DEPENDENCE OF THE MODULUS OF TORSION ON TENSION.

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IT is to be expected from the theory of elasticity that the coefficients of elasticity of a body should vary as it is deformed. If X_x, Y_y, Z_z denote the normal components of stress on planes perpendicular respectively to the X, Y and Z axes of coördinates; Y_x, Z_x, X_y , the tangential components on the same planes; and $x_x, y_y, z_z, y_x, z_x, x_y$, the corresponding components of strain; then the relation between stress and strain can be written in the general form

$$\begin{aligned} X_x &= f_1(x_x, y_y, z_z, y_x, z_x, x_y) \\ Y_y &= f_2(x_x, y_y, z_z, y_x, z_x, x_y) \end{aligned} \quad (1)$$

with similar formulas for the other components of stress. f_1, f_2 denote unknown functions. If these functions are developed according to Maclaurin's formula, we have

$$\begin{aligned} X_x &= c_{11}x_x + c_{12}y_y + c_{13}z_z + c_{14}y_x + c_{15}z_x + c_{16}x_y + d_{11}x_x^2 + d_{12}y_y^2 \\ &\quad + \cdots + d_{17} \cdot 2x_x y_y + \cdots + e_{11}x_x^3 + \cdots \\ Y_y &= c_{21}x_x + c_{22}y_y + \cdots + d_{21}x_x^2 + \cdots \end{aligned} \quad (2)$$

with similar equations for Z_z, Y_x, Z_x , and X_y ; the c 's, d 's, e 's, etc., denote constants. The constant term $f(0)$, which is usually placed at the beginning of Maclaurin's series, is omitted, because it is assumed that the stress disappears when all the strains disappear. The strains are exceedingly small quantities in comparison with the stresses; it is therefore customary to neglect their squares and products, and to break off the series (2) after the sixth term. The resulting equations form the basis of the ordinary theory of elasticity; they make the strains proportional to the stresses, the coefficients remaining unchanged as the strains increase indefinitely.

If, however, the strains are so large or so accurately known that their squares cannot be neglected, then the series (2) must include the terms of the second degree. This gives

$$\begin{aligned} X_x = & (c_{11} + d_{11}x_x + d_{17}y_y + d_{18}z_z + d_{19}y_z + d_{110}z_x + d_{111}x_y)x_x \\ & + (c_{12} + d_{12}y_y + d_{17}x_x + d_{112}z_z + d_{113}y_z + d_{114}z_x + d_{115}x_y)y_y \quad (3) \\ & + (c_{13} + \dots)z_z + (\dots)y_z + (\dots)z_x + (\dots)x_y \end{aligned}$$

and similar expressions for Y_y , Z_z , Y_z , Z_x , X_y . It is evident that the proportionality between stress and strain no longer subsists. In place of the ordinary coefficients of elasticity the c 's of our formulas, we have the expressions in () in (3); the latter fill exactly the place of the c 's in the ordinary theory of elasticity, and can therefore be regarded as new coefficients of elasticity which, however, depend on the strain and therefore vary as the body is deformed.

The theory of the dependence of the coefficients of elasticity on the deformation in the case of a homogeneous, isotropic body, has been worked out in two forms. Voigt,¹ assuming that the strains can be neglected in comparison to the size of the body, deduced formulas which represent the behavior of a homogeneous, isotropic body by means of three new constants of elasticity in addition to the two of the usual theory. Jos. Finger,² independently of Voigt, investigated the same problem, but did not neglect the strain in comparison to the size of the body. He arrived at formulas which contain four new constants, besides the two of the old theory of elasticity.

Experiments have verified the theory, and shown that the coefficients of elasticity depend in a slight degree on the deformation, even far within the elastic limit, and that, therefore, the strain is not exactly proportional to the stress. This seems to have been known as early as 1705, when James Bernouilli³ wrote: "Des fibres homogènes de même longueur et largeur, mais chargées de différents poids, ne s'étendent ni se compriment pas proportionnellement à ces poids ;

¹ W. Voigt, Wiedemann's Annalen, 52, p. 536, 1894.

² Jos. Finger, Wiener Berichte, Band 103, Abth. IIa, 163, 1894.

³ J. Bernouilli, Gesammelte Werke, Band II., 976-989. Geneva, 1744. Quoted in Todhunter and Pearson, History of the Theory of Elasticity and of the Strength of Materials. Cambridge, 1886.

mais l'extension ou la compression causée par le plus grand poids, est à l'extension ou la compression causée par le plus petit, en moindre raison que ce poids-là n'est à celui-ci." During the present century numerous experiments on the elasticity of extension, especially those of Eaton Hodgkinson¹ on structural iron and steel, and those of J. O. Thomson² on wires of various metals, have shown that for most substances the increase of length is not exactly proportional to the stretching force.

On the other hand, Voigt's careful experiments on the torsion and flexure of crystals failed to show any deviation from proportionality, although he carried the deformations almost to the breaking point. This result, however, follows from Voigt's development of the theory of the dependence of elastic constants on deformation; according to which the deviation from proportionality appears in a term of the first order in the case of extension, but in a term of the second order in the cases of torsion or flexure.

The modulus of torsion, then, according to theory, does not vary appreciably with the torsion; it does vary appreciably, however, with the extension in the direction of the axis of torsion. The theory enables us to express the modulus of torsion in terms of the tension in the direction of the axis. If m denote the modulus, P the tension, and α , β , γ be constants, then

$$m = \alpha + \beta P + \gamma P^2. \quad (4)$$

This formula follows from the theory, both under Voigt's assumption that the strains can be neglected with respect to the dimensions of the body, and under Finger's assumption that they cannot; only the constants have a somewhat different meaning in the two cases. In both cases α is the ordinary modulus of torsion, when no tension is present; β and γ are rather complicated functions of the five (according to Voigt) or six (according to Finger) elastic constants of the substance in question. The mathematical details of the deduction of the above formula may be left out here.³

¹ E. Hodgkinson, Report of the Commission appointed to enquire into the Application of Iron to Railway Structures. London, 1849, pp. 47-67.

² J. O. Thomson, Wiedemann's Annalen, 44, p. 555, 1891.

³ The mathematical deduction will be found in the Göttingen Dissertation—"Abhängigkeit der spezifischen Torsionswiderstandes einiger Metalldrähte von der Spannung." J. R. Benton, 1900.

The results of the experiments described below were found to agree well with this formula.

The experimental investigation of the changes of the modulus of torsion with the tension was suggested by Lord Kelvin¹ in 1878, but no experiments were made until those of Cantone and Michelucci² in 1897. They used wires of iron, copper, brass, nickel, and silver, which hung vertically, and were twisted by means of horizontal pulleys at the bottom of the wires. The wires were repeatedly loaded and unloaded, and the change of the angle of torsion at each change of load observed. Experiments were made with different values for the original angle of torsion. It was found that an increase of tension was accompanied by an increase of the angle of torsion, therefore by a decrease of the modulus of torsion. Nickel, however, behaved exactly oppositely to the other wires. Cantone and Michelucci did not attempt to formulate a law for the variation of the modulus of torsion; and they seem to have left out of account the influence of the change of shape of the wire when loaded; the fact of its becoming longer and thinner would change its amount of torsion, even if the modulus remained the same.

It was suggested to me by Professor Voigt to investigate the subject more exhaustively, using more exact methods. The experiments described here were made in his laboratory in the University of Göttingen.

The method used to determine the modulus of torsion under the various tensions consisted in comparing, by means of the method of coincidences, the period of torsional vibration of a disk fastened to the wire, with the period of a pendulum swinging in front of the disk.

Fig. 1.

¹ Lord Kelvin, Article "Elasticity" in the Encyclopedia Britannica.

² M. Cantone and E. Michelucci, Rendiconti della reale Accademia dei Lincei (5), 6, pp. 191-198, 1897.

A wire about three meters long was fastened firmly to the ceiling of the room by means of the clamp *a* (Fig. 1). The lower end of the wire was clamped in the end of the arm *b*, about 8 cm. long, which was free to turn about a horizontal axis *b'*. In this way the extension of the wire was not interfered with, while sidewise motion of the lower end was prevented, and the torsion was limited to the space between *a* and *b*. Exactly in the middle of this space a brass disk *c*, 18 cm. in diameter, was fastened. For wires of different thicknesses I used disks of different weights, ranging from 0.7 kg. to 2.2 kg.

The disk made torsional vibrations. The tension of the wire was produced by means of weights hung on below the arm *b*. I took special pains to see that the points at which the wire was clamped in *a* and *b* did not change when the wire contracted as a result of the tension.

On the edge of the disk *c* a little piece of sheet iron *d*, about 1 cm. high, and bent to a right angle, was set up vertically.

Fig. 2.

The two sides of the angle made equal angles with the radius of the disk, so that the resistance of the air was the same when the disk swung in one direction as when it swung in the other; one side was painted white so as to be distinctly visible, the other side was blackened. The white side was strongly illuminated (when the disk was in its position of rest) by the light of a gas burner concentrated by means of a lens. The whole angle *d* was movable along the edge of the disk, and could be set where it was wanted.

In front of the disk hung a pendulum, consisting of a lens-shaped piece of lead hung on a fine steel wire. At the same height as the angle *d*, this fine wire carried a little black cylinder *f*, of such a diameter that it just covered the white side of the angle *d*, when looked at in the horizontal line passing through the pendulum wire and the wire to be investigated. Between the disk and the pendulum wire was a black screen *g*, with a rectangular hole in it of just the same breadth as the diameter of the cylinder. When the pendulum and disk were at rest, the centers of the wire, the white half of the angle *d*, the hole in the screen, and the black cylinder

were all in the same straight line. A telescope was set up in that line at a distance of about three meters from the wire, so that the vibrations of the pendulum and disk could be observed from a distance.

The length of the pendulum was so adjusted that its period T was nearly equal to that of the torsional vibrations of the disk (or else $2T$ or $3T$ were made nearly equal to the period of the disk, in cases where it swung more slowly). The coincidences were observed through the telescope; a coincidence taking place when the black cylinder f passed the hole in the screen g at the same time (and in the same direction) as the illuminated side of the angle d , and hid the latter from view. A perfect coincidence rarely took place; the cylinder f usually left part of the bright angle uncovered; at one swing the bright part appeared on one side, at the next on the other side of the cylinder. It was possible, however, by watching a few swings before and after the coincidence, to estimate in tenths of a pendulum swing, the true time of the coincidence.

If θ is the interval between two successive coincidences, the period S of the disk is given by

$$S = \frac{\frac{\theta}{T}}{\frac{\theta}{T} \pm 1} \cdot T = \frac{\theta}{\theta \pm T} \cdot T \quad (5)$$

It will be seen that an error of the first order in T appears only in the second order in $\frac{\theta}{\theta \pm T}$, so that the period of the disk *in terms of* T can be determined very accurately when T is known only approximately. T was determined by measuring three times with a stop-watch (reading to 0.2 second) the time of 100 swings of the pendulum. The error of T may be taken as ± 0.002 second; that of $\frac{\theta}{\theta \pm T}$ is then $\mp \frac{0.002\theta}{(\theta \pm T)^2}$. If T remains the same for the different determinations of S under different tensions of the same wire, the values of S can be compared very accurately without the necessity of a very accurate determination of T .

The length of the pendulum was so adjusted that a coincidence occurred about every two and one-half minutes.

The pendulum lost its motion very slowly, and an amplitude of more than one and one-half degrees was never given to it, so that its period could be regarded as absolutely constant. The variations of its period during a series of observations must have amounted to less than a thousandth of one per cent. The period of the disk, on the contrary, varied considerably with the amplitude for most wires. The causes of this variation—resistance of the air, internal friction of the wire, adiabatic temperature changes due to the deformation, and a possible variation of the elasticity with the angle of torsion—were so complicated that it did not seem practicable to investigate them separately; the dependance of the period on the amplitude was therefore determined empirically. This was done by making a long series of observations as the amplitude died away. It was found that the time θ between two coincidences could be represented with very close approximation as a linear function of the amplitude A .

$$\theta = \theta_0 + bA. \quad (6)$$

This empirical result holds true up to about 50° , but the amplitudes in my experiments never exceeded 35° , and usually lay between 10° and 20° .

Thermometers were placed at the top, the middle, and the bottom of the wire, and the temperature was measured by them, and kept constant within about 2° by opening or closing a door which led to a cooler room. Sudden currents of air were avoided.

Each wire, soon after it was hung from the ceiling, was heated twice to dull redness by means of a Bunsen burner, under the greatest safe load, in order to make it hang straight. After it had cooled, it was loaded to somewhat more than the greatest load to be used in the experiments, and left hanging free for one to two days. Then it was clamped at the bottom, and the observations were begun.

In order to determine the period of the disk when the wire was under any given tension, a series of ten coincidences was observed. For some wires, however, the vibration died down so rapidly that I had to be satisfied with seven coincidences; while for some others the amplitude remained so nearly constant that the observations had to be continued for 15 or 20 coincidences, in order to determine the effect of the amplitude on the period. At each coincidence the

time was determined by means of a stop-watch (the difference of the observed, approximate coincidence from the true coincidence being estimated), and the amplitude of the disk was observed by means of a scale of angles marked on its edge. To determine the period at the amplitude zero, equations of the form (6) were set up, and the constants θ_0 and b determined from them by the method of least squares. θ_0 is the time between two coincidences at the amplitude zero; from it the period S of the disk was calculated, in terms of the period T of the pendulum.

The modulus of torsion m is given by the equation

$$m = \frac{2\pi K}{S^2} \cdot \frac{l}{r^4} \quad (7)$$

where K is the moment of inertia of the disk and wire, l is the length, and r the radius of the wire. To use this formula an accurate knowledge of T and K is necessary. But inasmuch as I was investigating only the *changes* of the modulus of torsion, and not its absolute value, I simply compared m with m_s , the modulus of torsion under a standard load; T and K , of course, remained the same. This gives

$$\frac{m}{m_s} = \frac{S_s^2 l_s^4}{S^2 l^4}, \quad \text{or} \quad m = \left(\frac{S_s}{S}\right)^2 \cdot \left(\frac{l}{l_s}\right) \cdot \left(\frac{r_s}{r}\right)^4 \cdot m_s. \quad (8)$$

The changes of l and r are those due to the linear extension and the lateral contraction of the wire when loaded. $\frac{l}{l_s}$ was determined by direct observation. A needle was fastened to the lower end of the wire and the distance through which it moved as the wire was loaded was read by means of a microscope with micrometer screw-head. The quantity $\frac{r_s}{r}$ was also determined directly, the contraction of the wire being measured by means of an interference apparatus. For a detailed account of the method used, see Article : "Determination of Poisson's Ratio by Means of an Interference Apparatus," PHYS. REV. $\frac{l}{l_s}$ and $\frac{r_s}{r}$ were very slightly different from unity, so that it was possible to assume Hooke's law, and regard the deformation as proportional to the tension.

Each wire was observed under five different tensions ; three series of observations were made ; first one beginning with the smallest tension, and increasing the tension continuously to the greatest ; then one beginning with the greatest and decreasing continuously to the smallest ; finally one beginning with the smallest again. Thus fifteen determinations were made with each wire, each determination consisting in the observation of a series of coincidences.

The results obtained are given in the tables below. The first column gives the substance of the wire, the second column its diameter in millimeters, the third the loads in kilograms, the fourth the tensions produced by them in kilograms per square millimeter ; the fifth and sixth columns give the modulus of torsion in terms of the modulus of torsion under (an assumed) standard tension, the fifth as observed and the sixth as calculated from the formula (4). The seventh column gives the approximate absolute value of the modulus of torsion.

Substance.	Diam-eter.	Load.	Ten-sion.	Modulus of Torsion.		
				Observed.	Calculat'd.	Absolute value.
Steel	0.685	2	5.43	1.00364	1.00362	7.523×10^{11}
		5	13.57	1.00219	1.00243	
		10	27.14	1.00000	0.99979	
		15	40.71	0.99608	0.99633	
		18	48.85	0.99386	0.99386	
Steel.....	1.000	10	12.73	1.00242	1.00247	8.077×10^{11}
		15	19.10	1.00120	1.00119	
		20	25.46	1.00000	0.99997	
		24.7	31.45	0.99897	0.99886	
		29.4	37.43	0.99772	0.99780	
Steel.....	1.405	10	6.45	1.00134	1.00142	8.030×10^{11}
		15	9.68	1.00059	1.00059	
		20	12.91	1.00000	0.99993	
		24.7	15.95	0.99949	0.99940	
		29.4	18.98	0.99894	0.99902	
Iron	0.883	3	4.89	1.00173	1.00213	5.671×10^{11}
		6	9.84	1.00185	1.00131	
		9	14.68	1.00000	0.99983	
		12	19.57	0.99771	0.99820	
		15	24.47	0.99600	0.99589	

Substance.	Diam-eter.	Load.	Ten-sion.	Modulus of Torsion.		
				Observed.	Calculat'd.	Absolute value.
Iron	1.190	5	4.50	1.00126	1.00126	6.372×10^{11}
		10	9.00	1.00058	1.00065	
		15	13.49	1.00000	0.99998	
		20	17.99	0.99932	0.99928	
		25	22.49	0.99859	0.99856	
Iron	1.578	10	5.00	1.00088	1.00090	6.732×10^{11}
		15	7.51	1.00063	1.00057	
		20	10.01	1.00000	1.00026	
		25	12.51	1.00042	0.99996	
		30	15.02	0.99963	0.99971	
Copper.....	0.947	2	2.84	1.00490	1.00501	7.057×10^{11}
		4	5.68	1.00400	1.00340	
		6	8.51	1.00000	1.00060	
		8	11.35	0.99659	0.99651	
		10	14.19	0.99058	0.99072	
Copper.....	1.390	2	1.32	0.99997	1.00000	6.139×10^{11}
		5	3.30	1.00027	1.00030	
		10	66.59	1.00000	0.99986	
		15	9.89	0.99803	0.99824	
		18	11.86	0.99680	0.99669	
Copper.....	1.728	10	4.24	1.00252	1.00122	5.342×10^{11}
		15	6.37	1.00077	1.00120	
		20	8.49	1.00000	1.00050	
		25	10.61	0.99753	0.99891	
		30	12.73	0.99760	0.99651	
Nickel	0.818	2	3.72	0.99581	0.99504	8.952×10^{11}
		5	9.29	0.99585	0.99610	
		10	18.58	1.00000	1.00000	
		15	27.87	1.00735	1.00644	
		18	33.44	1.01054	1.01130	
Nickel	0.963	2	2.74	0.99513	0.99489	9.026×10^{11}
		5	6.86	0.99649	0.99685	
		10	13.73	1.00000	1.00004	
		15	20.59	1.00352	1.00309	
		18	24.71	1.00463	1.00487	
Nickel	1.490	10	5.73	0.99734	0.99688	5.518×10^{11}
		15	8.60	0.99822	0.99862	
		20	11.47	1.00000	1.00014	
		25	14.34	1.00134	1.00147	
		30	17.20	1.00278	1.00258	
Brass	0.787	2	4.11	1.00384	1.00417	3.688×10^{11}
		5	10.28	1.00169	1.00173	
		8	16.44	1.00000	0.99917	
		11	22.61	0.99573	0.99641	
		14	28.77	0.99353	0.99343	

Substance.	Diam-eter.	Load.	Ten-sion.	Modulus of Torsion.		
				Observed.	Calculated.	Absolute value.
Brass	0.943	2	2.87	1.00110	1.00111	3.855×10^{11}
		4	5.74	0.99980	1.00050	
		6	8.60	1.00000	0.99993	
		8	11.47	0.99948	0.99933	
		10	14.34	0.99852	0.99870	
Brass	1.498	7	3.97	1.00057	1.00058	3.804×10^{11}
		10	5.67	1.00037	1.00038	
		15	8.51	1.00000	0.99999	
		20	11.35	0.99951	0.99953	
		22.8	12.94	0.99926	0.99925	
Nickelin	0.795	2	4.03	1.00946	1.00943	5.921×10^{11}
		5	10.07	1.00559	1.00571	
		10	20.14	1.00000	0.99987	
		15	30.21	0.99444	0.99449	
		18	36.24	0.99146	0.99147	
Nickelin	0.978	2	2.66	1.00410	1.00426	6.166×10^{11}
		5	6.65	1.00300	1.00270	
		10	13.31	1.00000	0.99998	
		15	19.96	0.99677	0.99711	
		18	23.96	0.99554	0.99531	
Nickelin	1.397	10	6.52	1.00144	1.00140	6.573×10^{11}
		15	9.78	1.00063	1.00071	
		20	13.05	1.00000	1.00011	
		25	16.31	0.99989	0.99961	
		30	19.57	0.99908	0.99921	
German silver	1.005	2	2.53	1.00417	1.00417	7.451×10^{11}
		5	6.33	1.00271	1.00270	
		10	12.67	1.00000	0.99999	
		15	19.00	0.99690	0.99694	
		18	22.81	0.99500	0.99496	
German silver	1.592	10	5.02	1.000526	1.000522	6.584×10^{11}
		15	7.53	1.000232	1.000245	
		20	10.05	1.000000	0.999986	
		25	12.56	0.999736	0.999742	
		30	15.07	0.999516	0.999515	
Brass with large per-centage of copper.	0.996	2	2.57	1.00318	1.00320	3.804×10^{11}
		5	6.42	1.00280	1.00310	
		10	12.84	1.00000	0.99943	
		15	19.26	0.99126	0.99140	
		18	23.11	0.98445	0.98460	
Bronze	0.902	2	2.82	1.00451	1.00437	9.077×10^{11}
		4	5.65	1.00158	1.00196	
		6	8.47	1.00000	0.99972	
		8	11.30	0.99762	0.99762	
		10	14.12	0.99565	0.99570	

From the above observations the following values of β and γ in the formula $m = a + \beta P + \gamma P^2$ result. 1 kg. per square mm. is taken as the unit of tension.

Substance.	Diameter of Wire.	Modulus of Torsion.	
		Observed.	Calculated.
Steel	0.685	-0.000104	-0.00000222
	1.000	-0.000218	+0.00000061
	1.405	-0.000367	+0.00000698
Iron	0.883	+0.000011	-0.0000105
	1.190	-0.000127	-0.0000007
	1.595	-0.000149	+0.0000013
Copper.	0.947	+0.000168	-0.0000843
	1.390	+0.000406	-0.0000545
	1.792	+0.001042	-0.0000895
Nickel	0.818	+0.000022	+0.0000135
	0.963	+0.000493	-0.0000013
	1.490	+0.000788	-0.0000126
Brass.....	0.787	-0.000339	-0.00000275
	0.943	-0.000360	-0.00000099
	1.498	-0.000082	-0.00000039
Nickelin.....	0.795	-0.000639	+0.00000220
	0.978	-0.000371	-0.00000175
	1.397	-0.000285	+0.00000449
German silver	1.005	-0.000305	-0.00000407
	1.592	-0.000126	+0.00000131
Brass with large percentage of copper.	0.996	+0.000439	-0.0000524
Bronze.	0.902	-0.001961	+0.0000397

These results differ for wires of different thickness of the same material. This must be due to the inhomogeneity of the surface, which would have a greater effect on thin wires than on thick ones. It is well known that working the surface of metals has a marked effect on their elasticity. The drawing of a wire must affect its surface; besides which the surface absorbs gases and vapors, and it is the seat of chemical action, the products of which it is not easy to entirely remove. The wires of different sizes give quite different values for the modulus of torsion, so that it is not surprising that the values of β and γ also vary.

The values found then, cannot be taken as universal constants for the substance in question, but must be regarded simply as constants which give the behavior of particular wires. As the thickness of the wires is increased, β and γ approach values which must be the universal ones for the substance in question. So that by a consideration of the change of β and γ with increasing diameter of the wire, it is possible to obtain a certain amount of knowledge about them. They may be either positive or negative; β probably never exceeds $0.001 m_s$ and is generally much less, γ is less than $0.00005 m_s$. The algebraic sign of β and γ for the different substances is as follows :

Steel.	$\beta -$	$\gamma +$
Iron.	$-$	$+$
Copper.	$+$	$-$
Nickel.	$+$	$-$
Brass.	$?$	$?$
Nickelin.	$-$	$+$
German silver.	$-$	$+$

It is not possible to investigate the dependence of the modulus of torsion on tension outside of rather narrow limits. The tension can not exceed the limit of elasticity on the one hand; on the other hand it is not practicable to carry on the investigation with negative tensions, as this would necessitate the use of short thick bars, where a uniform distribution of the stress over the end surface is almost unattainable. Between the tension zero and the elastic limit, the present investigation shows that the law

$$m = a + \beta P + \gamma P^2$$

holds true within the limits of error of the experiments, and that the different substances used behave in the following way: Both steel and iron have negative β and positive γ , which, however, is so small that β alone determines their behavior. Their modulus of torsion decreases therefore, as the tension increases. Copper, and brass with a large percentage of copper, have positive β and negative γ ; the latter is large enough to determine their behavior for moderate tensions. The modulus of torsion first increases a little, then decreases quickly. The modulus of torsion reaches a maximum when the tension is about 6 kg. per mm². Nickel has positive β and very small negative γ . Its modulus of torsion increases as the ten-

sion increases, as Cantone and Michelucci¹ also found. Nickel is the only one of the substances I investigated, which has this property up to the limit of elasticity. For brass both β and γ are very nearly zero, so that the modulus of torsion varies only slightly with the tension. Nickelin has negative β and very small positive γ ; its modulus of torsion decreases with increasing tension. For German silver β is negative, γ positive and small; its modulus of torsion also decreases with the tension. For bronze I have not enough observations to draw conclusions about the universal constants.

To sum up the results of this investigation we may say:

1. That a dependence of torsional elasticity on tension, which is to be expected from the theory, actually occurs.
2. That this dependence can be expressed, within the limits of error of the observations, by the law $m = a + \beta P + \gamma P^2$, where m is the modulus of torsion, P the tension, and a , β , γ are constants.
3. That on account of the inhomogeneity of the surface of the wires, no conclusion about the constants for the substance in question can be drawn from observations on a single wire; but the values of β and γ obtained from such observations represent merely the behavior of the individual wire.
4. That in spite of this, observations of a series of wires of the same substance and different diameters make it possible to draw conclusions about the universal constants for the substance. It is found accordingly that the modulus of torsion of steel, iron, brass (when it does not contain too much copper), nickelin and German silver decreases with increasing tension; that that of copper, and brass containing much copper, at first increases and then decreases; that that of nickel increases continuously. This holds between the tension zero and the limit of elasticity.

¹Rendiconti della reale Accademia dei Lincei (6), pp. 191-198, 1897.