Hitherto the wave-length has been supposed to be 2π , but, if we now take it to be $2\pi/k$, (24) becomes

$$V = \beta + 2\omega/k \cdot (1 + \frac{1}{4}k^2h^2) \dots\dots\dots(25),$$

as is evident by "dimensions." The velocity of propagation is that of the flow of the fluid in the steady motion at the place where

$$ky = 1 + \frac{1}{4}k^2h^2 \dots\dots\dots(26).$$

So far as the present investigation can reach, there is no sign of the amplitude of a wave tending spontaneously to increase.

Some Algebraical Theorems connected with the Theory of Partitions. By A. R. FORSYTH. Received October 28th, 1895. Read November 14th, 1895.

1. In a letter written to me last month (September), Major MacMahon suggested the consideration of the following problem:—

"Let the fraction, obtained by taking the reciprocal of

$$\begin{aligned} X &= (1-ax) \left(1 - \frac{x}{a}\right) \\ &\quad (1-abx^2) \left(1 - \frac{x^2}{ab}\right) \\ &\quad (1-abcx^3) \left(1 - \frac{x^3}{abc}\right) \\ &\quad \vdots \end{aligned}$$

where X contains n product-pairs, be expanded in ascending powers of x . In the expansion thus obtained, suppress every term containing a negative index for any one of the symbols $a, b, c, \dots$; and in the surviving terms let each of these symbols be made unity. The sum of the resulting series is required."

The result of the selective and summative operations, which are to be effected on X^{-1} for the present purpose, may be denoted by SlX^{-1} , and the like for more general fractions.

Special simple forms can easily be evaluated. Thus, when $n = 1$, it is necessary to consider the fraction

$$\frac{1}{(1-ax)\left(1-\frac{x}{a}\right)}.$$

Using a well-known theorem in trigonometry, we write

$$\frac{1-x^2}{(1-ax)\left(1-\frac{x}{a}\right)} = 1+x\left(a+\frac{1}{a}\right)+x^2\left(a^2+\frac{1}{a^2}\right)+\dots;$$

$$\begin{aligned} \text{whence } Sl \frac{1-x^2}{(1-ax)\left(1-\frac{x}{a}\right)} &= 1+x+x^2+\dots \\ &= \frac{1}{1-x}, \end{aligned}$$

$$\text{and therefore } Sl \frac{1}{(1-ax)\left(1-\frac{x}{a}\right)} = \frac{1}{(1-x)(1-x^2)}.$$

Similarly, by expanding the two parts of the fraction

$$\frac{1}{\left\{(1-ax)\left(1-\frac{x}{a}\right)\right\}\left\{(1-aby)\left(1-\frac{y}{ab}\right)\right\}},$$

taking the product, and effecting the same operations, it can be proved that

$$\begin{aligned} Sl \frac{1}{(1-ax)\left(1-\frac{x}{a}\right)(1-aby)\left(1-\frac{y}{ab}\right)} \\ = \frac{1-x^2y}{(1-x)(1-x^2)(1-xy)(1-y)(1-y^2)} \end{aligned}$$

so that, by taking $y = x^2$, we have

$$Sl \frac{1}{(1-ax)\left(1-\frac{x}{a}\right)(1-ax^2)\left(1-\frac{x^2}{ab}\right)} = \frac{1}{(1-x)(1-x^2)^2(1-x^3)}.$$

This method becomes almost insuperably laborious as the number of product-pairs is increased, and seems to be ineffective for the general case; but it suggests the form of the result, viz.,

$$\frac{1}{(1-x)(1-x^2)^2(1-x^3)^2 \dots (1-x^n)^2(1-x^{n+1})}$$

a form which Major MacMahon had inferred also from other considerations.

2. The following method leads to the general result.

The expansion of the fraction

$$\left[\left\{ (1-a_1x) \left(1 - \frac{x_1}{a_1} \right) \right\} \left\{ (1-a_1x_1a_2x_2) \left(1 - \frac{x_1x_2}{a_1a_2} \right) \right\} \dots \right]^{-1},$$

where there are s product-pairs, is the product of the series

$$\begin{aligned} & 1 + a_1x_1 + \dots + (a_1x_1)^{m_1} + \dots, \\ & 1 + \frac{x_1}{a_1} + \dots + \left(\frac{x_1}{a_1} \right)^{n_1} + \dots, \\ & 1 + a_1x_1a_2x_2 + \dots + (a_1x_1a_2x_2)^{m_2} + \dots \\ & 1 + \frac{x_1x_2}{a_1a_2} + \dots + \left(\frac{x_1x_2}{a_1a_2} \right)^{n_2} + \dots, \\ & \vdots \end{aligned}$$

where the integers m, n have all values from 0 to ∞ . Writing

$$x_1 = x_2 = \dots = x,$$

the general term in the expansion of

$$\left[\left\{ (1-a_1x) \left(1 - \frac{x}{a_1} \right) \right\} \left\{ (1-a_1a_2x^2) \left(1 - \frac{x^2}{a_1a_2} \right) \right\} \dots \right]^{-1}$$

has

$$a_1^{m_1-n_1+m_2-n_2+m_3-n_3+\dots} a_2^{m_2-n_2+m_3-n_3+\dots} a^{m_s-n_s+\dots} \dots,$$

for the coefficient of the power of x represented by

$$x^{m_1+n_1+2(m_2+n_2)+3(m_3+n_3)+\dots}.$$

This general term will survive if no one of the indices of the quantities $a_2, a_3, \dots$ be negative; hence, keeping the indices of $a_2, a_3, \dots$ not negative, the term survives for values of n_1 from 0 to

$m_1 + (m_2 - n_2) + (m_3 - n_3) + \dots$. In each such term we may now make a_1 unity; and then, taking their sum, we have

$$SlX^{-1} = Sla_2^{m_2 - n_2 + m_3 - n_3 + \dots} a_3^{m_3 - n_3 + \dots} \dots \\ \dots \frac{1}{1-x} \left\{ x^{m_1 + (2m_2 + 2n_2) + (3m_3 + 3n_3) + \dots} - x^{1 + 2m_1 + (3m_2 + n_2) + (4m_3 + 2n_3) + \dots} \right\}.$$

The integer m_1 now occurs only in the indices of the two powers of x ; and thus these two typical terms survive for all values of m_1 from 0 to ∞ . Summing for these, we have

$$SlX^{-1} = Sla_2^{m_2 - n_2 + m_3 - n_3 + \dots} a_3^{m_3 - n_3 + \dots} \dots \\ \dots \left\{ \frac{1}{(1-x)^2} x^{(2m_2 + 2n_2) + (3m_3 + 3n_3) + \dots} - \frac{x}{(1-x)(1-x^2)} x^{(3m_2 + n_2) + (4m_3 + 2n_3) + \dots} \right\}.$$

This concludes the first stage in the summation; its effect is to make a_1, m_1, n_1 disappear from the typical terms. And it is found that the summation can be effected by similar stages in succession, each consisting of two processes: the first is a summation of a limited number of terms, say for values of n_i ; the second is a summation of an unlimited number of terms, for values of m_i ; and the effect is to make a_i, m_i, n_i disappear from the typical terms.

The limits of n_2 are 0 to $m_2 + (m_3 - n_3) + \dots$ for terms that survive. Proceeding to take their sum when a_2 is made unity, we have

$$SlX^{-1} = Sla_3^{m_3 - n_3 + \dots} \dots \left[\frac{1}{(1-x)^2} \frac{1}{1-x^2} \left\{ x^{2m_2 + (3m_3 + 3n_3) + \dots} - x^{2 + 4m_2 + (5m_3 + n_3) + \dots} \right\} \right. \\ \left. - \frac{x}{(1-x)(1-x^2)} \frac{1}{1-x} \left\{ x^{3m_2 + (4m_3 + 2n_3) + \dots} - x^{1 + 4m_2 + 5m_3 + n_3 + \dots} \right\} \right] \\ = Sla_3^{m_3 - n_3 + \dots} \dots \frac{1}{(1-x)^2(1-x^2)} \left\{ x^{2m_2 + (3m_3 + 3n_3) + \dots} - x^{1 + 3m_2 + (4m_3 + 2n_3) + \dots} \right\}.$$

Now summing for m_2 for values 0 to ∞ , we have

$$SlX^{-1} = Sla_3^{m_3 - n_3 + \dots} \dots \left\{ \frac{x^{(3m_3 + 3n_3) + \dots}}{(1-x)^2(1-x^2)^2} - x \frac{x^{(4m_3 + 2n_3) + \dots}}{(1-x)^2(1-x^2)(1-x^3)} \right\},$$

being the result at the end of the second stage.

Proceeding similarly through the third stage, by summing first for n_3 from 0 to $m_3 + (m_4 - n_4) + \dots$, after making a_3 unity, and then

for m_s from 0 to ∞ , we find the result at the end of the stage in the form

$$SlX^{-1} = Sla_i^{m_s - n_s + \dots} \dots \left\{ \frac{x^{(4m_s + 4n_s) + \dots}}{(1-x)^3 (1-x^3)^3 (1-x^9)^3} \right. \\ \left. - x \frac{x^{(6m_s + 3m_s) + \dots}}{(1-x)^3 (1-x^3)^3 (1-x^9)^3 (1-x^4)} \right\}.$$

It is now clear that the result at the end of $i-1$ stages is

$$SlX^{-1} = Sla_i^{m_i - n_i + \dots} \dots \left[\frac{x^{(im_i + in_i) + \dots}}{(1-x)^3 (1-x^3)^3 \dots (1-x^{i-1})^3} \right. \\ \left. - x \frac{x^{[(i+1)m_i + (i-1)n_i] + \dots}}{(1-x)^3 (1-x^3)^3 \dots (1-x^{i-2})^3 (1-x^{i-1})(1-x^i)} \right],$$

the index of the power of x in the first term in the bracket being

$$\{im_i + in_i\} + \{(i+1)m_{i+1} + (i+1)n_{i+1}\} \\ + \{(i+2)m_{i+2} + (i+2)n_{i+2}\} + \dots,$$

and the index in the second term being

$$\{(i+1)m_i + (i-1)n_i\} + \{(i+2)m_{i+1} + in_{i+1}\} \\ + \{(i+3)m_{i+2} + (i+1)n_{i+2}\} + \dots.$$

The inductive establishment of the result is so simple after the preceding explanations that it need not be reproduced here.

At the end of s stages, there being s product-pairs, all the necessary summations have been effected, the symbolical coefficient involving the symbols a is now unity, and all the indices m_{s+1} , n_{s+1} , ... are zero. The end of the s^{th} stage gives the final result, so that we have

$$SlX^{-1} = \frac{1}{(1-x)^3 (1-x^3)^3 \dots (1-x^s)^3} \\ - \frac{x}{(1-x)^3 (1-x^3)^3 \dots (1-x^{s-1})^3 (1-x^s)(1-x^{s+1})} \\ = \frac{1}{(1-x)(1-x^3)^3 (1-x^9)^3 \dots (1-x^s)^3 (1-x^{s+1})},$$

which is the result required.

3. It may be added that, of course, the summation need not be taken in the order implied. Thus, dealing with a simple case for illustration—that of two product-pairs—we have

$$SLX^{-1} = Sla_1^{m_1-n_1+m_2-n_2} a_2^{m_2-n_2} x^{m_1+n_1+2m_2+2n_2},$$

and we may sum, first with regard to m_1 , and then with regard to n_1 . Taking $m_2-n_2 = \theta$, the typical term will survive when m_1 ranges from 0 to ∞ if $n_1 \leq \theta$, and, when m_1 ranges from $n_1-\theta$ to ∞ , if $n_1 \geq \theta+1$. Hence, summing for m_1 , we have

$$\begin{aligned} SLX^{-1} &= Sl a_2^{\theta} \frac{x^{n_1+4n_2+2\theta}}{1-x} \text{ for } n_1 = 0 \text{ to } \theta \\ &\quad + Sla_2^{\theta} \frac{x^{2n_1+4n_2+\theta}}{1-x} \text{ for } n_1 = \theta+1 \text{ to } \infty \\ &= Sl \frac{a_2^{\theta}}{1-x} \left\{ \frac{x^{4n_2+2\theta}-x^{1+4n_2+3\theta}}{1-x} + \frac{x^{2+4n_2+3\theta}}{1-x^3} \right\} \\ &= Sl \frac{a_2^{\theta}}{1-x} \left\{ \frac{x^{4n_2+2\theta}}{1-x} - x \frac{x^{4n_2+3\theta}}{1-x^3} \right\}, \end{aligned}$$

agreeing with the preceding form at the end of the first stage for the case $s = 2$.

In the present case, we may now sum for θ from 0 to ∞ , and for n_2 from 0 to ∞ , obtaining the result

$$SLX^{-1} = \frac{1}{(1-x)(1-x^3)^2(1-x^3)}.$$

But had there been more than two product-pairs, so that at the end of the first stage there would have been more than one symbol a_s , the summation would have had to be separated as above; viz., taking

$$m_s-n_s+\dots=\phi,$$

then m_s ranges from 0 to ∞ if $n_s \leq \phi$, and m_s ranges from $n_s-\phi$ to ∞ if $n_s \geq \phi+1$. And so on for other cases. A slight consideration will show that the previous sequence in summation is the more convenient, save at the last stage; an example of the latter is furnished in § 10 (*post*).

4. In another letter written a few days after the first, Major MacMahon propounded the similar problem for a more general case when there are s sets of products as above, but each product contains

$(1+r)$ factors (instead of two, as above). External considerations had led him to infer the form of the result in the limit when both s and r are infinite.

The preceding method can be applied: and some cases are solved in what follows. The general result is inferred, but it is not completely established, partly owing to the very laborious character of the algebra; and the limiting form, when both r and s are infinite, agrees with that which was inferred by Major MacMahon.

Let X denote the product

$$\begin{aligned} & (1-a_r x_1) \left(1 - \frac{a_{r-1}}{a_r} x_1\right) \left(1 - \frac{a_{r-2}}{a_{r-1}} x_1\right) \dots \left(1 - \frac{a_1}{a_2} x_1\right) \left(1 - \frac{x_1}{a_1}\right) \\ & (1-a_r x_1 b_r x_2) \left(1 - \frac{a_{r-1}}{a_r} x_1 \frac{b_{r-1}}{b_r} x_2\right) \dots \dots \dots \left(1 - \frac{a_1}{a_2} x_1 \frac{b_1}{b_2} x_2\right) \left(1 - \frac{x_1 x_2}{a_1 b_1}\right) \\ & (1-a_r x_1 b_r x_2 c_r x_3) \dots \dots \dots \left(1 - \frac{x_1 x_2 x_3}{a_1 b_1 c_1}\right) \\ & \vdots \end{aligned}$$

where there are s sets of symbols $a, b, c, \dots$, so that there are s lines; and each line contains $r+1$ factors. Then X^{-1} is to be expanded in positive powers of $x_1, x_2, \dots$; when all the variables $x_1, x_2, \dots$ are made equal to x , the general term has

$$\begin{array}{ccccccc} a_r^{r-l_{r-1}+m_r-m_{r-1}+n_r-n_{r-1}+\dots} & a_{r-1}^{l_{r-1}-l_{r-2}+m_{r-1}-m_{r-2}+n_{r-1}-n_{r-2}+\dots} & \dots & a_1^{l_1-l_0+m_1-m_0+n_1-n_0+\dots} & & & \\ \delta_r^{m_r-m_{r-1}+n_r-n_{r-1}+\dots} & \delta_{r-1}^{m_{r-1}-m_{r-2}+n_{r-1}-n_{r-2}+\dots} & \dots & \delta_1^{m_1-m_0+n_1-n_0+\dots} & & & \\ c_r^{n_r-n_{r-1}+\dots} & c_{r-1}^{n_{r-1}-n_{r-2}+\dots} & \dots & \dots & c_1^{n_1-n_0+\dots} & & \\ & \vdots & & & \vdots & & \vdots \end{array}$$

as the coefficient of the power of x represented by

$$x^{(l_0+l_1+\dots+l_r)+2(m_0+m_1+\dots+m_r)+3(n_0+n_1+\dots+n_r)+\dots}$$

In this expansion, all terms containing a negative exponent for any one (or for more than one) of the symbols $a, b, c, \dots$ are to be suppressed; in those which survive, each of the symbols $a, b, c, \dots$ is to be made unity; and the sum of the latter is required. The result, as before, is denoted by SlX^{-1} .

The summation is carried out in as many stages as there are sets

of symbols $a, b, c, \dots$, viz., in s stages. In each stage, there are $r+1$ summations; in the first stage, they are taken with regard to $l_0, l_1, \dots, l_r$ in respective succession; in the second stage, with regard to $m_0, m_1, \dots, m_r$ in respective succession; and so with the others.

Owing to the length of the expressions for the indices, and the coefficients in the general case, we shall first take a particular case, say $r=3$; then state the result for the case $r=4$, which can easily be established; it will then appear that the form of the result for the general case can be inferred, and can be inductively established, all these remarks applying to the first stage. Moreover, for the sake of brevity, we shall denote the quantity

$$\begin{aligned} & \lambda m_0 + \mu m_1 + \nu m_2 + \rho m_3 + \dots \\ & + (\lambda+1) n_0 + (\mu+1) n_1 + (\nu+1) n_2 + (\rho+1) n_3 + \dots \\ & + (\lambda+2) p_0 + (\mu+2) p_1 + (\nu+2) p_2 + (\rho+2) p_3 + \dots \\ & + \dots \end{aligned}$$

by the symbol $(\lambda, \mu, \nu, \rho, \dots)$; it will appear that the part of the index of x depending upon the integers $m, n, p, \dots$ always takes this form with different values of $\lambda, \mu, \nu, \rho, \dots$ from term to term, subject to the condition that the integer

$$\lambda + \mu + \nu + \rho + \dots$$

($= 2r+2$ in general) $= 8$ in the first special form considered. Further, also for the sake of brevity, it will be convenient to write

$$1-x^a = (a):$$

so that, for instance, $(1)^2(2)$ means $(1-x)^2(1-x^2)$, and so on.

5. Let Θ denote the part of the coefficient depending on the symbols $b, c, \dots$; it is the same for all the possible values of $l_0, l_1, \dots$, and is therefore unaffected by summation with regard to them.

Summing first for l_0 , the limits are 0—because of the origin of the term—to

$$l_1 + (m_1 - m_0) + (n_1 - n_0) + \dots;$$

higher values of l_0 give rise to terms that are to be suppressed. In the terms thus selected, we make a_1 equal to unity, and the result of summation gives

$$\begin{aligned} Sl X^{-1} &= Sl \Theta a_1^{l_1 - l_2 + \dots - m_2 + \dots} a_2^{l_2 - l_1 + m_2 - m_1 + \dots} \\ & \frac{1}{1-x} \{ x^{l_1 + l_2 + l_3 + (2, 2, 2, 2)} - x^{1 + 2l_1 + l_2 + (1, 3, 1, 2)} \}. \end{aligned}$$

The summation with regard to l_1 extends over terms given by values of l_1 from 0 to

$$l_3 + (m_3 - m_1) + \dots;$$

in these terms we make a_3 equal to unity, and find that, after summing, we have

$$SlX^{-1} = Sl\Theta x^{l_3 - l_2 + m_3 - m_2 + \dots}$$

$$\frac{1}{1-x} \left[\frac{x^{l_3 + l_2 + (2, 2, 2)} - x^{1 + 2l_2 + l_3 + (2, 1, 3, 2)}}{1-x} - \frac{x}{1-x^3} \{ x^{l_3 + l_2 + (1, 3, 2, 2)} - x^{2 + 3l_2 + l_3 + (1, 1, 4, 2)} \} \right].$$

The limits for l_2 that leave surviving terms are 0 to

$$l_3 + (m_3 - m_2) + \dots.$$

Taking these terms, making $a_3 = 1$, and summing, we have

$$\begin{aligned} SlX^{-1} = Sl\Theta & \left[\frac{1}{(1)^3} \left\{ \frac{x^{l_3 + (2, 2, 2, 2)} - x^{1 + 2l_2 + (2, 2, 1, 3)}}{(1)} \right\} \right. \\ & - \frac{x}{(1)^3} \left\{ \frac{x^{l_3 + (2, 1, 3, 2)} - x^{2 + 3l_2 + (2, 1, 1, 4)}}{(2)} \right\} \\ & - \frac{x}{(1)(2)} \left\{ \frac{x^{l_3 + (1, 3, 2, 2)} - x^{1 + 2l_2 + (1, 3, 1, 3)}}{(1)} \right\} \\ & \left. + \frac{x^3}{(1)(2)} \left\{ \frac{x^{l_3 + (1, 1, 4, 2)} - x^{3 + 4l_2 + (1, 1, 1, 5)}}{(3)} \right\} \right] \end{aligned}$$

In the particular case under consideration, there remains only a single operation in the first stage—summation with regard to l_3 from 0 to ∞ . Effecting this, we have as the result of the first stage, in the case of $r = 3$,

$$\begin{aligned} SlX^{-1} = Sl\Theta & \left[\frac{x^{(2, 2, 2, 2)}}{(1)^4} \right. \\ & - \frac{x}{(1)^3(2)} \{ x^{(2, 2, 1, 3)} + x^{(2, 1, 3, 2)} + x^{(1, 3, 2, 2)} \} \\ & + \frac{x^3}{(1)^3(2)^3} x^{(1, 3, 1, 3)} \\ & + \frac{x^3}{(1)^3(2)(3)} \{ x^{(2, 1, 1, 4)} + x^{(1, 1, 4, 2)} \} \\ & \left. - \frac{x^6}{1(2)(3)(4)} (x^{1, 1, 1, 5}) \right] \end{aligned}$$

6. To see what this becomes when there is only a single set of symbols a , it is sufficient to take all the indices $m, n, \dots$ zero. Each index of the form $(\lambda, \mu, \nu, \rho)$ is then zero; and we have

$$\begin{aligned} Sl \frac{1}{(1-a_3x)\left(1-\frac{a_3}{a_2}x\right)\left(1-\frac{a_1}{a_2}x\right)\left(1-\frac{x}{a_1}\right)} \\ = \frac{11}{(1)^4} - \frac{3x}{(1)^3(2)} + \frac{x^2}{(1)^2(2)^2} + \frac{x^3}{(1)^2(2)(3)} - \frac{x^4}{(1)(2)(3)(4)} \\ = \frac{1}{(1)(2)(3)(4)} \end{aligned}$$

after reductions,

$$= \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}.$$

7. In the case of $r=4$, the summation with regard to l_3 is from 0 to $l_4 + (m_4 - m_3) + \dots$; and the final operation is a summation with regard to l_4 from 0 to ∞ . The result is

$$\begin{aligned} SlX^{-1} = Sl\theta \left[\frac{x^{(2,2,2,2,2)}}{(1)^5} \right. \\ - \frac{x}{(1)^4(2)} \{x^{(2,2,2,1,3)} + x^{(2,2,1,3,2)} + x^{(2,1,3,2,2)} + x^{(1,3,2,2,2)}\} \\ + \frac{x^2}{(1)^3(2)^2} \{x^{(2,1,3,1,3)} + x^{(1,3,2,1,3)} + x^{(1,3,1,3,2)}\} \\ + \frac{x^3}{(1)^2(2)(3)} \{x^{(2,2,1,1,4)} + x^{(2,1,1,4,2)} + x^{(1,1,4,2,2)}\} \\ - \frac{x^4}{(1)^2(2)^2(3)} \{x^{(1,3,1,1,4)} + x^{(1,1,4,1,3)}\} \\ - \frac{x^5}{(1)^2(2)(3)(4)} \{x^{(2,1,1,1,5)} + x^{(1,1,1,5,2)}\} \\ \left. + \frac{x^{10}}{(1)(2)(3)(4)(5)} x^{(1,1,1,1,6)} \right]. \end{aligned}$$

8. The result of the first stage of operations in the general case of an unrestricted value of r can now be inferred. We remark:

(i.) The number of terms associated with $Sl\Theta_p$, each in the form of a fraction, is 2^r .

(ii.) Each term is composed of two parts—a coefficient, such as $\frac{x}{(1^4)(2)}$ above, and a power, such as $x^{(2,2,2,1,3)}$ above.

(iii.) The first fraction is of the form

$$\frac{1}{(1)^{r+1}} x^{(2,2,2,2,\dots)},$$

where 2 occurs $r+1$ times in the index of x .

(iv.) As regards the (fractional) coefficient of any term, its denominator is of the form

$$(1)^{\rho_1} (2)^{\rho_2} (3)^{\rho_3} (4)^{\rho_4} \dots,$$

where

$$\rho_1 + \rho_2 + \rho_3 + \rho_4 + \dots = r+1;$$

and its numerator is

$$(-1)^{\theta} x^{\rho_2 + 2\rho_3 + 3\rho_4 + \dots},$$

where θ is the sum of the integers $\rho_2, \rho_3, \rho_4, \dots$, and thus θ is $r+1-\rho_1$. Hence the (fractional) coefficient of a term is of the form

$$\frac{(-1)^{r+1-\rho_1} x^{\rho_2 + 2\rho_3 + 3\rho_4 + \dots}}{(1)^{\rho_1} (2)^{\rho_2} (3)^{\rho_3} (4)^{\rho_4} \dots},$$

subject to the condition

$$\rho_1 + \rho_2 + \rho_3 + \rho_4 + \dots = r+1.$$

(v.) As regards the index of the power of x , wherever 3 occurs it follows 1; wherever 4 occurs it follows 1, 1; wherever 5 occurs it follows 1, 1, 1, and so on.

When 13 occurs in the index there is a factor (2) in the denominator of the coefficient; when 114 occurs in the index there is a factor (3) in that denominator; when 1115, then a factor (4), and so on.

Two terms have the same coefficient, if the indices of the powers of x in them differ only by permutation of the symbols 2, 13, 114, 1115, Hence the total number of terms having the same coefficient is the same as the number of permutations of the symbols 2, 13, 114, 1115, ..., which occur in the index-symbol of any one;

and any one index-symbol can be written down from an inspection of the denominator. For example, an index-symbol to be associated with the coefficient in (iv.) is

$$2.2 \dots 13.13 \dots 114.114 \dots 1115.1115 \dots \dots,$$

where the numbers of the symbols 2, 13, 114, 1115, ... are $\rho_1, \rho_2, \rho_3, \rho_4, \dots$ respectively.

Every symbol is a partition of $2r+2$ into $r+1$ of the integers 1, 2, 3, ..., $r+2$, each repeated any proper number of times. Each such partition, in which the symbol 1 is properly associated with the symbols 3, 4, 5, ...—it is easy to see that the appropriate number of symbols 1 will be found in the partition—gives rise to an index-symbol; and the possible permutations are then to be formed as in the preceding paragraph.

The expression at the end of the last summation is now completely deducible in the general case.

Similar explanations apply as regards the expression before the end of the last summation; for the complete establishment, an inductive proof would be necessary in statement. It is not difficult—only lengthy—after what precedes.

9. There is interest in a special case, viz., $s = 1$, when there is only a single set of symbols a ; the first stage, as above, is the only one necessary, and the result—with all the symbols ($\lambda, \mu, \nu, \rho, \dots$) made zero—is the value of

$$U = Sl \left[(1 - a_r x) \left(1 - \frac{a_{r-1}}{a_r} x \right) \dots \left(1 - \frac{a_1}{a_2} x \right) \left(1 - \frac{x}{a_1} \right) \right]^{-1}.$$

It can be expressed in a simpler form, as follows:—

Let	α denote $\frac{1}{(1)},$
	$\beta \quad \quad \quad - \frac{x}{(1)(2)},$
	$\gamma \quad \quad \quad - \frac{x^2}{(1)(2)(3)},$
	$\delta \quad \quad \quad - \frac{x^3}{(1)(2)(3)(4)},$
	$\vdots \quad \quad \quad ,$

and so on. Form the product of the expressions

$$1 + zu\alpha + \frac{z^2}{2!}u^2\alpha^2 + \frac{z^3}{3!}u^3\alpha^3 + \dots, \text{ ad inf.,}$$

$$1 + zu^3\beta + \frac{z^2}{2!}u^4\beta^2 + \frac{z^3}{3!}u^6\beta^3 + \dots,$$

$$1 + zu^3\gamma + \frac{z^2}{2!}u^6\gamma^2 + \frac{z^3}{3!}u^9\gamma^3 + \dots,$$

⋮

After multiplying up, change z^m into $m!$; it is easy to see that the numerical coefficient of any particular term in the product is the number of permutations of the index-symbol in a corresponding term of the preceding investigation, and thus is the proper numerical coefficient to be associated with

$$\frac{(-1)^{r+1-p_1} 2^{p_2+2p_3+3p_4+\dots}}{(1)^{r_1} (2)^{r_2} (3)^{r_3} (4)^{r_4} \dots}$$

in the present case when the index-symbols are zero. It therefore follows* from all the preceding considerations that U is the coefficient of u^{r+1} in this product. But the product is

$$e^{zu\alpha + zu^3\beta + zu^3\gamma + \dots},$$

or, taking v to denote $u\alpha + u^3\beta + u^3\gamma + \dots$, the product is

$$e^{zv} \\ = 1 + zv + \frac{z^2}{2!}v^2 + \frac{z^3}{3!}v^3 + \dots,$$

whatever z may be. Now realizing, the product is

$$\begin{aligned} & 1 + v + v^2 + v^3 + \dots \\ &= \frac{1}{1-v} \\ &= \frac{1}{1 - u\alpha - u^3\beta - u^3\gamma - \dots}, \end{aligned}$$

so that U is the coefficient of u^{r+1} in

$$\frac{1}{1 - \frac{u}{(1)} + \frac{u^2x}{(1)(2)} - \frac{u^3x^2}{(1)(2)(3)} + \frac{u^4x^3}{(1)(2)(3)(4)} - \dots \text{ ad inf.}},$$

* This was suggested by Jeffery's method of finding the number of permutations of a number of things when they are the same in sets; see Todhunter's *Algebra*, p. 537.

that is, in

$$\frac{1}{(1-u)(1-xu)(1-x^2u)(1-x^3u) \dots \text{ad inf.}}$$

that is, in

$$1 + \frac{u}{(1)} + \frac{u^2}{(1)(2)} + \frac{u^3}{(1)(2)(3)} + \frac{u^4}{(1)(2)(3)(4)} + \dots,$$

that is, it is

$$\frac{1}{(1)(2) \dots (r+1)}$$

Hence

$$\begin{aligned} Sl \left[(1-a_r x) \left(1 - \frac{a_{r-1}}{a_r} x \right) \dots \left(1 - \frac{a_1}{a_2} x \right) \left(1 - \frac{x}{a_1} \right) \right]^{-1} \\ = \frac{1}{(1-x)(1-x^2)(1-x^3) \dots (1-x^{r+1})}. \end{aligned}$$

10. This particular result can also be established as follows:—In the expansion of the fraction, the general term is of the form

$$a_r^{p_r - p_{r-1}} a_{r-1}^{p_{r-1} - p_{r-2}} a_{r-2}^{p_{r-2} - p_{r-3}} \dots a_2^{p_2 - p_1} a_1^{p_1 - p_0} x^{p_r + p_{r-1} + p_{r-2} + \dots + p_1 + p_0}.$$

Summing first with regard to p_r , the limits are p_{r-1} to ∞ for terms that survive; the result is

$$\frac{1}{1-x} a_{r-1}^{p_{r-1} - p_{r-2}} a_{r-2}^{p_{r-2} - p_{r-3}} \dots a_2^{p_2 - p_1} a_1^{p_1 - p_0} x^{2p_{r-1} + p_{r-2} + p_{r-3} + \dots + p_1 + p_0}.$$

Summing next for p_{r-1} , the limits are p_{r-2} to ∞ for terms that survive; the result is

$$\frac{1}{(1-x)(1-x^2)} a_{r-2}^{p_{r-2} - p_{r-3}} \dots a_2^{p_2 - p_1} a_1^{p_1 - p_0} x^{3p_{r-2} + p_{r-3} + \dots + p_1 + p_0}.$$

Proceeding in this way, the result of r summations—the last being from p_0 to ∞ —is

$$\frac{1}{(1-x)(1-x^2) \dots (1-x^r)} x^{(r+1)p_0}.$$

And p_0 can now have any value from 0 to ∞ ; hence a last summation leads to the result as stated above.

11. Proceeding to the second stage, the summation can be carried out in a similar manner, either for the aggregate of terms together,

or for each term separately. The latter was the mode adopted with the following result.

Denoting by Φ the part of the symbolical coefficient Θ which is independent of $c, d, \dots$, and by $|\lambda, \mu, \nu, \rho|$ the expression

$$\begin{aligned} & \lambda n_0 + \mu n_1 + \nu n_2 + \rho n_3 \\ & + (\lambda + 1)p_0 + (\mu + 1)p_1 + (\nu + 1)p_2 + (\rho + 1)p_3 \\ & + \dots, \end{aligned}$$

it can be proved that

$$\begin{aligned} SlX^{-1} = Sl\Phi & \left[\frac{x^{[3,3,3,3]}}{(1)^4(2)^4} \right. \\ & - \frac{x}{(1)^4(2)^3(3)} \{x^{[3,3,2,4]} + x^{[3,2,4,3]} + x^{[2,4,3,3]}\} \\ & + \frac{x^3}{(1)^4(2)^2(3)^2} x^{[2,4,2,4]} \\ & + \frac{x^3}{(1)^3(2)^3(3)^2} \{x^{[3,1,4,4]} + x^{[1,4,4,3]}\} \\ & + \frac{x^3}{(1)^4(2)^2(3)(4)} \{x^{[3,2,2,5]} + x^{[2,2,5,3]}\} \\ & - \frac{x^4}{(1)^3(2)^3(3)(4)} \{x^{[3,1,3,5]} + x^{[1,3,5,3]}\} \\ & - \frac{x^5}{(1)^3(2)^2(3)^2(4)} \{x^{[1,4,2,5]} + x^{[2,1,5,4]}\} \\ & - \frac{x^6}{(1)^4(2)(3)(4)(5)} x^{[2,2,2,5]} \\ & + \frac{x^7}{(1)^3(2)^2(3)(4)(5)} \{x^{[2,1,3,5]} + x^{[1,3,2,5]}\} \\ & + \frac{x^8}{(1)^3(2)^3(3)^2(4)^2} x^{[1,1,5,5]} \\ & \left. - \frac{x^9}{(1)(2)^2(3)^2(4)(5)} x^{[1,1,4,5]} \right], \end{aligned}$$

which is the result at the end of the second stage.

12. In particular, if only two sets of symbols occur in the original expression for X^{-1} , we have $\Phi = 1$; and all the symbolic indices $|\lambda, \mu, \nu, \rho|$ are zero. Then

$$\begin{aligned}
 SlX^{-1} &= \frac{1}{(1)^4(2)^4} - \frac{3x}{(1)^4(2)^3(3)} + \frac{x^2}{(1)^4(2)^2(3)^2} + \frac{2x^3}{(1)^3(2)^3(3)^2} \\
 &\quad + \frac{2x^3}{(1)^4(2)^2(3)(4)} - \frac{2x^4}{(1)^3(2)^3(3)(4)} - \frac{2x^5}{(1)^2(2)^3(3)^2(4)} \\
 &\quad - \frac{x^6}{(1)^4(2)(3)(4)(5)} + \frac{2x^7}{(1)^3(2)^2(3)(4)(5)} \\
 &\quad + \frac{x^8}{(1)^2(2)^2(3)^2(4)^2} - \frac{x^9}{(1)^2(2)^2(3)^2(4)(5)} \\
 &= \frac{1}{(1)(2)^2(3)^2(4)^2(5)}, \text{ multiplied by}
 \end{aligned}$$

x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
1	3	8	14	22	27	30	27	22	14	8	3	1
	-3	-9	-21	-36	-51	-60	-60	-51	-36	-21	-9	-3
		1	3	6	10	13	14	13	10	6	3	1
			2	4	9	12	14	14	12	8	4	2
			2	6	12	18	22	22	18	12	6	2
				-2	-4	-8	-10	-12	-10	-8	-4	-2
					-2	-4	-6	-8	-8	-6	-4	-2
						-1	-3	-5	-6	-5	-3	-1
							2	4	6	6	4	2
								1	1	1	1	1
									-1	-1	-1	-1
1	± 3	± 9	± 21	± 38	± 57	± 73	± 79	± 76	± 61	± 41	± 21	± 9

$$= \frac{1}{(1)(2)^2(3)^2(4)^2(5)};$$

and therefore

$$Sl \left\{ \frac{1}{(1-a_3x) \left(1 - \frac{a_2}{a_3}x\right) \left(1 - \frac{a_1}{a_2}x\right) \left(1 - \frac{x}{a_1}\right)} \right. \\ \left. \times \frac{1}{(1-a_3b_3x^3) \left(1 - \frac{a_2b_2}{a_3b_3}x^3\right) \left(1 - \frac{a_1b_1}{a_2b_2}x^3\right) \left(1 - \frac{x^3}{a_1b_1}\right)} \right\} \\ = \frac{1}{(1-x)(1-x^3)^2(1-x^9)^2(1-x^{27})^2(1-x^{81})^2}.$$

13. From the results obtained, the general expression at the end of the second stage in the case of s sets of symbols and $r+1$ factors involving any one set does not present itself as an obvious generalizable result from the particular case. But the last expression, and corresponding expressions which I have obtained in simpler cases, suggest (but, of course, do not prove) that in the general case the required sum is the reciprocal of the product of the expressions

$$\begin{aligned} (1)(2)(3)(4) \dots (r+1), \\ (2)(3)(4) \dots (r+2), \\ (3)(4) \dots (r+2)(r+3), \\ \dots \dots \dots \dots \\ s(1+s)(2+s) \dots (r+s). \end{aligned}$$

14. Further, we have at once

$$Sl \frac{1}{(1-a_1x)(1-a_1a_2x^2) \dots (1-a_1a_2 \dots a_nx^n)} = \frac{1}{(1-x)(1-x^2) \dots (1-x^n)},$$

and we have proved that

$$Sl \frac{1}{(1-a_{n-1}x) \left(1 - \frac{a_{n-2}}{a_{n-1}}x\right) \dots \left(1 - \frac{a_1}{a_2}x\right) \left(1 - \frac{x}{a_1}\right)} \\ = \frac{1}{(1-x)(1-x^2) \dots (1-x^n)}.$$

We have also proved that

$$Sl \frac{1}{(1-a_1x)\left(1-\frac{x}{a_1}\right)(1-a_1a_2x^2)\left(1-\frac{x^2}{a_1a_2}\right) \dots n \text{ product-pairs}}$$

$$= \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4) \dots (1-x^n)(1-x^{n+1})},$$

and we have inferred that the latter expression is also the value of

$$Sl \left\{ \frac{1}{(1-a_{n-1}x)\left(1-\frac{a_{n-2}}{a_{n-1}}x\right) \dots \left(1-\frac{a_1}{a_2}x\right)\left(1-\frac{x}{a_1}\right)} \right.$$

$$\times \frac{1}{(1-a_{n-1}b_{n-1}x^2)\left(1-\frac{a_{n-2}b_{n-2}}{a_{n-1}b_{n-1}}x^2\right) \dots \left(1-\frac{a_1b_1}{a_2b_2}x^2\right)\left(1-\frac{x^2}{a_1b_1}\right)} \Bigg\}.$$

Comparing these results with the diagram of the general expression given in § 13, it can be inferred that there is a symmetry of value between two expressions, one having m rows and n factors in each row, the other having n rows and m factors in each row. The establishment of this symmetry can be effected with the aid of the theory of partitions, by which subject, indeed, all these fractions were primarily suggested.

2nd October, 1895.

Note on Matrices. By J. BRILL, M.A. Received October 14th, 1895. Communicated November 14th, 1895.

My object in the following short communication is to obtain the most general form of the differential of a matrix which admits of its being commutative with the matrix itself.

If m be the matrix, and $\lambda_1, \lambda_2, \dots, \lambda_n$ its latent roots, which we shall suppose to be all different, then m satisfies the equation

$$(m-\lambda_1)(m-\lambda_2) \dots (m-\lambda_n) = 0 \dots\dots\dots(1).$$

D 2