hence the tangents at A' and O' will also meet on OB' produced; that is to say, $A'B' = B'O'$.

Hence, with S (or S') as centre, and the corresponding polar as axis of perspective, the harmonic polygon $ABC\dots$ projects harmonically into a regular polygon $A'B'O'\dots$; which is the same as saying that the vertices of the one invert into the vertices of the other, with either S or S' for centre of inversion.

The connexion between a harmonic and a regular polygon can also be seen by reciprocating, with respect to k , the ellipse Σ of Fig. 2 into a circle, in which case $p_1 p_2 p_3 \dots$ reciprocates into a regular polygon.

[To the memoirs cited in Art. 1 must now be added another by MM. Tarry and Neuberg, published by the *Association française pour l'avancement des sciences* (*Congrès de Nancy, séance du 13 août, 1886*), entitled "Sur les polygones et les polyèdres harmoniques," and which contains interesting and ingenious extensions of some of the foregoing properties to the case of solid figures, by the method of harmonic projection. It may here be mentioned that, although the *géométrie brocardienne* of the triangle was only rediscovered and amplified in England, having previously originated on the continent (see the brief historical sketch on pp. 180–184 of "Companion to Weekly Problem Papers"), all the fundamental properties of harmonic polygons are of British origin. As to the present paper, the theorems of Arts. 2, 4, 5, 6, 7, 14, 15 (last two paragraphs), 16, 19, 24, 26, are new; and, of the remaining propositions and formulæ, many had been previously assumed by extension from the triangle, without rigorous demonstration.]

On Relations between Circles and Algebraic Curves, with Applications to Dynamics. By Prof. R. W. GENESE, M.A.

[Read April 7th, 1887.]

1. Let $u_n + u_{n-1} + u_{n-2} + \&c. = 0 \dots\dots\dots (1)$

be the equation to a curve of the n^{th} degree, where u_n , as usual, denotes a homogeneous expression of r dimensions in x, y .

Let $P(x, y)$ be one of the points of intersection of (1) with a circle of radius ρ about the origin O ; θ the inclination of OP to the axis of x . Also, let

$$\lambda = \rho (\cos \theta + i \sin \theta),$$

then
$$\frac{\rho^3}{\lambda} = \rho (\cos \theta - i \sin \theta),$$

$$\left. \begin{aligned} \frac{1}{2} \left(\lambda + \frac{\rho^3}{\lambda} \right) &= \rho \cos \theta = x \\ \frac{1}{2i} \left(\lambda - \frac{\rho^3}{\lambda} \right) &= \rho \sin \theta = y \end{aligned} \right\} \dots\dots\dots(2).$$

Substituting these values of x, y in (1), we shall obtain, *in general*, an equation of degree $2n$ in λ .

The sum of the roots will be $\Sigma \rho \cos \theta + i \Sigma \rho \sin \theta$, *i.e.*, the real part will be the sum of the $2n$ abscissæ of intersections, the imaginary part the sum of the ordinates.

The product of the roots will be, by De Moivre's theorem,

$$\rho^{2n} (\cos \Sigma \theta + i \sin \Sigma \theta).$$

Now, let

$$\begin{aligned} u_n &= ax^n + bx^{n-1}y + cx^{n-2}y^2 + \dots, \\ u_{n-1} &= a_1x^{n-1} + b_1x^{n-2}y + \dots, \\ &\&ampc. \end{aligned}$$

Then, using (2), and multiplying by λ^n , we get

$$\begin{aligned} &\frac{1}{2^n} \left\{ a (\lambda^3 + \rho^3)^n + \frac{b}{i} (\lambda^3 + \rho^3)^{n-1} (\lambda^3 - \rho^3) + \dots \right\} \\ &+ \frac{\lambda}{2^{n-1}} \left\{ a_1 (\lambda^3 + \rho^3)^{n-1} + \frac{b_1}{i} (\lambda^3 + \rho^3)^{n-2} (\lambda^3 - \rho^3) + \dots \right\} \\ &+ \&ampc. = 0 \dots\dots\dots(3). \end{aligned}$$

Whence,
$$\text{sum of roots} = -2 \frac{a_1 + b_1 i^{-1} + c_1 i^{-2} + \dots}{a + b i^{-1} + c i^{-2} + \dots} \dots\dots\dots(4),$$

and
$$\text{product of roots} = \rho^{2n} \frac{a - b i^{-1} + c i^{-2} - \&ampc.}{a + b i^{-1} + c i^{-2} + \&ampc.} \dots\dots\dots(5).$$

2. From (4) we infer that

"The centre of mean position of the $2n$ points of intersection of any algebraic curve with a circle is independent of the radius of the circle if its centre be given."(A).

This theorem (which admits of simpler proof) was given by M. d'Ocagne in the *Nouvelles Annales de Mathématiques* for 1886, page 295; and a more general theorem by Liouville, in the *Journal de Mathématiques pures et appliquées*, 1841.

3. From (5) it is clear that $\cos \Sigma \theta$ and $\sin \Sigma \theta$ depend only on the
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coefficients of u_n , which do not change with change of origin, and therefore that

The sum of the inclinations to a given axis of the radii vectores common to a given algebraic curve and a circle is independent both of the radius of the circle and of the position of its centre.(B).

4. Since, in general, u_n and u_{n-1} determine the asymptotes of (1), we see that the point in theorem (A), and the sum of angles in (B), may be obtained by using the asymptotes in place of the curve (1). Both theorems are obvious geometrically in the case in which (1) degenerates into straight lines. With respect to (B), it may be noted that, though pairs of lines given by $u_n = 0$ may be imaginary, yet the bisectors of the angles between them are real.

5. Let the centre of mean position for circles of centre O be denoted by $G(\bar{x}, \bar{y})$. Then there is a simple connection between O and G which may be thus investigated.

Let the origin be moved to the point $O'(\alpha, \beta)$; then u_n is unaltered and u_{n-1} is changed by a linear function of α, β , and the constants a, b, c , &c. Thus, if $(\bar{x}', \bar{y}')$ be the new position of G referred to the old axes,

$$\left. \begin{aligned} \bar{x}' &= \alpha + \bar{x} + p\alpha + q\beta \\ \bar{y}' &= \beta + \bar{y} + r\alpha + s\beta \end{aligned} \right\} \dots\dots\dots(6)$$

where p, q, r, s are constants.

Therefore α, β are expressible linearly in terms of $\bar{x}', \bar{y}'$, and the locus of G is of the same degree as the locus of O .

6. Now, let the radius ρ receive a small increment $d\rho$, and in consequence let the point of intersection P move to P' . By (B), $\Sigma\theta$ is constant, therefore

$$\Sigma d\theta = 0 \dots\dots\dots(7).$$

Also ρ and $d\rho$ are the same for all the points P , therefore

$$\Sigma \frac{\rho d\theta}{d\rho} = 0,$$

i.e.,

$$\Sigma \tan \phi = 0 \dots\dots\dots(8),$$

where ϕ is the angle the tangent to the given curve at P makes with ρ .

hence
$$\sum \frac{\sin OGP}{\sin OPQ} = 0,$$

i.e.,
$$\sum \frac{OP}{OQ} = 0,$$

or, since OP is the same for all,

$$\sum \frac{1}{OQ} = 0 \dots\dots\dots(11).$$

It is worthy of remark that the axis of x is any line whatever through O . The polygon formed by the normals at the $2n$ points P has then the property that the polar line of O with respect to it is the line at infinity. Thus if the equation to the sides be

$$(1-a_1x-b_1y)(1-a_2x-b_2y) \dots = 0,$$

we have
$$\sum a = 0, \text{ and } \sum b = 0 \dots\dots\dots(12).$$

This property has been noticed by Liouville.

8. The ordinary method of reciprocation applied to the preceding theorem, yields interesting results. Thus we have, from (B),

If common tangents be drawn to a curve of any class and a variable circle of given centre, the sum of their inclinations to any fixed axis is constant.(C).

Care must be taken to consider the direction of a tangent as that leaving the circle in the positive sense of rotation round the centre. It is clear that the constant is twice the sum of the inclinations of the tangents from the concentric point-circle.

9. From (8),

The sum of the reciprocals of the common tangents to a curve and a circle, with due regard to sign, is zero.(D),

A few words of explanation may be added, viz., the corresponding ϕ 's of reciprocal curves are equal, and $\tan \phi = \frac{p}{t}$, where p is the radius of the circle and t the common tangent. Or, the theorem may be deduced independently of (8); thus, if (p, α) be the polar coordinates of the point of contact of a common tangent with the circle, by (C),

$$\sum \alpha = \text{constant},$$

therefore
$$\sum da = 0,$$

therefore $\Sigma \frac{da}{dp} = 0,$

therefore $\Sigma \frac{1}{t} = 0.*$

From (10)... If θ be the vectorial angle of the point of contact of a common tangent with the curve,

$$\Sigma \sin \theta \sec \phi = 0 \dots\dots\dots(13).$$

From (11)... If OT be the polar subtangent of the point of contact of a common tangent with the curve

$$\Sigma OT = 0 \dots\dots\dots(14).$$

10. The preceding theory holds, as stated, in general. If, however, the curve of n^{th} degree be circular, or the curve of n^{th} class have a focus at the centre of the variable circle, the theorems either fail or need modification. For,

1°. If the curve (1) be simply circular, then u_n contains the factor $x^2 + y^2$ or ρ^2 , and its terms become mixed in the equation for λ with those of u_{n-2} . In fact, if

$$u_n = (x^2 + y^2)(a'x^{n-2} + b'x^{n-3}y + \&c.),$$

the equation (3) is replaced by

$$\frac{1}{2^{n-1}} \{ a_1 (\lambda^2 + \rho^2)^{n-1} + b_1 (\lambda^2 + \rho^2)^{n-2} (\lambda^2 - \rho^2) + \&c. \} \\ + \frac{\lambda}{2^{n-2}} \{ (a'\rho^2 + a_2)(\lambda^2 + \rho^2)^{n-2} + \&c. \} + \&c. = 0.$$

The sum of the $2n-2$ values of λ now contains ρ^2 , and theorem (A) fails, theorem (B) remains true for concentric circles, but $\Sigma \theta$ varies with the centre of the system.

If, however, the given curve (1) have a point-circle asymptote at O , i.e., if u_{n-1} also be divisible by $x^2 + y^2$, then both theorems fail. In this case O is a quadruple focus.

2°. If the given curve be bicircular [i.e., if u_n contain $(x^2 + y^2)^2$ as a factor, and u_{n-1} contain $(x^2 + y^2)$], theorem (A) fails, and theorem (B) holds only for concentric circles.

The same conclusion holds if the curve have the circular points at

* The convention of signs may be explained thus: let $T'PT$ be a tangent to the circle, and consider PT positive and PT' negative. Now let the circle rotate in its plane round the centre. When PT touches the given curve we have a positive tangent, when PT' a negative.

infinity, I, J , as singular points of any degree of multiplicity, provided the line IJ itself do not touch the curve at I and J .

3°. If I, J be multiple points of the curve (1), and if the curve touch the line at infinity at I, J , then theorem (A) again obtains, and also theorem (B) in the modified form.

The conclusion seems to be that theorem (A) is not true for circular curves unless they touch the line at infinity; and that theorem (B) in its modified form for concentric circles is true in all cases except one, viz., when the curve is singly circular, and the centre of the cutting circle is a quadruple focus of the curve.

It follows that the reciprocal theorems (C), (D) [obtained by reciprocating (B) with respect to a circle, centre O] fail in one case only, viz., when the given curve is circular with a single focus at the centre of the variable circle.

Applications to Dynamics.

11. Theorem (A) states that, if equal weights be placed at the intersections of a curve of n^{th} degree and a circle of given centre O , the position of the weight centre G is independent of the radius of the circle. Applying the method of reciprocation (explained *Proceedings Lond. Math. Soc.*, Vol. xvii., p. 409), and using a circle of centre O , we learn that

If equal forces act along the $2n$ common tangents to a curve of the n^{th} class and a circle of fixed centre, the line of action of the resultant is independent of the radius of the circle. (E).

12. Again, theorem (A) may be re-stated,

If the radii vectores to the points of intersection above represent forces, their resultant is independent of the radius of the circle.

Now let P, P' (see Fig. Art. 6) be intersections for radii ρ and $\rho + d\rho$, then a series of forces PP' acting at O would be in equilibrium. They would also be in equilibrium if acting along PP' , &c., for

the sum of the moments round $O = 2$ sum of areas $POP' = 0$, by (9).

Also
$$\frac{PP'}{d\rho} = \sec \phi.$$

Hence the following theorem,

If along the tangents to a curve, at its intersections with a circle, forces act proportional to the secants of the angles made with the radii, they will be in equilibrium..... (F).

13. Reciprocating with respect to the centre of the circle, we find that

If, at the points of contact with a curve of the common tangents to the curve and any circle, weights act inversely proportional to the tangents, with their proper sign, they will balance.(G).

14. Again, returning to the figure of Art. 6, theorem (F) shows that a set of forces of the type PT balance.

Therefore forces represented by OT , &c. have the same resultant as forces represented by OP , &c., i.e., a fixed resultant ($2n OG$).

Therefore $\Sigma OT \sin(\theta + 90^\circ)$ is constant,
or, $\Sigma \rho \tan \phi \cos \theta$ „ ;
so $\Sigma \rho \tan \phi \sin \theta$ „ ;
and $\Sigma \tan \phi = 0$.

Hence

Weights acting at the intersections of a curve and circle proportional to the polar subtangents to the curve (the centre of the circle being pole) are equivalent to a couple.(H).

Reciprocally,

If forces act along the common tangents to a curve and a circle inversely proportional to their lengths, their resultant passes through the centre of the circle.(I).

(I) may be verified by taking moments round the centre, and using (D).

15. The point O and the polygon of Art. 7 are thus related. If forces act along the perpendiculars from O on the sides, and inversely proportional to the lengths of the perpendiculars, they are in equilibrium.

For, if $ax + by = 1$ be expressed in the form

$$x \cos \alpha + y \sin \alpha = p,$$

$$a = \frac{\cos \alpha}{p},$$

$$b = \frac{\sin \alpha}{p},$$

therefore, by (12), $\Sigma \frac{1}{p} \cos \alpha$ and $\Sigma \frac{1}{p} \sin \alpha$
both vanish.

Note on the Interpretation of Formula (4) in the preceding paper.

Consider n straight lines through the origin, making angles $\theta_1, \theta_2, \dots, \theta_n$ with the axis of x ; their equation is

$$(x \sin \theta_1 - y \cos \theta_1)(x \sin \theta_2 - y \cos \theta_2) \dots (x \sin \theta_n - y \cos \theta_n) = 0.$$

In the sinister, put $y = ix$; where $i = \sqrt{-1}$, it becomes

$$x^n (\sin \theta_1 - i \cos \theta_1)(\sin \theta_2 - i \cos \theta_2) \dots,$$

$$\text{or} \quad (-ix)^n (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \dots,$$

$$\text{or} \quad (-ix)^n (\cos \Sigma \theta + i \sin \Sigma \theta).$$

Hence, if the result of writing $y = ix$ in the sinister of the equation to n straight lines be $P + iQ$, and

1°. if n be even, then

$$\frac{Q}{P} = \tan \Sigma \theta,$$

2°. if n be odd,

$$-\frac{P}{Q} = \tan \Sigma \theta;$$

$$\text{whence, in any case,} \quad \frac{Q}{P} = \tan \left(\Sigma \theta + \frac{n\pi}{2} \right).$$

Thus, in the case of two straight lines

$$ax^2 + 2hxy + by^2 = 0,$$

$$\text{we have, at once,} \quad \tan (\theta_1 + \theta_2) = \frac{2h}{a-b}.$$

$$\text{Again,} \quad \frac{\cos \theta + i \sin \theta}{\cos \phi + i \sin \phi} = \cos (\theta - \phi) + i \sin (\theta - \phi).$$

With this help we are enabled to construct the *direction* of the line OG of the preceding paper, viz., if θ_1, θ_2 , &c. be the inclinations to the axis of x of the asymptotes of the original curve, ϕ_1, ϕ_2 , &c. those

of the asymptotes of the first polar of O , γ that of OG ,

$$\begin{aligned}\cos \gamma + \epsilon \sin \gamma &= \frac{a_1 + b_1 \epsilon^{-1} + c_1 \epsilon^{-2} + \dots}{a + b \epsilon^{-1} + c \epsilon^{-2}} \text{ to a constant factor près} \\ &= \frac{a + b \epsilon + c \epsilon^2}{a_1 + b_1 \epsilon + c_1 \epsilon^2 + \dots} \quad " \quad " \\ &= \frac{(-\epsilon)^n (\cos \Sigma \theta + \epsilon \sin \Sigma \theta)}{(-\epsilon)^{n-1} (\cos \Sigma \phi + \epsilon \sin \Sigma \phi)} \quad " \quad " \\ &= -\epsilon \{ \cos (\Sigma \theta - \Sigma \phi) + \epsilon \sin (\Sigma \theta - \Sigma \phi) \},\end{aligned}$$

whence

$$\Sigma \theta - \Sigma \phi = \gamma + \frac{\pi}{2},$$

neglecting revolutions.

In the particular case of the conic, if we take the axis of x parallel to an axis of the conic, $\Sigma \theta$ is zero, $\Sigma \phi + \frac{\pi}{2}$ is the angle made by a perpendicular ON on the polar of O with the axis. Hence OG and the perpendicular on the polar of O are equally inclined to an axis.

Another form of the theory may be useful. Let

$$x \cos \alpha + y \sin \alpha - p = 0,$$

$$x \cos \beta + y \sin \beta - q = 0,$$

be two straight lines; the ratio of the perpendiculars upon them from a circular point at infinity ($x = \infty$, $y = ix$) is

$$\frac{\infty (\cos \alpha + \epsilon \sin \alpha) - p}{\infty (\cos \beta + \epsilon \sin \beta) - q} = \cos \overline{\alpha - \beta} + \epsilon \sin \overline{\alpha - \beta},$$

and determines the angle between the lines.

The introduction of constant multipliers will clearly not affect the ratio of the real and imaginary parts.

In biangular coordinates, for example, the coordinates of a circular point at infinity are simply $(\epsilon, -\epsilon)$; let

$$a\lambda + b\mu + c = 0, \quad a'\lambda + b'\mu + c' = 0$$

be two straight lines, the angle (δ) between them is determined by the

$$\text{ratio} \quad \frac{a\epsilon + b(-\epsilon) + c}{a'\epsilon + b'(-\epsilon) + c'} = \frac{\{(a-b)\epsilon + c\} \{\epsilon' - (a' - b')\epsilon\}}{c'^2 + (a' - b')^2},$$

therefore

$$\tan \delta = \frac{c'(a-b) - c(a'-b')}{cc' + (a-b)(a'-b')}.$$