"Bibliografia Achille Sannia — Lezioni di Geometria Proiettiva dettate nella Università di Napoli." (Presented with the "Giornale di Matematica," July, August, 1885.)

"United States Coast and Geodetic Survey," J. E. Hilgard, Superintendent,— "Methods and Results," Appendix No. 15, Report for 1884, "Gravity Research, use of the Noddy for measuring the Swaying of a Pendulum support;" Appendix No. 16, Report for 1884, "Gravity Research, effect of the Flexure of a Pendulum upon its period of Oscillation;" Washington, 1885.

On a Theorem in Kinematics. By Mr. J. J. WALKER.

[Read Dec. 10th, 1885.]

The principle on which the composition of rotations, of any magnitudes, about axes in space, is founded, is Hamilton's Theorem (*Proc. R. I. A.*, 1834), obtained and demonstrated* by the then new Quaternion method—that two successive rotations through angles 2θ , $2\theta'$ about two intersecting axes OA, OB , regarded as radii of a sphere, are equivalent to a single rotation, through an angle 2ϕ , about a third radius OC , determined by making the angles BAC, ABC of the spherical triangle ABC equal to θ, θ' respectively; ϕ being equal to the supplement of the third angle ACB of the triangle.

I have recently noticed the following connected theorem, which seems of interest in itself, and the geometrical proof of which contains a demonstration of Hamilton's theorem; viz., In the successive rotations, of any magnitude, of a rigid body about two intersecting axes, any line in the body whose direction passes through their intersection describes portions of two right cones: *these meet again in a side which is common also to the cone of the equivalent single rotation.*

Let α, β, ρ be the unit vectors OA, OB, OR , the last rotating about the first two through angles $2\theta, 2\theta'$ respectively, so as to coincide with ρ_1, ρ_2 (OR_1, OR_2) successively; also, let γ be OC , the axis of the single equivalent rotation 2ϕ , which shall at once make ρ coincide with ρ_2 . If the cones on which ρ moves in its rotations, and of which ρ_1 is one common side, meet again in σ (OS), and the angle $R_1AS = 2\psi$, then

$$\sigma = \rho + 2 \sin(\theta + \psi) \{ V\alpha\rho \cos(\theta + \psi) + \alpha V\alpha\rho \sin(\theta + \psi) \} + \dots (1),$$

* Since verified by conceiving a pyramid to roll round another fixed equal pyramid, which is the reflexion or image of the first relative to their common face.

† This form of σ expresses that it is the resultant of ρ and a vector equal and parallel to the chord of the arc described by the term of ρ in rotating.

and, as the condition that this vector shall lie on the right cone about γ as axis and containing ρ as a side, it is necessary that

$$S\gamma\sigma = S\gamma\rho,$$

$$\text{i.e. (1), } S\gamma \{ V\alpha\rho \cos(\theta + \psi) + \alpha V\alpha\rho \sin(\theta + \psi) \} = 0 \dots\dots\dots(2).$$

$$\text{Now, } \gamma \sin \phi = \alpha \sin \theta \cos \theta' + \beta \sin \theta' \cos \theta + V\beta\alpha \sin \theta \sin \theta',$$

and, since the substitution of α , multiplied by any scalar, plainly satisfies (2), it is to be shown that

$$(S \cdot \beta V\alpha\rho \cos \theta + S \cdot V\beta\alpha V\alpha\rho \sin \theta) \cos(\theta + \psi) \\ + (S \cdot \beta\alpha V\alpha\rho \cos \theta - S \cdot \beta V\alpha\rho \sin \theta) \sin(\theta + \psi) = 0 \dots(3).$$

Let r, c be the angles which ρ, β make with α ; then, since

$$S \cdot \beta\alpha V\alpha\rho = S \cdot V\beta\alpha V\alpha\rho = -S \cdot V\alpha\beta V\alpha\rho = \sin c \sin r \cos(2\theta + \psi),$$

$2\theta + \psi$ being the angle BAR between the planes of $\alpha\rho, \alpha\beta$, the expression (3) reduces to

$$\cos(2\theta + \psi) \{ S \cdot \beta V\alpha\rho + \sin c \sin r \sin(2\theta + \psi) \},$$

which vanishes, $V\alpha\rho$ being the vector (δ) to the pole (D) of AR multiplied by $\sin r$, and $S\beta\delta$ being equal to

$$-\cos BD = -\sin AB \cos BAD = -\sin AB \sin BAR \\ = -\sin c \sin(2\theta + \psi).$$

It may be convenient, to avoid reference, to verify here that γ has the vector value assigned in the above proof. It is the vector which remains unchanged by the two rotations round α, β ; i.e.,

$$\gamma = \beta^{\frac{r}{2}} \alpha^{\frac{c}{2}} \gamma \alpha^{-\frac{r}{2}} \beta^{-\frac{c}{2}},$$

$$\text{or } \gamma \beta^{\frac{r}{2}} \alpha^{\frac{c}{2}} = \beta^{\frac{r}{2}} \alpha^{\frac{c}{2}} \gamma,$$

$$\text{or } V \cdot \gamma V \beta^{\frac{r}{2}} \alpha^{\frac{c}{2}} = 0,$$

$$\text{whence } \gamma V \beta^{\frac{r}{2}} \alpha^{\frac{c}{2}} = \text{a scalar,}$$

$$\text{i.e., } V \beta^{\frac{r}{2}} \alpha^{\frac{c}{2}} = \gamma \text{ (to a scalar factor),}$$

$$\text{or } k\gamma = \beta \sin \theta' \cos \theta + \alpha \sin \theta \cos \theta' + V\beta\alpha \sin \theta \sin \theta',$$

and the particular value of k , which may be easily verified to be $\sin \phi$, does not concern the application just made.

Having thus verified the theorem of the cointersection of the three cones by the Quaternion method, I proceed to show how it, and the original theorem, may be derived from the elementary geometry of the sphere.

Let a radius OR of a sphere rotate about another OA into the position OR_1 , from which it is carried by a second rotation about another radius OB into the position OR_2 . Let the small circles, arcs of which RR_1 , R_1R_2 are described in the above rotations, meet again in S_1 ; and finally, let any third small circle, about C as pole, be described through R , R_2 , cutting the first pair again in S , S_2 . Joining the

points on the sphere by arcs of great circles, as in the figure, the angles referred to being dihedral angles contained by planes of great

circles, $\angle RAS = 2 \angle CAS$, $\angle R_1AS_1 = 2 \angle BAS_1$,

therefore $\angle RAR_1 + \angle SAS_1 = 2 \angle CAS - 2 \angle BAS_1$
 $= 2 \angle BAC + 2 \angle SAS_1$,

i.e., $\angle RAR_1 - \angle SAS_1 = 2 \angle BAC$.

Similarly, $\angle R_1BR_2 - \angle S_1BS_2 = 2 \angle ABC$.

Also $\angle RCS = 2 \angle ACS$, $\angle R_2CS_2 = 2 \angle BCS_2$,

therefore $2\pi - \angle RCS - \angle R_2CS_2$,

or $\angle RCR_2 + \angle SCS_2 = 2(\pi - \angle ACB + \angle SCS_2)$,

i.e., $\angle RCR_2 - \angle SCS_2 = 2(\pi - \angle ACB)$.

If the point S_1 fell outside the circle through RR_2 of which C is pole, the signs connecting the pairs of angles in the foregoing results would change; viz., they would be

$$\angle RAR_1 + \angle SAS_1 = 2 \angle BAC, \quad \angle R_1BR_2 + \angle S_1BS_2 = 2 \angle ABC,$$

$$\angle R_1CR_2 + \angle S_1CS_2 = 2(\pi - \angle BCA).$$

But, if the circle through RR_2 about C as pole passed through S_1 , viz., if SS_1S_2 coincided in S , then

$$\angle BAC = \angle RAR_1, \quad \angle ABC = \angle R_1BR_2, \quad \text{and} \quad \angle R_1CR_2 = 2(\pi - \angle BCA),$$

so that in this case, and this only, would the axis OC and $\angle R_1CR_2$ be the same for all radii whatever; i.e., would the whole body, and not merely a given vector or line in it, perform a rotation about OC equivalent to the two of magnitudes 2θ , $2\theta'$ about OA , OB successively; and that rotation would be through an angle equal to $2(\pi - \angle ACB)$.

The axis (OC) of the resultant rotation, in this treatment of Hamilton's theorem, is thus fixed in a manner wholly free from any ambiguity; viz., it is the connector of the poles of the small circle determined, on any sphere about the fixed point O as centre, by the initial (OR) and final (OR_2) positions of a radius, and the reflexion (OS) of its intermediate position (OR_1) relatively to the plane of the axes (OA , OB) of the component rotations.

Thursday, January 14th, 1886.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Mr. J. B. Colgrove, M.A., Loughborough Grammar School, was elected a member; and Mr. S. O. Roberts was admitted into the Society.

The President (Mr. Walker, Vice-President, in the Chair) communicated a question, proposed by Mr. H. M. Taylor, as to the novelty of an elementary result in Plane Conics, and the extension of the property to other curves; several members made suggestions as to possible sources of information. The President having resumed the chair, Mrs. Bryant read a paper on "Logarithms in General Logic"; a long discussion ensued, in which Mr. Kempe, Prof. Sylvester, the President, Mr. S. Roberts, and Mrs. Bryant, took part.

Mr. J. Hammond (Prof. Sylvester, Vice-President, in the Chair) read a paper "On a Class of Integrable Reciprocants." Captain Macmahon made a short communication on a Certain Property of Seminvariants.

The following presents were received :—

- "Proceedings of the Royal Society," Vol. xxxix., No. 239.
- "Educational Times," for January.
- "Mathematical Papers, chiefly connected with the q -Series in Elliptic Functions," 1883—1885, 8vo; by J. W. L. Glaisher, F.R.S.; Cambridge, 1885.
- "Manchester Literary and Philosophical Society—Memoirs," Vol. viii., Third Series, 8vo, London, 1884; "Proceedings," Vols. xxiii. and xxiv., 8vo, Manchester, 1884 and 1885.
- "On the Discovery of the Periodic Law, and on Relations among the Atomic Weights," by J. A. R. Newlands; 8vo, London, 1884.
- "Johns Hopkins University Circulars," Vol. v., No. 45.
- "Report of the National Academy of Sciences," for the years 1883 and 1884; 8vo, Washington, 1884 and 1885.
- "Proceedings of the National Academy of Sciences," Vol. i., Part ii., 8vo; Washington, 1884.
- "Memoirs of the National Academy of Sciences," Vol. iii., Part i., 4to, 1884; Washington, 1885.
- "Beiblätter zu den Annalen der Physik und Chemie," B. ix., St. 12.
- "Jornal de Sciencias Mathematicas e Astronomicas," publicado pelo Dr. F. Gomes Teixeira, Vol. vi., No. 4; Coimbra, 1885.
- "Atti della R. Accademia dei Lincei—Rendiconti," Vol. i., F. 26 and 27.
- "Journal für Mathematik," Bd. xcix., Heft 3.
- "Les Fonctions d'une Seule Variable à un Nombre quelconque de Périodes," par F. Casorati.