I may mention that, in constructing such solvable cases, depending, in the language of elliptic functions, on the division of a period by seven, a cubic equation ($Z = 0$) turns up requiring solution, which may be illustrated geometrically by a certain ternary homogeneous equation representing a non-unicursal cubic curve.

Professor Smith mentions that the solution in integers of such an equation depends on some criterion not yet discovered; although, as he says, it ought not to lie beyond the present scope of analysis.

But, as the Theosophist would say, "this is a detail of which our mathematicians know so little."

Mr. G. B. Mathews is fortunately interested at the present time in Abel's general theory, and I have great hopes that he will succeed in this problem, and at the same time present a general theory.

This episode encourages me to make the suggestion that the modern principle of the Subdivision of Labour might with advantage be employed in mathematical research; and that by Collaboration on new ideas our progress would be accelerated.

Prof. Tait makes a feeling complaint somewhere that in looking back at the time spent on working out a new development, the greater part of it is occupied on analysis that might well be turned over to a trained mathematical student.

Most of us, I expect, will echo Prof. Tait's opinion; but, so far as I

am aware, the only attempt at the present day to introduce collaboration into mathematical work is to be found in Prof. Klein's *Seminär* at Gottingen.

In his wonderful lectures on the ikosahedron, modular, hyper-elliptic, and Abelian functions, Prof. Klein reserves his powers for a general comprehensive view of the subject; and the detailed algebraical analytical development is handed over to his pupils to work up, at the same time stimulating their interest.

The system appears to work very well, judging by the published results in the *Mathematische Annalen*; as in all scientific work, there is no pretence at concealment.

We are not like the artist, say Rubens, or the writer, say Dumas, who are reported to have passed off, as their own original work, the parts elaborated by pupils and assistants.

An advantage of such collaboration lies in the admission of the outside world to the order of development of new ideas. We see the structure growing, and can trace the scaffolding employed in its erection.

The perfect geometrical form of the *Principia* was for a long time a hindrance to the formation of a school in this country to follow up Newton's discoveries, as no successors arose who could wield this great weapon of geometry with facility. But the communications brought before this Society by Mr. Ball and Dr. Glaisher, on the rough notes of Newton's calculations, preserved in his manuscripts, are of great value and encouragement to us, in showing that Newton himself, for purposes of discovery, adopted an analytical method similar to that we employ nowadays; and that the form of the results was subsequently recast in the Greek geometrical style.

I presume we are all agreed, however, as to the value of a study of the *Principia*, as a chastening influence on the luxuriance of symbols and analysis. For my part I should prefer to see the whole three books prescribed in the Cambridge course, to be studied in the original Latin.

In two hundred years no real advance has been made to complete the speculations as to the cause of gravity with which the *Principia* concludes; nor has any exception been observed, in the solar system, to Kepler's Laws and the Law of Gravitation.

The first report of the newly discovered satellite of Jupiter gave an incompatible distance and period; these numbers have been subsequently corrected; but any discrepancy, if confirmed, would lead to

most interesting speculations as to the causes modifying the universal applicability of Newton's Law.

"Heresy in science is the author of all progress," and to discover and theorize on the smallest discrepancies with simple acknowledged rules is certain to lead to important development.

In Prof. Smith's words, "an apparent contradiction (as distinct from a mere misunderstanding) is always to be regarded as an indication of some undiscovered truth."

In Electricity, the "law of the inverse square" is capable of indirect experimental verification with great accuracy; but we were indebted to Maxwell, in the Mathematical Tripos of 1877, for a very beautiful question, of a somewhat heretical nature, in which he showed that, if we assume the law to vary as the negative power $(2+p)$, where p is small, the value of p could be inferred by means of a formula from a simple electrical experiment. So far the value of p has been found insensible; but it will be interesting, when more detailed reports are to hand, to employ Maxwell's method on Jupiter's new satellite.

Newton's Third Law, that "action and reaction are equal and opposite"—according to Maxwell's interpretation, the definition of a *stress*—implies in its generality that the inertia of the medium by which the stress is propagated is insensible.

Maxwell, in his *Electricity and Magnetism*, acting as the interpreter of Faraday's views, first taught us to look for the explanation of electrical phenomena in the stress in the medium or dielectric which propagates the action to a distance.

The effect of quasi-electrical inertia, as exhibited in the Herz oscillations, has only recently attracted attention; and perhaps we shall find that the corresponding inertia of space, in which gravity is propagated, may either slightly modify the laws of gravity, or, on the other hand, be the means of accounting for their existence.

Formerly, the mathematician, when called upon to justify the utility of his subject, would point to its astronomical achievements, if only for their bearing on Navigation.

Nowadays I think we could instance the applications of Electricity, as the creation of men whose genius was essentially mathematical.

Much of the theory which served in physical astronomy was found to be immediately applicable in electrical science; and now, in return, it seems that the minute manifestations of electricity will help to throw light on the mysterious laws which govern the motion of our universe.

Returning to the question of Collaboration, the correspondence of

Newton and Cotes, we find, is principally concerned with the preparation of a new edition of the *Principia*. The laborious part of the work was undertaken by Cotes, and it seems a pity that Newton did not employ him as a collaborator earlier.

Mr. Ball read to us an interesting paper on "Newton's Classification of Cubic Curves," in which he showed us that most of the theorems rediscovered of late years were already known to Newton.

In my opinion, Newton should, for his genius, have been otherwise occupied; but doubtless he became fascinated, in a manner with which I presume we are most of us familiar, with the algebraical details of some calculations in connexion with an astronomical or dynamical question, while the great theory of gravitation was taking shape in his mind.

So, too, Maxwell informs us that he was lured away into the *Analysis Situs*, from being compelled to develop it for his electromagnetic theory; and even Rankine, the mathematician of the engineer, confesses that he was once attracted by the theory of numbers, and wasted much time over it, most agreeably, no doubt, but to no purpose. "Le temps le mieux employé est celui que l'on perd."

During the past year the Society has suffered great loss, in Dr. Hirst, Professor Wolstenholme, and two foreign members, Prof. Kronecker and Signor Betti.

Dr. Hirst was an original member of our Society, and its first Treasurer; he took great interest in our meetings, and he filled the office of President. The Society is indebted to him for the bequest of his valuable library.

His work on the *Correlation of Space* brought him into correspondence with the Italian mathematicians of the school of Cremona; he contributed often to foreign journals, in which his linguistic knowledge showed to advantage.

Signor Betti has favoured us with communications on the subject of Elasticity, in which the principles are presented in an elegant and condensed form.

Professor Wolstenholme's mathematical fame rests on his wonderful collection of original mathematical problems, a production peculiarly characteristic of the British school. The influence of these problems on the formation of the taste and ability of a student cannot be over-estimated. It was my good fortune to be his colleague for some time at Cooper's Hill, and to keep up a continuous acquaintance.

Prof. Kronecker's work, chiefly concerned with the Theory of

Numbers, is too great and extensive to receive more than mere mention here; for details we must refer to a memoir of him in the *Comptes Rendus*, January, 1892, written by M. Hermite.

Looking through the communications to the Society in the past two years, we find papers on all branches of mathematical research; and it would be invidious to single out any one for special commendation, when we consider the different tastes and proclivities of those who cultivate our science, arising from the specialization which is now a necessity on our part.

Speaking for myself, however, I must express the pleasure I experienced in reading Mr. Burnside's communications on "Functions derived from their Discontinuities," and on "Automorphic Functions." In these he works out the purely analytical developments of Klein, Weber, and Poincaré, with reference to the physical standpoint of electrical and hydrodynamical applications. This physical interpretation, to my mind, throws light and interest on the complicated results which arise, and gives a hold by which the mind can obtain a first grasp of the subject.

In conclusion I should like to say a few words about our Society.

In the twenty-seven years of its existence we can look back upon the active membership of those mathematicians who will be remembered by posterity as the leaders of thought of this generation; and the London Mathematical Society has already amply justified its existence, in the works of past members, such as De Morgan, Spottiswoode, Stephen Smith, Hirst, Maxwell, Wolstenholme, Clifford, and Buchheim.

I think we may congratulate ourselves on its present condition intellectually, financially (thanks to our President), and numerically.

It has been said that we can never be a numerous body; but, by judicious recruiting, our numbers could doubtless be still further added to.

I leave it to our junior members to come forward in this work, and also in otherwise keeping up the vitality of our subject and Society; so that posterity may hold the same high opinion of them as we do of our predecessors.

Note on the Equation $y^2 = x(x^4 - 1)$. By W. BURNSIDE.

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The Cartesian co-ordinates of a point on a curve whose deficiency is greater than one, cannot be expressed as rational or as elliptic functions of a single parameter; but it has recently been shown that, in a large number of cases, including certainly all curves whose equations are of hyper-elliptic form with real coefficients, the co-ordinates of a variable point on the curve can be expressed as uniform functions of a single parameter with an infinite number of essentially singular points, the functions being automorphic with respect to a certain group of substitutions.

Although, however, the possibility of this mode of representing the coefficients has been proved, no instance has ever been given, so far as I know, in which the representation has actually been carried out for a particular equation. Indeed, to do this it would generally be necessary to find a group having the required relation to the equation, and there is, I believe, no known method of doing this. If, however, from independent considerations, the relation in question between a given equation and a known group has once been established, it will become a matter of calculation to carry out the process referred to.

In the case of the equation

$$y^2 = x(x^4 - 1) \dots\dots\dots(i.),$$

the actual expressions for x and y , as functions of a single parameter, can, by such considerations as those just mentioned, be obtained with remarkably simple analysis, and the result is perhaps of sufficient interest to justify me in communicating it to the Society.

It is convenient to begin by giving two well-known formulæ in elliptic functions that will be required. Those are

$$\wp 2u - e = \left[\frac{\wp^2 u - 2e\wp u - e^2 - e'e''}{\wp' u} \right]^2,$$

and

$$\wp(u + \omega) - e = \frac{(e - e')(e - e'')}{\wp u - e};$$

whence, by differentiation,

$$\frac{\wp'(u + \omega)}{\wp' u} = - \frac{(e - e')(e - e'')}{(\wp u - e)^2}.$$

If $2\omega, 2\omega'$ are any pair of primitive periods of the elliptic functions, these formulæ are equivalent to

$$\wp 2u - \wp \omega = \left\{ \frac{\left(\wp u - \wp \frac{\omega}{2} \right) \left[\wp u - \wp \left(\omega' + \frac{\omega}{2} \right) \right]}{\wp'(u)} \right\}^2 \dots\dots(ii.)$$

and
$$\frac{\wp'(u + \omega)}{\wp' u} = - \left\{ \frac{\wp \frac{\omega}{2} - \wp \omega}{\wp u - \wp \omega} \right\}^2 \dots\dots\dots(iii.).$$

In Klein-Fricke, *Theorie der Elliptischen Modulfunctionen*, i., p. 652, it is shown that the x and y of equation (i.) are expressible as modular functions for that sub-group of the modular group which is formed of all substitutions

$$\omega, \left(\frac{a\omega + \beta}{\gamma\omega + \delta} \right), a\delta - \beta\gamma = 1,$$

such that $\begin{pmatrix} a, \beta \\ \gamma, \delta \end{pmatrix}$ is congruent to modulus 8 with one of the forms

$$\begin{pmatrix} 1, 0 \\ 0, 1 \end{pmatrix}, \begin{pmatrix} 1, 4 \\ 4, 1 \end{pmatrix}, \begin{pmatrix} 5, 4 \\ 0, 5 \end{pmatrix}, \begin{pmatrix} 5, 0 \\ 4, 5 \end{pmatrix};$$

and moreover that x is then that modular function which Prof. Klein calls the octahedral irrationality. It does not, however, enter into the plan of Prof. Klein's work to exhibit x and y explicitly as functions of ω , which is the object of the present note.

There is no difficulty in showing that the twenty-four values of Prof. Klein's octahedral irrationality are given in terms of elliptic functions of submultiples of the periods by the expression

$$\frac{\wp \frac{\omega}{2} - \wp \omega}{\wp \frac{\omega'}{2} - \wp \omega},$$

and the twenty-three expressions derivable from it by the substitutions

$$\omega_1 = \omega + \omega', \quad \omega'_1 = \omega'$$

and

$$\omega_2 = -\omega', \quad \omega'_2 = \omega,$$

If now

$$x = \frac{\wp \frac{\omega}{2} - \wp \omega}{\wp \frac{\omega'}{2} - \wp \omega},$$

then, by (iii.),

$$x^3 = - \frac{\wp' \left(\frac{\omega'}{2} + \omega \right)}{\wp' \frac{\omega'}{2}}.$$

The addition equation gives

$$\wp\omega' + 2\wp \frac{\omega'}{2} = \frac{1}{4} \frac{\wp'^3 \frac{\omega'}{2}}{\left(\wp\omega' - \wp \frac{\omega'}{2} \right)^3},$$

and

$$\wp\omega' + 2\wp \left(\frac{\omega'}{2} + \omega \right) = \frac{1}{4} \frac{\wp'^3 \left(\frac{\omega'}{2} + \omega \right)}{\left[\wp\omega' - \wp \left(\frac{\omega'}{2} + \omega \right) \right]^3},$$

or since, as may be deduced at once from (ii.),

$$2\wp\omega' = \wp \frac{\omega'}{2} + \wp \left(\frac{\omega'}{2} + \omega \right),$$

$$\wp\omega' + 2\wp \left(\frac{\omega'}{2} + \omega \right) = \frac{1}{4} \frac{\wp'^3 \left(\frac{\omega'}{2} + \omega \right)}{\left[\wp\omega' - \wp \frac{\omega'}{2} \right]^3}.$$

Hence

$$x^4 = \frac{\wp\omega' + 2\wp \left(\frac{\omega'}{2} + \omega \right)}{\wp\omega' + 2\wp \frac{\omega'}{2}},$$

and

$$x^4 - 1 = 2 \frac{\wp \left(\frac{\omega'}{2} + \omega \right) - \wp \frac{\omega'}{2}}{\wp\omega' + 2\wp \frac{\omega'}{2}}.$$

$$= 4 \frac{\wp\omega' - \wp \frac{\omega'}{2}}{\wp\omega' + 2\wp \frac{\omega'}{2}}.$$

$$= 16 \frac{\left(\wp\omega' - \wp \frac{\omega'}{2} \right)^3}{\wp'^3 \frac{\omega'}{2}}.$$

Therefore $x(x^4 - 1) = -16 \frac{\left(\wp \frac{\omega'}{2} - \wp \omega\right) \left(\wp \frac{\omega'}{2} - \wp \omega'\right)^2}{\left(\wp \frac{\omega'}{2} - \wp \omega\right) \wp'^2 \frac{\omega'}{2}}.$

Taking account of equation (ii.), this can be written in the form

$$x(x^4 - 1) = -16 \left\{ \frac{\left(\wp \frac{\omega'}{2} - \wp \omega'\right) \left(\wp \frac{\omega'}{4} - \wp \frac{\omega'}{2}\right) \left[\wp \frac{\omega'}{4} - \wp \left(\frac{\omega'}{2} + \omega\right)\right]}{\left(\wp \frac{\omega'}{4} - \wp \frac{\omega}{2}\right) \left[\wp \frac{\omega'}{4} - \wp \left(\frac{\omega}{2} + \omega'\right)\right] \wp' \frac{\omega}{4} \wp' \frac{\omega'}{2}} \right\}^2;$$

and therefore, finally, if

$$y = 4i \frac{\left(\wp \frac{\omega'}{2} - \wp \omega'\right) \left(\wp \frac{\omega'}{4} - \wp \frac{\omega'}{2}\right) \left[\wp \frac{\omega'}{4} - \wp \left(\frac{\omega'}{2} + \omega\right)\right] \times \left(\wp \frac{\omega}{4} - \wp \frac{\omega}{2}\right) \left[\wp \frac{\omega}{4} - \wp \left(\frac{\omega}{2} + \omega'\right)\right]}{\left(\wp \frac{\omega'}{4} - \wp \frac{\omega}{2}\right) \left[\wp \frac{\omega'}{4} - \wp \left(\frac{\omega}{2} + \omega'\right)\right] \wp' \frac{\omega}{4} \wp' \frac{\omega'}{2}},$$

and

$$x = \frac{\wp \frac{\omega}{2} - \wp \omega}{\wp \frac{\omega}{2} - \wp \omega},$$

then

$$y^2 = x(x^4 - 1).$$

Taking account of homogeneity, x and y are thus expressed as uniform functions of the parameter ω'/ω , and they each have the real axis for an essentially singular line.