hence $\mathfrak{J}_1, \mathfrak{J}_2$ are roots of the equation

$$\{1 + \phi(\mathfrak{J})\} t^2 - pt^2 + qt - r = 0;$$

and, since $\frac{1}{\mathfrak{J}_1} + \frac{1}{\mathfrak{J}_2} + \frac{1}{\mathfrak{J}} = 0$, if t is the third root of this equation, $\frac{1}{\mathfrak{J}_1} + \frac{1}{\mathfrak{J}_2} + \frac{1}{t} = \frac{q}{r} = \frac{1}{t} - \frac{1}{\mathfrak{J}}$, we have $t = \frac{r\mathfrak{J}}{r + q\mathfrak{J}}$, and $\mathfrak{J}$ is determined by the cubic

$$(pq - 2r)\mathfrak{J}^3 + (q^2 + 2pr)\mathfrak{J}^2 + qr\mathfrak{J} + 2r^2 = 0.$$

If t_1, t_2, t_3 are the roots of this equation, the parameters of the node corresponding to t_1 will be the roots of $t^2 - u_1t - u_1t_1 = 0$; this quartic will, therefore, be in general trinodal.

$$\text{Since } \phi(\mathfrak{J}_1) - \phi(\mathfrak{J}_2) = 0, \quad \phi(\mathfrak{J}_2) - \phi(\mathfrak{J}_3) = 0,$$

$$\text{or } p\mathfrak{J}_1^2\mathfrak{J}_2^2 - q\mathfrak{J}_1\mathfrak{J}_2(\mathfrak{J}_1 + \mathfrak{J}_2) + r(\mathfrak{J}_1^2 + \mathfrak{J}_2^2 + \mathfrak{J}_1\mathfrak{J}_2) = 0,$$

the locus of the intersections of the opposite sides of the quadrilateral (13), (24) will be a cuspidal cubic. If we consider the tangents corresponding to the roots of the equation $\mathfrak{J}^2\phi(\mathfrak{J}) = 0$, we see that the points of contact of these tangents and the vertices of the triangle formed by them lie on the quartic.

Of course, corresponding to all the theorems concerning polygons circumscribed about a cuspidal cubic, we shall have, by reciprocation, similar theorems concerning polygons inscribed in a cuspidal cubic.

The Algebraic Solution of the Modular Equation for the Septic Transformation. By G. S. ELY, Fellow in Johns Hopkins University, U.S.

[Read April 6th, 1882.]

$$\text{To transform } \frac{dx}{\sqrt{1-x^2} \cdot 1-\kappa^2x^2} = \frac{Mdy}{\sqrt{1-y^2} \cdot 1-\lambda^2y^2},$$

Prof. Cayley writes for the septic transformation

$$\frac{1-y}{1+y} = \frac{1-x}{1+x} \left(\frac{1-ax+\beta x^2-\gamma x^3}{1+ax+\beta x^2+\gamma x^3} \right)^3,$$

and shows that the conditions necessary for the change of x into

$\frac{1}{\kappa x}$, and y into $\frac{1}{\lambda y}$, are

$$\left. \begin{aligned} \gamma &= \Omega, \\ \kappa^3 (\beta^3 + 2\beta\gamma + 2a\gamma) &= \Omega (2\beta + 2a + a^3), \\ \kappa^4 (2\beta + 2a\beta + 2\gamma + a^3) &= \Omega (\beta^3 + 2a\beta + 2\gamma + 2a\gamma), \\ \kappa^5 (1 + 2a) &= \Omega (\gamma^3 + 2\beta\gamma), \end{aligned} \right\} \dots\dots\dots (I.),$$

where $\Omega^3 = \frac{\kappa^7}{\lambda}$. Writing $\kappa = u^4$, $\lambda = v^4$, we have $\Omega = \frac{u^{14}}{v^3}$, $\gamma = \sqrt{\Omega} = \frac{u^7}{v}$, and, by making x and y indefinitely small,

$$M^{-1} = 2a + 1 \dots\dots\dots (a).$$

From the fourth of equations (I.),

$$2\beta = u^3 v^3 M^{-1} - \gamma \dots\dots\dots (b),$$

and from the second and third of equations (I.),

$$v^3 (\beta^3 + 2\beta\gamma + 2a\gamma) = u^5 (2\beta + 2a + a^3) \dots\dots\dots (c),$$

$$u^2 v^3 (2\beta + 2a\beta + 2\gamma + a^3) = \beta^3 + 2a\beta + 2\gamma + 2a\gamma \dots\dots\dots (d).$$

Then, from equations (a), (b), (c), (d), and the equation $\gamma = \frac{u^7}{v}$, a , β , γ , and M are to be eliminated, and the resultant when reduced will be the modular equation desired.

Eliminating a , β , and γ , we obtain the two equations in M ,

$$Av^3 M^{-2} - 2Bv^3 M^{-1} + Cv = 0 \dots\dots\dots (e),$$

$$Bv^4 M^{-2} + Dv M^{-1} + E = 0 \dots\dots\dots (f),$$

where

$$A = v^3 - 1,$$

$$B = (1 + uv) (1 - uv)^3,$$

$$C = 4u^7 - 4uv^3 + 3v (1 - u^5),$$

$$D = 2u (v^4 - u^4) (1 + u^3 v^3) - 2v^3 (1 - u^5),$$

$$E = v^4 - u^{13} - 6u^5 v (1 - u^3 v^3).$$

Then, eliminating M^{-2} and M^{-1} from equations (e) and (f), we have the following determinant—

$$\begin{vmatrix} Av^3 & -2Bv^3 & Cv & 0 \\ 0 & Av^3 & -2Bv^3 & Cv \\ Bv^4 & Dv & E & 0 \\ 0 & Bv^4 & Dv & E \end{vmatrix} = 0 = \begin{vmatrix} A & -2Bv & Cv & 0 \\ 0 & A & -2Bv & C \\ Bv^3 & D & E & 0 \\ 0 & Bv^3 & Dv & E \end{vmatrix}$$

$$= A \{AE^2 + D(2BEv + CDv) - 2BCEv^3\}$$

$$+ B^2 v^4 (4BE + 2DC + C^2 v^3) = 0.$$

Substituting the values of A, B, C, D , and E , and expanding, we have

$$\begin{aligned}
 & u^{24} + v^{24} - 8(u^{23}v^7 + u^7v^{23} + u^{17}v + uv^{17}) + 82(u^{20}v^4 + u^4v^{20}) \\
 & + 20(u^{23}v^6 + u^6v^{23} + u^{18}v^3 + u^3v^{18}) - 56(u^{31}v^5 + u^5v^{31} + u^{19}v^8 + u^8v^{19}) \\
 & + 64(u^{21}v^{13} + u^{13}v^{21} + u^{11}v^5 + u^5v^{11}) - 208(u^{20}v^{13} + u^{13}v^{20} + u^{12}v^4 + u^4v^{12}) \\
 & + 352(u^{19}v^{11} + u^{11}v^{19} + u^{18}v^5 + u^5v^{18}) - 272(u^{18}v^{10} + u^{10}v^{18} + u^{14}v^6 + u^6v^{14}) \\
 & - 160(u^{17}v^9 + u^9v^{17} + u^{15}v^7 + u^7v^{15}) + 463(u^{16}v^8 + u^8v^{16}) \\
 & - 128(u^8v^5 + u^{19}v^{19}) + 448(u^6v^6 + u^{18}v^{18}) - 512(u^7v^7 + u^{17}v^{17}) \\
 & - 224(u^8v^8 + u^{16}v^{16}) + 1232(u^9v^9 + u^{15}v^{15}) - 968(u^{10}v^{10} + u^{14}v^{14}) \\
 & - 976(u^{11}v^{11} + u^{13}v^{13}) + 2236u^{13}v^{13} = 0 \dots\dots\dots(II.)
 \end{aligned}$$

In the *Philosophical Transactions* for 1874, p. 397, Prof. Cayley has for the septic transformation these equations

$$\left. \begin{aligned}
 \kappa^2 \alpha^3 &= \Omega \delta^3, \\
 \kappa(2\alpha\gamma + 2\alpha\beta + \beta^3) &= \Omega(\gamma^2 + 2\gamma\delta + 2\beta\delta), \\
 \gamma^3 + 2\beta\gamma + 2\alpha\delta + 2\beta\delta &= \Omega\kappa(2\alpha\gamma + 2\beta\gamma + 2\alpha\delta + \beta^3), \\
 \delta^3 + 2\gamma\delta &= \Omega\kappa^3(\alpha^3 + 2\alpha\beta),
 \end{aligned} \right\} \dots\dots(III.)$$

Apparently there is one more constant in these equations than in equations (I.) But if in these equations we divide by $\alpha^3\Omega$, and then make $\frac{\beta}{\alpha} = \alpha$, $\frac{\gamma}{\alpha} = \beta$, $\frac{\delta}{\alpha} = \gamma$, and $\frac{\kappa^3}{\Omega} = \Omega$, we obtain equations (I.)

Now, in equations (III.), making $\alpha = 0$ and $\delta = 0$, the equations are

$$\begin{aligned}
 \kappa\beta^3 &= \Omega\gamma^3, \\
 \gamma^3 + 2\beta\gamma &= \Omega\kappa(2\beta\gamma + \beta^3);
 \end{aligned}$$

whence dividing by $\beta^3\Omega$, and making $\frac{\kappa}{\Omega} = \Omega$, $\frac{\gamma}{\beta} = \alpha$, we obtain

$$\begin{aligned}
 \alpha^3 &= \Omega, \\
 \kappa^2(2\alpha + 1) &= \Omega(\alpha^3 + 2\alpha),
 \end{aligned}$$

which are exactly the equations for the cubic transformation. Therefore equation (II.) is divisible by the left-hand member of the modular equation for the cubic transformation, that is, by

$$u^4 - v^4 + 2uv(1 - u^3v^3).$$

Again, in equations (III.), making $\gamma = \kappa\alpha$, $\delta = \kappa\beta$, the equations become

$$\begin{aligned}
 \kappa\alpha^3 &= \Omega\beta^3, \\
 2\kappa^2\alpha^3 + \kappa(2\alpha\beta + \beta^3) &= \Omega\kappa^3(\alpha^3 + 2\alpha\beta) + 2\Omega\kappa\beta^3, \\
 \kappa^2\alpha^3 + 2\kappa(\beta^3 + 2\alpha\beta) &= 2\Omega\kappa^3(\alpha^3 + 2\alpha\beta) + \Omega\kappa\beta^3, \\
 \beta^3 + 2\alpha\beta &= \Omega\kappa(\alpha^3 + 2\alpha\beta).
 \end{aligned}$$

The first and last of these being satisfied, the second and third are satisfied identically. Divide the first and last by $\alpha^3\Omega$, and make

$$\frac{\kappa}{\Omega} = \Omega \text{ and } \frac{\beta}{\alpha} = \alpha; \text{ whence}$$

$$\alpha^3 = \Omega,$$

$$\kappa^3(2\alpha+1) = \Omega(\alpha^3+2\alpha).$$

Therefore equation (II.) is again divisible by

$$u^4 - v^4 + 2uv(1 - u^2v^2).$$

But we observe that equation (II.) is symmetrical in u and v , and therefore, as it is divisible by

$$\{u^4 - v^4 + 2uv(1 - u^2v^2)\}^3,$$

it must also be divisible by

$$\{v^4 - u^4 + 2uv(1 - u^2v^2)\}^3.$$

Then, actually performing the division, we find equation (II.) to be

$$\begin{aligned} &\{u^4 - v^4 + 2uv(1 - u^2v^2)\}^3 \{v^4 - u^4 + 2uv(1 - u^2v^2)\}^3 \\ &\times \{(1 - u^3)(1 - v^3) - (1 - uv)^3\} = 0. \end{aligned}$$

This is the product of five factors, and, as the first four belong to cubic transformations, the last only can belong to the septic transformation. We have thus, as the modular equation for the septic transformation,

$$(1 - u^3)(1 - v^3) = (1 - uv)^3,$$

a result which I believe has never been obtained before algebraically.

On a Symmetric Determinant connected with Lagrange's Interpolation Problem. By THOMAS MUIR, M.A., F.R.S.E.

[Read April 6th, 1882.]

1. In Lagrange's interpolation-problem it is required to find the values of $A_1, A_2, \dots, A_n$, so that

$$A_1 \sin x + A_2 \sin 2x + \dots + A_n \sin nx$$

may coincide in value with an assigned function when x has the values $\pi/(n+1), 2\pi/(n+1), \dots, n\pi/(n+1)$. The data lead to a set of n equa-