On Congruences of the Third Order and Class.

By Dr. T. ARCHER HIRST, F.R.S.

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1. In the *Monats Bericht* of the Academy of Berlin, for the 17th of January, 1878, Kummer had occasion to study the properties of two congruences of the third order and class, having one and the same focal surface of the eighth order and class. A year ago, when engaged on Cremonian Congruences,* I was led to consider two associated ones of the third order and class, which, at first sight, appeared to be identical with those of Kummer. Further examination, however, soon proved them to be varieties of the latter which have not hitherto, I believe, been noticed. I propose to examine them in the present paper, and, in particular, to show in what manner they presented themselves to me as degenerate cases of Cremonian Congruences of the fifth order and fourth class.

2. Such a congruence (5, 3) is generated by joining the corresponding points of two, arbitrarily situated, planes α and β , between which a Cremonian Correspondence of the third order has been established.† The planes α and β are singular ones, containing congruence-curves a_4 and b_4 , respectively, of the sixth order and fourth class.‡ The intersection $\overline{a\beta}$ is a triple congruence-ray and, at the same time, a triple tangent of each of the above congruence-curves; the points of contact, in each plane, being the intersections of $\overline{a\beta}$ with the fourth-class cubic, in the other plane, which corresponds to that line.§

The double points A_1 and B_1 of the above cubics $\overline{a^3}$ and $\overline{b^3}$ are at the principal double points of the cubic correspondence. They are the vertices of quadric congruence-cones A_1^2 and B_1^2 , passing through all the five principal points of the correspondence, which are situated in β and α , respectively.|| The four pairs of associated principal single points of this correspondence—say, $A'_1, B'_1; A''_1, B''_1; A'''_1, B'''_1$, and A''''_1, B''''_1 —are centres of four pairs of congruence-pencils; the planes of each pair intersect in the line passing through both centres, and pass through the principal double points. The plane of each of the eight pencils, in short, passes through the principal line which corresponds to its centre.

3. Besides α and β and the eight, above described, there are two

* *Proceedings of the London Mathematical Society*, Vol. xiv., pp. 259—301. Hereafter the articles of this paper will be referred to in the abridged form, C. C.

† C. C., Arts. 3 and 6. ‡ C. C., Art. 7. § C. C., Art. 8. || C. C., Art. 9.

other singular planes, each of which passes through *both* the principal double points A_3 and B_3 , and intersects α and β in a pair of lines which correspond to one another, point by point. Each of these planes, therefore, contains a congruence-conic which touches α and β at points on $\bar{a}^3$ and $\bar{b}^3$.

4. The several tangents of these two congruence-conics, and the rays of each of the above four pairs of congruence-pencils, belong, in reality, to a system of quadric reguli,* whose generators, in the aggregate, constitute the congruence (5, 3) under consideration. The two directrices of each regulus of the system being corresponding rays of the pencils $A_3(\alpha)$ and $B_3(\beta)$, one generator thereof lies in α , and another in β ; these generators join, obviously, the intersections, with $\bar{a}\bar{\beta}$, of the two directrices of the regulus, with their corresponding points on $\bar{a}^3$ and $\bar{b}^3$. The quadric surfaces, therefore, on which the several reguli of the system are situated, all pass through the principal points A_3 and B_3 , and touch the planes α and β at points which lie on the curves $\bar{a}^3$ and $\bar{b}^3$.

5. The second, or conjugate system of reguli, situated on the above quadrics, form, in the aggregate, another congruence (5, 3), which has the same singular points and planes—in short, the same focal surface as the one already described. But *this* congruence is not Cremonian, that is to say, its rays do not determine a bi-rational correspondence between any two of its singular planes; A_3 and B_3 are here simply centres of congruence-pencils situated in α and β .†

6. The common focal surface of the above two congruences is the envelope of the system of quadric-surfaces already referred to. It is of the twelfth order and eighth class; touches the planes α and β along the cubics $\bar{a}^3$ and $\bar{b}^3$, and, besides this, cuts them in the fourth class, sextic congruence-curves a_4 and b_4 .‡

7. A_3 and B_3 are, in general, quadruple points of the focal surface; the tangent cone at each being the envelope of the plane which connects a ray of $A_3(\alpha)$, or $B_3(\beta)$, with the point in which its corresponding ray of $B_3(\beta)$, or $A_3(\alpha)$, intersects, for the second time, the principal conic in β , or α .§ This cone is well known to be of the third class and fourth order; it touches the plane α , or β , twice, and has triple contact with the quadric congruence-cone A_3^2 , or B_3^2 .

It is important for what follows to notice that, by the formulæ of Plucker, each of these cones possesses *three cuspidal edges*.

8. Without further discussing the properties of the above focal

* Conf. C. C., Art. 32.

† Conf. C. C., Art. 36.

‡ C. C., Arts. 10, 13, and 15.

§ Conf. C. C., Art. 36 (c).

surface, I pass, at once, to the degenerate case of the congruence (5, 3) which was referred to in Art. 1.

It presents itself when the planes α and β are so placed that each of two points O and D , on $\overline{\alpha\beta}$, becomes self-correspondent. Two congruences (1, 0), consisting of rays issuing from these points, may then be detached from the congruence (5, 3). This done, the residual congruence (3, 3), say $\mathbf{C}$, will still have the two quadric congruence-cones A_1^2 and B_1^2 , of Art. 2, and the four pairs of congruence-pencils with centres $A'_1, B'_1; A''_1, B''_1; A'''_1, B'''_1$; and A_1''''', B_1''''' . The planes α and β , however, will now contain congruence-conics a_1 and b_1 , having $\overline{\alpha\beta}$ for a common tangent, and, instead of the two congruence-conics, described in Art. 3, we shall now have two congruence-pencils, with centres O' and D' , supplemented by two other pencils, with centres O and D , respectively, and situated in planes γ and δ , each of which touches *both* the conics a_1 and b_1 .* Included in the system of quadric reguli whose generators, in the aggregate, constitute the congruence $\mathbf{C}$ under consideration, we have now, in short, two new pencil-pairs,—one with the centres O and O' and the planes γ and (A_1CB_1) , the other with the centres D and D' and the planes δ and (A_1DB_1) , respectively.

9. The reguli, conjugate to those of the system just referred to, constitute, in the aggregate, a second congruence (3, 3), say $\mathbf{C}_1$, having the same singular points and planes as $\mathbf{C}$.

In passing from $\mathbf{C}$ to $\mathbf{C}_1$, however, the planes of the pencil-pairs whose centres are $A'_1, B'_1; A''_1, B''_1; A'''_1, B'''_1; A_1''''', B_1'''''; O, O'; D, D'$ are interchanged, and α and β are now merely planes of congruence-pencils having their centres at A_1 and B_1 , respectively.† The congruence $\mathbf{C}_1$, moreover, is no longer Cremonian; its rays establish, in fact, a (2, 2) correspondence between any two of its singular planes.

10. Each of the congruences $\mathbf{C}$ and $\mathbf{C}_1$ is self-reciprocal, as is also their common focal surface. The latter, which is, of course, the envelope of the system of quadric surfaces upon which the above-mentioned conjugate reguli are situated, is of the eighth order and class.

Besides touching the planes α and β along the fourth-class cubics $\overline{a^3}$ and $\overline{b^3}$, the focal surface likewise cuts them in the conics a_1 and b_1 ; and, correlatively, A_1 and B_1 are not only quadruple points of the focal surface, the cones of contact at which are of the third class (Art. 7),

* C. C., Art. 20.

† Conf. C. C., Art. 36, (c) and (d).

but they are likewise vertices of quadric cones, A_1^2 and B_1^2 , circumscribed to the surface elsewhere.

The points of each of the six point-and-plane pairs, included in the system of enveloping quadrics, are nodes of the focal surface, the quadric tangent-cones at which touch both the planes of the pair; and correlatively these planes touch the focal surface along conics which pass through both the points of the pair. These conics form, obviously, the intersection of the point-and-plane pair with the next succeeding quadric of the system.

11. Generally, every two consecutive quadrics of the system referred to in the last Art. intersect, in a quartic curve—the characteristic—which passes through A_1 and B_1 and touches α and β . The tangent at each of these two points is a generator of one of the third class, quartic cones referred to in Art. 7, and in each of these two planes the characteristic touches one of the fourth-class cubics $\bar{\alpha}^3$ and $\bar{\beta}^3$.

12. Two consecutive characteristics, in other words, three consecutive quadrics of the system in Art. 10, intersect, not only in A_1 and B_1 , but in six other points situated on the *curve of regression* of the focal surface. At each of these six points the curve in question has three-point contact with one of the quadrics S^2 of the system. The three points, in fact, are the intersection of S^2 with (1) the two quadrics immediately preceding it, (2) the quadrics immediately preceding and following it, and (3) the two quadrics immediately succeeding it. The above six intersections of two consecutive characteristics, therefore, count as eighteen intersections of S^2 and the curve of regression.

But the points A_1 and B_1 , as will be presently shown, are triple points on the curve in question; so that the latter intersects S^2 , on the whole, in twenty-four points, and is consequently, like Kummer's curve of regression, of the *twelfth order*.

13. That A_1 and B_1 are, in reality, triple points on our curve of regression follows most readily from the fact, alluded to at the end of Art. 7, that the tangent-cone to the focal surface at each of these, its quadruple points, has three cuspidal edges. For the generators of these cones are clearly tangents, at A_1 and B_1 , respectively, of the several characteristics passing through these points, and the existence of a cuspidal edge thereof obviously implies that of a characteristic of a similar stationary character,—that is to say, of one which intersects its consecutive characteristic in six points whereof one has come to coincide with A_1 , or B_1 .

14. The curve of regression of the focal surface, besides having triple points at A_1 and B_1 , has, as we have seen in Art. 12, sextuple,

three-pointic contact with every quadric of the system in Art. 10. Six of the latter, however, break up into point-and-plane pairs. The planes of each of four of these degenerate quadrics pass through two associated principal single points of the cubic correspondence between α and β , as do also the two conics along which these planes touch the focal surface. One of these conics, moreover, passes through A_1 and the other through B_2 ; and each of them likewise passes through the three points in each of which its plane osculates the curve of regression.

The planes of each of the remaining two point-and-plane pairs of the system touch the focal surface along a pair of conics intersecting in C and C' , in the one case, and in D and D' , in the other. In both cases, one of the conics of the pair passes through the two principal points, A_1 and B_2 , while the other, situated in γ or δ , passes through the four points in each of which its plane osculates the curve of regression.

15. The preceding congruence (3, 3) differs from that described by Kummer in having the two singular points A_1 and B_2 , and the two singular planes α and β . There is, however, a still more special congruence (3, 3) which merits a passing notice.

When the correspondence between α and β is such that the distance between two of the four principal single points in the former is equal to that between their associates in the latter, both pairs may be brought to coincide on $\overline{a\beta}$; A_1''' and B_1''' , for instance, in C , and A_1'''' and B_1'''' in D . When the generating planes are thus placed, however, each of the cubics $\overline{a^3}$ and $\overline{b^3}$, corresponding to $\overline{a\beta}$, breaks up into three right lines, viz.: $\overline{OA_1}$, $\overline{DA_1}$, and $\overline{A_1'A_1''}$ in α , and $\overline{OB_1}$, $\overline{DB_1}$, $\overline{B_1'B_1''}$ in β . To the several points of $\overline{OA_1}$ and $\overline{OB_1}$ corresponds the point C solely, and to those of $\overline{DA_1}$ and $\overline{DB_1}$ the point D ; while between $\overline{a\beta}$ and $\overline{A_1'A_1''}$, as well as between $\overline{a\beta}$ and $\overline{B_1'B_1''}$, a point-to-point correspondence exists, whereby the two congruence-conics a_2 and b_2 are generated. Each of these conics touches $\overline{a\beta}$, as well as the three right lines into which the cubic of its plane has broken up.

16. The planes γ and δ of the congruence-pencils whose centres are C and D respectively, since they touch both the conics a_2 and b_2 , (Art. 8), now coincide, respectively, with the planes (A_1CB_2) and (A_2DB_1) ; and at the same time the centres, C' and D' , of the congruence-pencils which these planes formerly contained, now coincide, respectively, with C and D . In short, in place of two of the congruence pencil-pairs described in Art. 8, we have now two *doubled* pencils $\overline{A_1CB_2}$ and $\overline{A_2DB_1}$.

These, as degenerate quadric reguli belonging to the system described in Arts. 4 and 8, coincide, moreover, with their conjugate reguli. In other words, the above doubled pencils, like the tangents of the two conics described in Art. 3, are common to the two associated congruences referred to in Arts. 5 and 9.

17. The system of quadric surfaces on which the above-mentioned reguli are situated, and whose envelope is the focal surface common to the congruences $\mathbf{C}$ and $\mathbf{C}_1$, now includes the two doubled planes (A_1OB_2) and (A_1DB_2) , each bounded by a pair of right lines,— $\overline{A_1O}$, $\overline{B_2O}$ and $\overline{A_1D}$, $\overline{B_2D}$ respectively — which lie wholly on the focal surface.

On this surface the lines $\overline{A_1A_1'}$ and $\overline{B_1B_1'}$ also lie; along them, in fact, the surface touches the planes α and β , respectively, since each quadric of the enveloping system touches these planes at points situated on these lines (Art. 4).

18. The focal surface being, like the congruences $\mathbf{C}$ and $\mathbf{C}_1$, self-reciprocal, we might at once infer from the above that the tangent-planes to it at A_1 , as well as those at B_1 , are co-axial, the axes being generators, respectively, of the congruence-cones A_1^2 and B_1^2 .

These results, however, are easily verified directly. The tangent-planes in question each connect a ray of A_1 (α) or B_1 (β) with a generator of the congruence-cone A_1^2 or B_1^2 (Art. 7), and these rays and generators are now not only in (1, 1) correspondence with each other, but in each of the lines $\overline{A_1O}$ and $\overline{A_1D}$, or $\overline{B_1O}$ and $\overline{B_1D}$, two corresponding elements coincide. By a well-known theorem, therefore the planes of all other pairs of corresponding elements are co-axial. The axis, of course, is the intersection of the plane $(A_1A_1'B_1')$ with $(A_1A_1'B_1'')$, or of $(B_1A_1'B_1')$ with $(B_1A_1'B_1'')$, and from elementary considerations it can be readily shown that this intersection lies on the congruence-cone $A_1^2 \equiv A_1(B_1B_1'B_1''OD)$, or $B_1^2 \equiv B_1(A_1A_1'A_1''OD)$.

Finally, it may be observed that this axis, and the lines $\overline{A_1O}$, $\overline{A_1D}$, or $\overline{B_1O}$, $\overline{B_1D}$, represent, in the present case, the three stationary edges of the tangent-cone at A_1 , or B_1 , to the focal surface—which edges were found in Art. 13 to be likewise tangents, at the triple points, of the curve of regression on this focal surface.