Next, to effect the partition $p = c^2 + 2d^2$. In the course of the tentative work above, the form

$$p = 3251^2 - 2 \,.\, 30^2$$

was obtained (at the start): this, and the now known form $(a^2 + b^2)$, are suitable base-forms (since $a^2 + b^2$, $c^2 + 2d^2$, $e^2 - 2f^2$ are a quadratic triad), so that

$$p = 2415^2 + 2176^2 = 3251^2 - 2 \,.\, 30^2.$$

This falls under Case i. Hence, by (28),

$$p = \frac{(2176 \,.\, 3251)^2 + 2\,(2415 \,.\, 30)^2}{2176^2 + 2 \,.\, 30^2} = \frac{(2415 \,.\, 15)^2 + 2\,(544 \,.\, 3251)^2}{15^2 + 2 \,.\, 544^2} = \frac{F}{f}$$

(the factor 8 having been cancelled out of F, f).

Here $f = 592{,}097 = 11 \,.\, 19 \,.\, 2833 = (3^2 + 2 \,.\, 1^2)(1^2 + 2 \,.\, 3^2)(41^2 + 2 \,.\, 24^2)$,

so that the *safe* procedure is to divide conformally by each factor separately (writing $f_1 = 11$, $f_2 = 19$, $f_3 = 2833$), as in Art. 20. Omitting details, the steps are

$$F_1 = \frac{F}{f_1} = \frac{(2415 \,.\, 15)^2 + 2\,(544 \,.\, 3251)^2}{11 = (3^2 + 2 \,.\, 1^2)} = 331{,}433^2 + 2 \,.\, 479{,}037^2,$$

$$F_2 = \frac{F_1}{f_2} = \frac{331{,}433^2 + 2 \,.\, 479{,}037^2}{19 = (1^2 + 2 \,.\, 3^2)} = 133{,}831^2 + 2 \,.\, 77{,}544^2,$$

$$F_3 = \frac{F_2}{f_3} = \frac{133{,}831^2 + 2 \,.\, 77{,}544^2}{833 = (41^2 + 2 \,.\, 24^2)} = 623^2 + 2 \,.\, 2256^2 = p,$$

the required partition.

On a Series of Cotrinodal Quartics. *By* H. M. TAYLOR, M.A.

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1. Many well-known theorems in the geometry of the triangle relate to straight lines drawn at right angles to the sides of the triangle.

In attempting to generalize these theorems we are led to consider in what cases and in what circumstances it is possible by orthogonal projection to project a triangle and a given point in its plane into a triangle and its orthocentre.

2. A well-known theorem given by Lazare Carnot states that, if the angular points A, B, C of a triangle be joined to any two points O, O' in its plane by straight lines which cut the opposite sides of the triangle in the points D, D', E, E', F, F', then these six points lie on a conic. From the well-known properties of the nine-point circle it

appears that, if the point O were the centre of gravity of the triangle ABC, and if the conic $DD'EE'FF'$ were an ellipse, then the orthogonal projection which would project the ellipse into a circle would project the triangle ABC and the point O' into a triangle and its orthocentre.

It can be proved that, when the point O is the centre of gravity of the triangle ABC, the conic $DD'EE'FF'$ will be a parabola if the point O' lie on a side of the triangle or a side produced; the conic will be an ellipse if the point O' lie within the triangle or in a part of the plane reached by crossing two of the sides; and the conic will be a hyperbola if the point O' lie within a part of the plane reached by crossing one of the sides.

3. We shall now consider the more general problem to find, when one of the points O, O' is given, the locus on which the other point must lie in order that the conic $DD'EE'FF$ may have a given eccentricity.

If the terms of the second degree in the equation of a conic in rectangular coordinates be

$$Ax^2+2Hxy+By^2,$$

the eccentricity e of the conic is given by the equation

$$\frac{1-e^2}{(2-e^2)^2}=\frac{AB-H^2}{(A+B)^2},$$

or the equivalent equation

$$\frac{e^4}{(2-e^2)^2}=\frac{(A-B)^2+4H^2}{(A+B)^2}.$$

4. If the equation of a conic in trilinear coordinates

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\beta\gamma+2w'\alpha\beta=0$$

be transformed to rectangular coordinates by the usual transformation

$$\alpha=p_1-x\cos a-y\sin a,$$

$$\beta=p_2-x\cos\beta-y\sin\beta,$$

$$\gamma=p_3-x\cos\gamma-y\sin\gamma,$$

then

$$x^2\left\{u\cos^2 a+v\cos^2\beta+w\cos^2\gamma+2u'\cos\beta\cos\gamma+2v'\cos\gamma\cos a+2w'\cos a\cos\beta\right\}$$

$$+xy\left\{2u\cos a\sin a+2v\cos\beta\sin\beta+2w\cos\gamma\sin\gamma+2u'(\cos\beta\sin\gamma+\sin\beta\cos\gamma)\right.$$

$$\left.+2v'(\cos\gamma\sin a+\cos a\sin\gamma)+2w'(\cos a\sin\beta+\cos\beta\sin a)\right\}$$

$$+y^2\left\{u\sin^2 a+v\sin^2\beta+w\sin^2\gamma+2u'\sin\beta\sin\gamma+2v'\sin\gamma\sin a+2w'\sin a\sin\beta\right\}$$

$$+\text{terms of first degree}=\text{const.}$$

If this be written

$$Ax^2+2Hxy+By^2+\text{terms of first degree}=\text{const.},$$

then

$$A+B=u+v+w+2u'\cos(\beta-\gamma)+2v'\cos(\gamma-\alpha)+2w'\cos(\alpha-\beta)$$
$$=u+v+w-2u'\cos A-2v'\cos B-2w'\cos C;$$

$$A-B=u\cos 2\alpha+v\cos 2\beta+w\cos 2\gamma+2u'\cos(\beta+\gamma)+2v'\cos(\gamma+\alpha)+2w'\cos(\alpha+\beta)$$
$$=u\cos 2\alpha+v\cos(2\alpha-2C)+w\cos(2\alpha+2B)+2u'\cos(2\alpha+B-C)-2v'\cos(2\alpha+B)-2w'\cos(2\alpha-C)$$
$$=\cos 2\alpha\,\{u+v\cos 2C+w\cos 2B+2u'\cos(B-C)-2v'\cos B-2w'\cos C\}$$
$$+\sin 2\alpha\,\{v\sin 2C-w\sin 2B-2u'\sin(B-C)+2v'\sin B-2w'\sin C\};$$

$$2H=u\sin 2\alpha+v\sin 2\beta+w\sin 2\gamma+2u'\sin(\beta+\gamma)+2v'\sin(\gamma+\alpha)+2w'\sin(\alpha+\beta)$$
$$=u\sin 2\alpha+v\sin(2\alpha-2C)+w\sin(2\alpha+2B)+2u'\sin(2\alpha+B-C)-2v'\sin(2\alpha+B)-2w'\sin(2\alpha-C)$$
$$=\sin 2\alpha\,\{u+v\cos 2C+w\cos 2B+2u'\cos(B-C)-2v'\cos B-2w'\cos C\}$$
$$+\cos 2\alpha\,\{-v\sin 2C+w\sin 2B+2u'\sin(B-C)-2v'\sin B+2w'\sin C\}.$$

Therefore

$$(A-B)^2+4H^2=u^2+\&c.+4u'^2+\&c.+2uv\cos 2(\alpha-\beta)+\&c.$$
$$+2u'v'\cos(\alpha-\beta)+\&c.+4uu'\cos(2\alpha-\beta-\gamma)+\&c.$$
$$+2uv'\cos(\alpha-\gamma)+\&c.$$
$$=u^2+\&c.+4u'^2+\&c.+2uv\cos 2C+\&c.-8u'v'\cos C+\&c.$$
$$+4uu'\cos(B-C)+\&c.-4uv'\cos B+\&c.$$

If we take the triangle ABC as the triangle of reference, then, if the coordinates of O be α, β, γ, and those of O' be α', β', γ', and x, y, z be current coordinates, the equation of the conic $DD'EE'FF'$ is

$$\frac{x^2}{\alpha\alpha'}+\frac{y^2}{\beta\beta'}+\frac{z^2}{\gamma\gamma'}-yz\left(\frac{1}{\beta\gamma'}+\frac{1}{\beta'\gamma}\right)-zx\left(\frac{1}{\gamma'\alpha}+\frac{1}{\gamma\alpha'}\right)-xy\left(\frac{1}{\alpha\beta'}+\frac{1}{\alpha'\beta}\right)=0.$$

Comparing this equation with the equation

$$ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy=0,$$

we see that each of the expressions $A+B$, $A-B$, $2H$ is of -1

dimension in $\alpha\beta\gamma$, and that therefore the equation

$$(A-B)^2+4H^2 = \frac{e^4}{(2-e^2)^2}(A+B)^2$$

admits only terms of the form

$$\frac{1}{\alpha^2},\ \frac{1}{\beta^2},\ \frac{1}{\gamma^2},\ \frac{1}{\beta\gamma},\ \frac{1}{\gamma\alpha},\ \frac{1}{\alpha\beta};$$

it follows therefore that, if the point O' $(\alpha', \beta', \gamma')$ be fixed, and the eccentricity e be given, the locus of the point O is a trinodal quartic having A, B, C for nodes.

It appears also from the equation

$$AB-H^2 = \frac{1-e^2}{(2-e^2)^2}(A+B)^2$$

that the locus $AB-H^2=0$ separates the parts of the plane in which O must lie in the two cases: (1) when the conic is an ellipse, and (2) when it is a hyperbola.

It may be remarked that, if the conic be a circle, the locus of the point O is reduced to a point.

5. The remaining part of this paper is concerned more especially with the case when the conic is a parabola. It is somewhat more convenient in discussing this problem to use triangular coordinates.

The condition that the triangular equation

$$\frac{x^2}{\alpha\alpha'}+\frac{y^2}{\beta\beta'}+\frac{z^2}{\gamma\gamma'}-yz\left(\frac{1}{\beta\gamma'}+\frac{1}{\beta'\gamma}\right)-zx\left(\frac{1}{\gamma'\alpha}+\frac{1}{\gamma\alpha'}\right)-xy\left(\frac{1}{\alpha\beta'}+\frac{1}{\alpha'\beta}\right)=0$$

should represent a parabola is

$$\frac{4}{\beta\beta'\gamma\gamma'}+\frac{4}{\gamma\gamma'\alpha\alpha'}+\frac{4}{\alpha\alpha'\beta\beta'}+\frac{4}{\alpha\alpha'}\left(\frac{1}{\beta\gamma'}+\frac{1}{\beta'\gamma}\right)+\frac{4}{\beta\beta'}\left(\frac{1}{\gamma\alpha'}+\frac{1}{\gamma'\alpha}\right)$$
$$+\frac{4}{\gamma\gamma'}\left(\frac{1}{\alpha\beta'}+\frac{1}{\alpha'\beta}\right)-\left(\frac{1}{\beta\gamma'}+\frac{1}{\beta'\gamma}\right)^2-\left(\frac{1}{\gamma\alpha'}+\frac{1}{\gamma'\alpha}\right)^2-\left(\frac{1}{\alpha\beta'}+\frac{1}{\alpha'\beta}\right)^2$$
$$+2\left(\frac{1}{\gamma\alpha'}+\frac{1}{\gamma'\alpha}\right)\left(\frac{1}{\alpha\beta'}+\frac{1}{\alpha'\beta}\right)+2\left(\frac{1}{\alpha\beta'}+\frac{1}{\alpha'\beta}\right)\left(\frac{1}{\beta\gamma'}+\frac{1}{\beta'\gamma}\right)$$
$$+2\left(\frac{1}{\beta\gamma'}+\frac{1}{\beta'\gamma}\right)\left(\frac{1}{\gamma\alpha'}+\frac{1}{\gamma'\alpha}\right)=0,$$

so that

$$\left\{\frac{4}{\beta'\gamma'}+\frac{4}{\beta'\alpha'}+\frac{4}{\gamma'\alpha'}-\frac{2}{\beta'\gamma'}+\frac{2}{\alpha'^2}+\frac{2}{\alpha'\beta'}+\frac{2}{\gamma'\alpha'}\right\}\frac{1}{\beta\gamma}+\&c.$$
$$=\left(\frac{1}{\beta'}-\frac{1}{\gamma'}\right)^2\frac{1}{\alpha^2}+\left(\frac{1}{\gamma'}-\frac{1}{\alpha'}\right)^2\frac{1}{\beta^2}+\left(\frac{1}{\alpha'}-\frac{1}{\beta'}\right)^2\frac{1}{\gamma^2}.$$

Therefore $8\left\{\frac{1}{\gamma'a}+\frac{1}{a'\beta}\right\}\frac{1}{\beta\gamma}+\&c.$

$$=\left\{\left(\frac{1}{\beta'}-\frac{1}{\gamma'}\right)\frac{1}{a}+\left(\frac{1}{\gamma'}-\frac{1}{a}\right)\frac{1}{\beta}+\left(\frac{1}{a'}-\frac{1}{\beta'}\right)\frac{1}{\gamma}\right\}^2.$$

Therefore $8a'\beta'\gamma'a\beta\gamma\left\{(\beta'+\gamma')\,a+(\gamma'+a')\,\beta+(a'+\beta')\,\gamma\right\}$

$$=\left\{a'(\beta'-\gamma')\beta\gamma+\beta'(\gamma'-a')\gamma a+\gamma'(a'-\beta')a\beta\right\}^2,$$

an equation which gives the locus of the point O for any given position $a'\beta'\gamma'$ of the point O', when the conic $DD'EE'FF'$ is a parabola. The equation may for convenience be written $8a'\beta'\gamma'a\beta\gamma L = S^2$.

6. It may be noticed that the line $L = 0$ is the polar of the point O' with respect to the conic $\beta\gamma+\gamma a+a\beta = 0$, which is the circumscribing ellipse of minimum area.

The conic $S = 0$ is a hyperbola circumscribing the triangle ABC and passing through its centre of gravity G and also through the point O'.

The conic $S = 0$ is completely determined by the position of the point O', but any position of the point O' on the conic $S = 0$ would equally well determine the same conic $S = 0$. On the other hand, every different position of the point O' on the conic $S = 0$ gives rise to a different quartic $8a'\beta'\gamma'a\beta\gamma L = S^2$. It seems convenient, in discussing the different forms of the quartic, to assume the conic $S = 0$ determined by some point on it and then to trace the changes in the quartic curve for different positions of the point O', while it traces out the conic $S = 0$.

This has been done for me by my friend Mr. W. H. Blythe, who has kindly drawn a series of diagrams illustrative of the manner in which the shape and the position of the quartic curve change with the position of the point O'.

A paper by Mr. Blythe exhibiting these diagrams accompanies this paper.

7. It is easy to prove that the straight line $\beta+\gamma = 0$ meets the quartic twice at the point A, and is a tangent to the curve at the point of intersection with

$$a\,(a'\beta'+a'\gamma'+2\beta'\gamma')-\beta a'\,(\beta'-\gamma') = 0,$$

that is, at the point

$$\frac{a}{a'(\beta'-\gamma')}=\frac{+\beta}{a'\beta'+a'\gamma'+2\beta'\gamma'}=\frac{-\gamma}{a'\beta'+a'\gamma'+2\beta'\gamma'}=\frac{1}{a'(\beta'-\gamma')}.$$

Similarly, the straight line $\frac{\beta}{\beta'}+\frac{\gamma}{\gamma'}=0$ meets the curve twice at the point A, and it touches the quartic curve at the point where it meets

$$\alpha\,(2\alpha'+\beta'+\gamma')+\beta\,\frac{\alpha'}{\beta'}\,(\beta'-\gamma')=0,$$

that is, at the point

$$\frac{\frac{\alpha}{\alpha'}}{\beta'-\gamma'}=\frac{-\frac{\beta}{\beta'}}{2\alpha'+\beta'+\gamma'}=\frac{\frac{\gamma}{\gamma'}}{2\alpha'+\beta'+\gamma'}$$

$$=\frac{1}{\alpha'(\beta'-\gamma')-(\beta'-\gamma')(2\alpha'+\beta'+\gamma)}$$

$$=\frac{-1}{(\beta'-\gamma')(\alpha'+\beta'+\gamma')}=-\frac{1}{\beta'-\gamma}$$

8. We will now discuss in which of the portions of space, into which the plane is divided by the three sides of the triangle ABC and the line L, the curve $8\alpha'\beta'\gamma'\alpha\beta\gamma L=S^2$ lies.

It is clear from the form of the equation that the portions of space in which the curve lies are to be found from each other by crossing *two* or *four* of the straight lines $\alpha=0$, $\beta=0$, $\gamma=0$, $L=0$.

If, therefore, for any given position of the point $\alpha'\beta'\gamma'$, we determine the position of the line L and also the position of some point on the curve in one of the parts of the plane, we shall have determined in which parts of the plane the curve lies.

First, let us take the case where $\beta'+\gamma'$, $\gamma'+\alpha'$, and $\alpha'+\beta'$ are all positive, that is, where the point $\alpha'\beta'\gamma'$ lies within the triangle $A'B'C'$ formed by drawing through the angular points of the triangle ABC parallels to the opposite sides; then the expression L or

$$\alpha\,(\beta'+\gamma')+\beta\,(\gamma'+\alpha')+\gamma\,(\alpha'+\beta')$$

is positive if α, β, γ be all positive. Therefore, in this case, if α', β', γ' be all positive, there must be some part of the curve within the triangle ABC, but, if one of the three quantities α', β', γ' be negative, there is no part of the curve within the triangle ABC.

In this case, the triangle is not cut by the line L.

Next, let us consider the case where $\alpha'+\beta'$ and $\alpha'+\gamma'$ are both positive and $\beta'+\gamma'$ is negative; in this case, the line L will cut the sides AC and AB both internally in some points Q and R, and the quantity L will be negative for points on the same side of QR as A, and positive for points on the opposite side. It follows therefore that, if the product $\beta'\gamma'$ be positive, some part of the curve lies within the quadrilateral $RBCQ$, and, if the product $\beta'\gamma'$ be negative, no part lies within the quadrilateral $RBCQ$. Again, in the case where $\beta'+\gamma'$ is positive and both $\alpha'+\beta'$ and $\alpha'+\gamma'$ are negative, the line L cuts both the sides AC and AB internally in some points Q and R; and the quantity L is positive for points on the same side of QR as the point A, and negative for points on the opposite side. It follows therefore that some part of the curve lies within the quadrilateral $RBCQ$.

The tangents at the double point A are

$$8\beta\gamma\alpha\,\beta'\gamma'(\beta'+\gamma')=\gamma^2\beta'^2(\gamma'-\alpha')^2+\beta^2\gamma'^2(\alpha'-\beta')^2+2\beta\gamma\beta'\gamma'(\gamma'-\alpha')(\alpha'-\beta').$$

These tangents will be real if

$$\{\beta'\gamma'(\gamma'-\alpha')(\alpha'-\beta')-4\alpha'\beta'\gamma'(\beta'+\gamma')\}^2-\beta'^2\gamma'^2(\gamma'-\alpha')^2(\alpha'-\beta')^2$$

is positive, that is, if $\alpha'(\beta'+\gamma')(\gamma'+\alpha')(\alpha'+\beta')$ is positive; that is, if the point $\alpha'\beta'\gamma'$ lies within the quadrilateral $BCB'C'$ or in any of the spaces reached from within the quadrilateral by crossing two of its sides.

9. For convenience, we will take the equation of the quartic to be

$$a^2\beta^2\gamma^2+b^2\gamma^2\alpha^2+c^2\alpha^2\beta^2=2\alpha\beta\gamma(l\alpha+m\beta+n\gamma).$$

When this equation is written in the form

$$(a\beta\gamma+b\gamma\alpha+c\alpha\beta)^2=2\alpha\beta\gamma\,\{(l+bc)\,\alpha+(m+ca)\,\beta+(n+ab)\,\gamma\},$$

we see that the straight line

$$(l+bc)\,\alpha+(m+ca)\,\beta+(n+ab)\,\gamma=0$$

is a double tangent to the quartic touching at the points where it meets the conic

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=0.$$

As the above form of the equation of the quartic is equally true when the sign of one of the letters a, b, c is changed, it follows that the straight lines

$$(l+bc)\,\alpha+(m-ac)\,\beta+(n-ab)\,\gamma=0,$$

$$(l-bc)\,\alpha+(m+ac)\,\beta+(n-ab)\,\gamma=0,$$

$$(l-bc)\,\alpha+(m-ac)\,\beta+(n+ab)\,\gamma=0$$

are double tangents, their points of contact being the points where they meet the conics

$$-a\beta\gamma+b\gamma\alpha+c\alpha\beta=0,$$
$$a\beta\gamma-b\gamma\alpha+c\alpha\beta=0,$$
$$a\beta\gamma+b\gamma\alpha-c\alpha\beta=0,$$

respectively.

The eight points of contact of these four double tangents lie upon the conic

$$(l^2-b^2c^2)\,\alpha^2+(m^2-c^2a^2)\,\beta^2+(n^2-a^2b^2)\,\gamma^2+2mn\beta\gamma+2nl\gamma\alpha+2lm\alpha\beta=0\,;$$

and the equation of the quartic may be written

$$\{(l+bc)\,\alpha+(m+ac)\,\beta+(n+ab)\,\gamma\}\,\{(l+bc)\,\alpha+(m-ac)\,\beta+(n-ab)\,\gamma\}$$
$$\times\{(l-bc)\alpha+(m+bc)\beta+(n-ab)\gamma\}\,\{(l-bc)\alpha+(m-ac)\beta+(n+ab)\gamma\}$$
$$-\{(l^2-b^2c^2)\,\alpha^2+(m^2-c^2a^2)\beta^2+(n^2-a^2b^2)\,\gamma^2+2mn\beta\gamma+2nl\gamma\alpha+2lm\alpha\beta\}^2$$
$$\equiv -a^2b^2c^2\{a^2\beta^2\gamma^2+b^2\gamma^2\alpha^2+c^2\alpha^2\beta^2-2a\beta\gamma\,(l\alpha+m\beta+n\gamma)\}=0.$$

Of the four conics

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=0,$$
$$-a\beta\gamma+b\gamma\alpha+c\alpha\beta=0,$$
$$a\beta\gamma-b\gamma\alpha+c\alpha\beta=0,$$
$$a\beta\gamma+b\gamma\alpha-c\alpha\beta=0,$$

three are hyperbolas and one is an ellipse. They touch in pairs at the angular points of the triangle ABC.

10. When we compare this form with the earlier form

$$8\alpha'\beta'\gamma'\alpha\beta\gamma L=S^2,$$

we have

$$a=\alpha'(\beta'-\gamma'),\quad b=\beta'(\gamma'-\alpha'),\quad c=\gamma'(\alpha'-\beta'),$$
$$l=4\alpha'\beta'\gamma'(\beta'+\gamma')+\beta'\gamma'(\alpha'-\beta')(\alpha'-\gamma'),$$
$$l+bc=4\alpha'\beta'\gamma'(\beta'+\gamma'),$$
$$\begin{aligned}l-bc&=4\alpha'\beta'\gamma'(\beta'+\gamma')+2\beta'\gamma'(\alpha'-\beta')(\alpha'-\gamma')\\&=2\beta'\gamma'\{2\alpha'\beta'+2\alpha'\gamma'+\alpha'^2-\alpha'\beta'-\alpha'\gamma'+\beta'\gamma'\}\\&=2\beta'\gamma'\{\alpha'^2+\alpha'\beta'+\alpha'\gamma'+\beta'\gamma'\}\\&=2\beta'\gamma'(\alpha'+\beta')(\alpha'+\gamma').\end{aligned}$$

Therefore $l^2-b^2c^2=8\alpha'\beta'^2\gamma'^2(\beta'+\gamma')(\gamma'+\alpha')(\alpha'+\beta')$.

The equation of the conic through the eight points of contact of the four double tangents therefore is

$$4\alpha'\beta'\gamma'\left\{(\beta'+\gamma')\,\alpha+(\gamma'+\alpha')\,\beta+(\alpha'+\beta')\,\gamma\right\}$$
$$\times 2\left\{\beta'\gamma'(\alpha'+\beta')(\alpha'+\gamma')\alpha+\gamma'\alpha'(\beta'+\alpha')(\beta'+\gamma')\beta+\alpha'\beta'(\gamma'+\alpha')(\gamma'+\beta')\gamma\right\}$$
$$+2\alpha'\beta'\gamma'(\beta'-\gamma')(\gamma'-\alpha')(\alpha'-\beta')\left\{\alpha'(\beta'-\gamma')\,\beta\gamma+\beta'(\gamma'-\alpha')\,\gamma\alpha\right.$$
$$\left.+\gamma(\alpha'-\beta')\,\alpha\beta\right\}=0;$$

or

$$4(\beta'+\gamma')(\gamma'+\alpha')(\alpha'+\beta')\left\{(\beta'+\gamma')\,\alpha+(\gamma'+\alpha')\,\beta+(\alpha'+\beta')\,\gamma\right\}$$
$$\times\left\{\frac{\beta'\gamma'\alpha}{\beta'+\gamma'}+\frac{\gamma'\alpha'\beta}{\gamma'+\alpha'}+\frac{\alpha'\beta'\gamma}{\alpha'+\beta'}\right\}+(\beta'-\gamma')(\gamma'-\alpha')(\alpha'-\beta')$$
$$\times\left\{\alpha'(\beta'-\gamma')\,\beta\gamma+\beta'(\gamma'-\alpha')\,\gamma\alpha+\gamma'(\alpha'-\beta')\,\alpha\beta\right\}=0.$$

The equations of the four double tangents of the quartic are

$$(\beta'+\gamma')\,\alpha+(\gamma'+\alpha')\,\beta+(\alpha'+\beta')\,\gamma=0,$$
$$2\beta'\gamma'\alpha+\gamma'\,(\alpha'+\beta')\,\beta+\beta'\,(\alpha'+\gamma')\,\gamma=0,$$
$$\gamma'\,(\alpha'+\beta')\,\alpha+2\alpha'\gamma'\beta+\alpha'\,(\beta'+\gamma')\,\gamma=0,$$
$$\beta'\,(\alpha'+\gamma')\,\alpha+\alpha'\,(\beta'+\gamma')\,\beta+2\alpha'\beta'\gamma=0.$$

The condition that the quartic should degenerate into two conics is identical with the condition that the conic

$$8\alpha'\beta'\gamma'\left\{\beta\gamma\,(\beta'+\gamma')+\gamma\alpha\,(\gamma'+\alpha')+\alpha\beta\,(\alpha'+\beta')\right\}$$
$$=\left\{\alpha\alpha'(\beta'-\gamma')+\beta\beta'(\gamma'-\alpha')+\gamma\gamma'(\alpha'-\beta')\right\}^2$$

should degenerate into two straight lines.

This is

$$\begin{vmatrix} \alpha'^2(\beta'-\gamma')^2, & \alpha'\beta'(\beta'-\gamma')(\gamma'-\alpha') & \alpha'\gamma'(\beta'-\gamma')(\alpha-\beta') \\ & \quad -4\alpha'\beta'\gamma'(\alpha'+\beta'), & \quad -4\alpha'\beta'\gamma'(\gamma'+\alpha') \\ \alpha'\beta'(\beta'-\gamma')(\gamma'-\alpha') & \beta'^2(\gamma'-\alpha')^2, & \beta'\gamma'(\gamma'-\alpha')(\alpha'-\beta') \\ \quad -4\alpha'\beta'\gamma'(\alpha'+\beta'), & & \quad -4\alpha'\beta'\gamma'(\beta'+\gamma') \\ \alpha'\gamma'(\beta'-\gamma')(\alpha'-\beta') & \beta'\gamma'(\gamma'-\alpha')(\alpha'-\beta') & \gamma'^2(\alpha'-\beta')^2 \\ \quad -4\alpha'\beta'\gamma'(\gamma'+\alpha'), & \quad -4\alpha'\beta'\gamma'(\beta'+\gamma'), & \end{vmatrix}=0,$$

which reduces to

$$\alpha'^2\beta'^2\gamma'^2(\beta'+\gamma')^2(\gamma'+\alpha')^2(\alpha'+\beta')^2 = 0.$$

The quartic therefore splits up into two conics only when the point $\alpha'\beta'\gamma'$ lies on a side, or a side produced, of one of the two triangles ABC, $A'B'C'$.

If $\alpha' = 0$, the equation of the quartic becomes

$$\beta'^2\gamma'^2\alpha^2(\beta-\gamma)^2 = 0,$$

that is, the quartic reduces to the two straight lines BC, AG, repeated twice.

If $\beta'+\gamma' = 0$, the equation of the quartic becomes

$$\{2\alpha'\beta'\beta\gamma+\beta'(\alpha'-\beta')\gamma\alpha+\beta'(\alpha'+\beta')\alpha\beta\}^2 = 0.$$

The quartic therefore reduces to the conic

$$2\alpha'\beta\gamma+(\alpha'-\beta')\gamma\alpha+(\alpha'+\beta')\alpha\beta = 0,$$

repeated twice. This conic circumscribes the parallelogram $ABA'C$, and has its centre at the middle point of BC. It is an ellipse, a degenerate parabola, or a hyperbola, according as the point $\alpha'\beta'\gamma'$ lies between the points B' and C', coincides with one of them, or lies outside them.

The quartic degenerates into a straight line and a cubic when the point $\alpha'\beta'\gamma'$ lies in one of the straight lines AG, BG, CG; thus, if $\beta' = \gamma'$, the equation of the quartic admits of the factor α; the quartic therefore in this case reduces to the line BC and the cubic

$$8\alpha'\beta\gamma\{2\beta'\alpha+(\alpha'+\beta')\beta+(\alpha'+\beta')\beta\} = \alpha\{(\alpha'+\beta')\beta-(\alpha'-\beta')\gamma\}^2.$$

The form of this cubic will change as the point O' moves along the line AG.

11. The following diagrams illustrate these results.* Let

$$S \equiv \beta\gamma\alpha'(\beta-\gamma)+\gamma\alpha\beta'(\gamma-\alpha)+\alpha\beta\gamma'(\alpha-\beta),$$

$$L \equiv \alpha(\beta'+\gamma')+\beta(\gamma'+\alpha')+\gamma(\alpha'+\beta').$$

A point P $[\alpha', \beta', \gamma']$ moves along a conic of the form S; it is required to trace the changes in the form of the quartic

$$S^2 = 8\alpha'\beta'\gamma'\alpha\beta\gamma L.$$

Take the triangle of reference as equilateral.

* The rest of the paper is due to Mr. W. H. Blythe; see § 6.

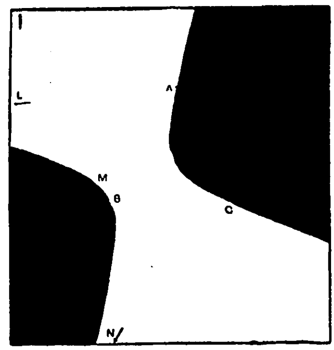


Figure 1 represents the locus of P, a hyperbola passing through A, B, C, the angular points of the triangle of reference, and through its centre of gravity.

This locus cuts the lines $\beta+\gamma=0$, $\gamma+\alpha=0$, $\alpha+\beta=0$ in the points L, M, N.

[The points L and N are too distant to be shown in the figure; their position is indicated by letters placed at the side of the diagram.]

In taking different positions of P on the locus given in Figure 1, we obtain all possible forms of the quartic; but the hyperbola in Figure 1 will in certain cases reduce to two straight lines, namely, a side of the triangle and its corresponding median, *i.e.*, $\alpha=0$ and $\beta=\gamma$. When P moves along $\alpha=0$, the quartic reduces to two coincident pairs of straight lines $\alpha=0$ and $\beta=\gamma$.

When P moves along the straight line $\beta=\gamma$, the quartic reduces to the straight line $\alpha=0$, together with a cubic having a node at A. The forms of these cubics will be given later on; see Figures 16 and 17.

It is shown in Salmon's *Higher Plane Curves* that each quartic represented by the above equation has a corresponding conic, such that, if α, β, γ be a point Q on the quartic, and x, y, z the corresponding point q on the conic, then

$$\alpha : \beta : \gamma :: yz : zx : xy.$$

If one of these points be given, the other can be found from the fact that QA and qA make equal angles with

$$\beta+\gamma=0,$$

QB and qB with $\qquad \gamma+\alpha=0,$

and QC and qC with $\qquad \alpha+\beta=0.$

The position of the corresponding conic can also be determined geometrically, for it touches the six straight lines

$$\beta+\gamma=0, \quad \gamma+\alpha=0, \quad \alpha+\beta=0,$$

$$\alpha\alpha'+\beta\beta'=0, \quad \beta\beta'+\gamma\gamma'=0, \quad \gamma\gamma'+\alpha\alpha'=0.$$

It is only necessary to know the straight line

$$\alpha\alpha'+\beta\beta'+\gamma\gamma' = 0.$$

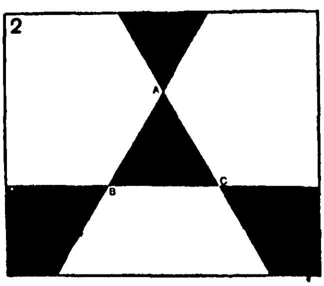
This line cuts the sides of the triangle ABC in the points at which

$$\alpha\alpha'+\beta\beta' = 0, \quad \beta\beta'+\gamma\gamma' = 0, \quad \gamma\gamma'+\alpha\alpha' = 0$$

meet them.

It will be found that the line at infinity corresponds to the circumscribing circle of ABC.

Figure 2.—When P is at the centre of gravity of the triangle, the corresponding conic is the circumscribing circle. The quartic is represented by the three sides of the triangle ABC and the line at infinity.



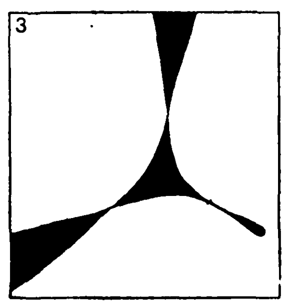


Figure 3.—Next, let P move to a point between the centre and the angle C.

The corresponding conic is an ellipse cutting all the sides of the triangle internally, and the circumscribing circle in four points, two on the arc CA and two on the arc CB. The quartic has therefore four asymptotes corresponding to the four points at which the conic meets the circle.

Two infinite branches pass through the angle B and two through A. Each pair have a corresponding hyperbolic branch having the same asymptotes.

These branches, which touch the lines

$$\alpha+\beta = 0, \quad \beta+\gamma = 0, \quad \text{and} \quad \gamma+\alpha = 0,$$

are too distant to appear in the figure, but are shown in Figures 4 and 5.

The part of the conic lying outside the triangle ABC between AB

and the circumscribing circle corresponds to a loop of the quartic outside the angle C.

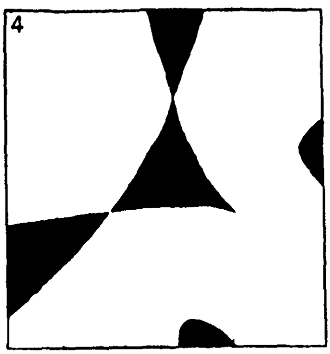
Since the conic touches

$$\alpha\alpha'+\beta\beta'=0, \quad \beta\beta'+\gamma\gamma'=0, \quad \gamma\gamma'+\alpha\alpha'=0,$$

the quartic touches

$$\alpha\beta'+\beta\alpha'=0, \quad \beta\gamma'+\gamma\beta'=0, \quad \gamma'\alpha+\alpha'\gamma=0.$$

Figure 4.—When P moves to the angle C, the corresponding conic, still an ellipse, touches the side AB. The loop of the quartic at C disappears, and we have a cusp at C.



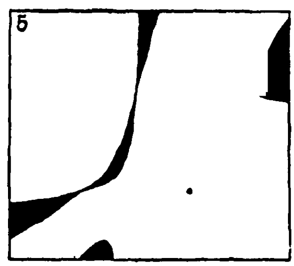
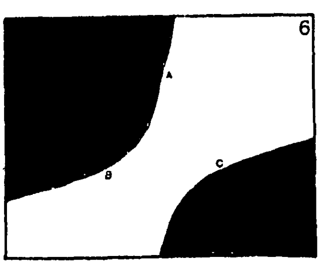


Figure 5.—As P continues to move in the same direction along its locus, the corresponding conic does not cut AB; therefore we find an acnode at C.

Figure 6.—When P coincides with L, the corresponding conic becomes two coincident straight lines; therefore the quartic reduces to two coincident conics.



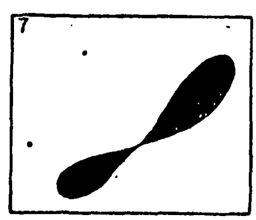
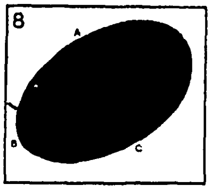


Figure 7.—As P continues to move in the same direction, the corresponding conic, now a hyperbola, does not cut BC, CA, or the circumscribing circle, but cuts AB externally. The quartic therefore has acnodes at A and B, and consists of two finite loops having a node at C, both external to the triangle.

Figure 8.—When P coincides with M, the corresponding conic becomes two coincident straight lines that do not cut the circumscribing circle. The quartic reduces to two coincident ellipses that circumscribe ABC.



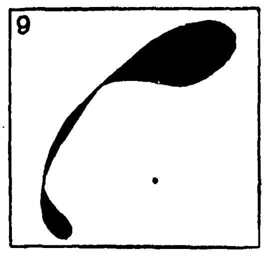


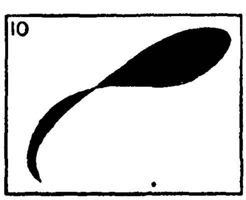
Figure 9.—When P lies between M and B, the corresponding conic may be either an ellipse or a hyperbola, for, when the locus of P cuts the circumscribing circle, the corresponding conic becomes a parabola.

This fact, however, does not affect the form of the quartic, for it only implies that it touches the circumscribing circle.

The corresponding conic cuts the sides AC and BC externally, but not the side AB.

The form of the quartic as shown by the figure is such that it has an acnode at C.

Figure 10.—When P coincides with B, the loop beyond B disappears, and is replaced by a cusp.



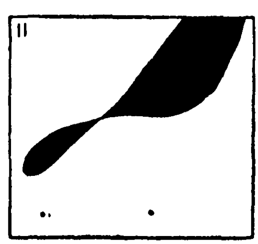
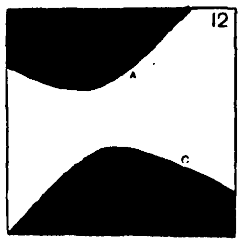
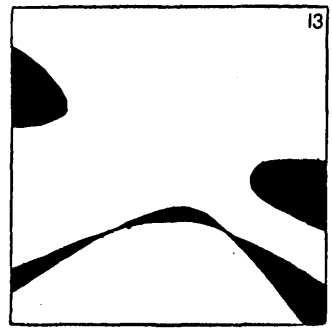
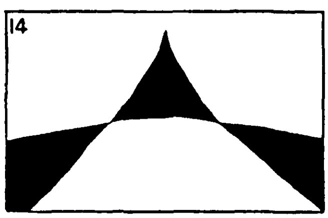


Figure 11.—When P continues to move towards N, the corresponding conic ceases to cut AC; we now find acnodes at B and C. The quartic assumes the form of Figure 7, one loop being too elongated to be entirely shown.

Figures 12, 13, 14, and 15 are similar in form to 6, 5, 4, and 3, if we interchange the angles A and C.







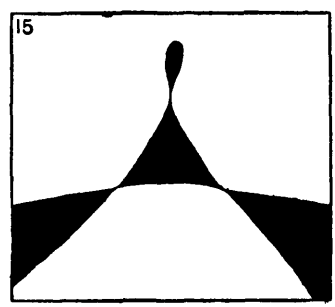
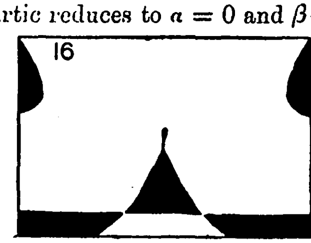


Figure 16.—Take the special case in which P moves along the line $\beta = \gamma$. At the centre of gravity, we again obtain Figure 3. Now let P move towards the angle A. The quartic reduces to the straight line $\alpha = 0$ and a cubic.

These together form a figure as given, analogous to Figure 15.

When P coincides with A, we again obtain a cusp.

When P moves up to the intersection of $\alpha+\beta = 0$ and $\alpha+\gamma = 0$, the quartic reduces to $\alpha = 0$ and $\beta+\gamma = 0$, each repeated twice.



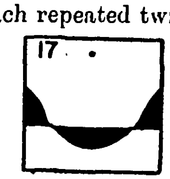


Figure 17.—We next get a figure somewhat similar in form to Figure 9.

The changes that follow can be traced from figures already given.