PERPETUANT SYZYGIES

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THE methods used in the following discussion are entirely based on the symbolical notation, perpetuants linear in the coefficients of each quantic concerned being alone considered (perpetuant types). The symbolical forms of perpetuant types (which had previously all been identified by other methods) were calculated by Grace.* His result—"Any perpetuant linear in the coefficients of each of δ quantics, denoted by the letters $a_1, a_2, \dots, a_\delta$, can be expressed linearly in terms of products and of perpetuants $(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_\delta)^{\lambda_{\delta-1}}$, where $\lambda_1 \geq 2^{\delta-2}$, $\lambda_2 \geq 2^{\delta-3}$, ..., $\lambda_{\delta-1} \geq 1$, the order of the letters being fixed beforehand"—is fundamental to our present purpose and will for convenience be quoted as "The Perpetuant Type Theorem."

The difficulty of dealing with the actual syzygies is avoided as follows: all possible products of irreducible forms, for a given degree and weight, are arranged in a predetermined sequence, so that each individual product has a definite place in that sequence. Any syzygy may then be regarded as expressing that one of its products, which comes first in the predetermined sequence, in terms of products which come after it. The first product will be called reducible, by virtue of its being linearly expressible in terms of later products.

Thus, instead of actually finding syzygies, we seek to discover what products are reducible and what products are irreducible. Now all products of perpetuant types of degree δ can be expressed in terms of the forms

$$(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_\delta)^{\lambda_{\delta-1}}, \dots;$$

where the indices λ are only restricted to be positive integers or zeros, the order of the letters being fixed; such forms are all linearly independent, for, if there is a relation between them, there must be a purely algebraic relation between the symbolical products written down, and this is

* *Proc. London Math. Soc.*, Vol. xxiv,

easily* shown to be impossible. The generating function for all linearly independent perpetuant types and products of perpetuant types of degree δ is therefore $\frac{1}{(1-x)^{\delta-1}}$; it is known† that the generating function for actually irreducible forms of degree δ is $\frac{x^{2^{\delta-1}-1}}{(1-x)^{\delta-1}}$. So the generating function for irreducible products must be

$$\frac{1-x^{2^{\delta-1}-1}}{(1-x)^{\delta-1}}.$$

It is our object to identify these irreducible products: when this has been done for any degree, all the independent syzygies of this degree will have been identified, there being one such syzygy for each reducible product of irreducible forms. Now there cannot be fewer irreducible forms (types) and products of degree δ than the number enumerated by the generating function $\frac{1}{(1-x)^{\delta-1}}$; hence, when we have reduced all forms except this number, we shall have a proof of the irreducibility of the remaining products, as well as a new proof of the irreducibility of the forms already classified as perpetuant types.

Our results are only complete as far as degree 8: a large class of products has been discussed in general, and, though it cannot be asserted that the generating functions obtained for these products are exact, yet the fact that the particular cases of them are exact as far as degree 8 establishes a strong presumption that such is always the case.

The syzygies are all derived from those of degree 4 due to Stroh,‡ and the Jacobian identity

$$(bc) + (ca) + (ab) \equiv 0.$$

The syzygies of degree 4 have been completely discussed already§ by a different method, but we have discussed them here fully in illustration of the general methods employed in the present paper.

* For, if there is a relation, let $(a_1a_2)^{\lambda_1} \Sigma N(a_1a_3)^{\lambda_2} \dots (a_1a_t)^{\lambda_{t-1}}$ be those terms for which the index of (a_1a_2) is lowest; then we may divide out by $(a_1a_2)^{\lambda_1}$ and put $a_1 = a_2$ after division; we get $\Sigma N(a_1a_3)^{\lambda_2} \dots (a_1a_t)^{\lambda_{t-1}} = 0$. Proceeding in this way, we ultimately see that every coefficient N is zero.

† The irreducibility of these forms has been finally demonstrated by Wood, "On the Irreducibility of Perpetuant Types," *Proc. London Math. Soc.*, Vol. 1, Ser. 2.

‡ *Math. Ann.*, Bd. xxxiii., S. 61-107, § 18.

§ Wood, *Proc. London Math. Soc.*, Vol. 2, Ser. 2,

The above methods are apparently insufficient to obtain all the syzygies for the reduction of products of three forms each of degree 3, and here arises the difficulty in the treatment of products of degree 9.

It seems probable that the generating function for all irreducible products and types of degree δ , where the products have no factor of degree less than $(n+1)$, is

$$\frac{x^{\binom{\delta}{1} + \binom{\delta}{2} + \dots + \binom{\delta}{\kappa}}}{(1-x)^{\delta-1}}, \quad \kappa \leq \frac{\delta}{2};$$

and this result has been proved true for the cases $\kappa = 1, 2$, and $\kappa \geq \frac{\delta}{3}$. The question will be discussed at the end of Section VI.

The paper has been divided into sections as follows:—

CONTENTS.

Introduction.

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II. Syzygies of Degrees 3 and 4.

III. Definition of Reducibility. General Methods of Reduction by Differential Operators.

IV. Syzygies of Degree 5.

V. (i.) Products having no Factor of Degree less than 2. (ii.) Products having no Factor of Degree less than 3.

VI. Syzygies for the Reduction of the Product Forms:—(i.) C_x^2 . (ii.) $C_x C_m$ ($\kappa < m < 2\kappa$). (iii.) $C_x C_{2x}$.

VII. Syzygies of Degrees 6, 7, and 8. Note on Syzygies of Degree 9.

Previous Literature on Perpetuant Syzygies.

Previously published papers, of which none date since 1887, differ fundamentally from the present in three respects: (1) by the use of the literal notation and the theory of partitions; (2) by making the discovery of the syzygies a basis for the enumeration of the irreducible perpetuants; (3) by actual expression, apart from mere enumeration, of the syzygies.

The present paper starts from a knowledge of the symbolical form of the perpetuants and deduces a means of enumerating the syzygies; their actual expressions can be obtained from Section III. All the papers quoted are to be found in the *American Journal of Mathematics*.

Sylvester, Vol. v., "On Subinvariants."

This paper was the starting point of all investigations on perpetuants and treated of certain syzygies of degrees 5, 6, and 7: an error in degree 7 was subsequently corrected by *Hammond*.

Cayley, Vol. vii., "A Memoir on Seminvariants."

This deals with a method of suitably expressing certain syzygies of degrees 5 and 6.

Hammond, Vol. v., "On the Solution of the Differential Equation of Sources."

This paper obtains the results of Sylvester afresh, and corrects the error in the treatment of degree 7.

—— Vol. viii., "On Perpetuants, with Applications to the Theory of Finite Quantics."

This paper sketches a general method of classifying and expressing certain syzygies, but gives no definite result for any degree greater than 8.

MacMahon, Vol. x., "Expression of Syzygies among Perpetuants by means of Partitions."

The paper contains, *inter alia*, the actual expression of a number of sextic syzygies enumerated by the generating function

$$\frac{x^6 + x^9 + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17}}{(1-x^2)(1-x^4)(1-x^6)}.$$

Several sextic syzygies, however, remain unexpressed.

I. EXTENSION OF THE PERPETUANT TYPE THEOREM.

1. Consider any perpetuant

$$(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_k)^{\lambda_{k-1}} \equiv C_k.$$

We assume throughout that the letters are taken in a definite order fixed beforehand. If C_k appears as a factor of a product P , we shall find certain conditions, ultimately affecting the indices λ , for the "irreducibility" of P ; here we use the word "irreducibility" in a perfectly general sense, as implying that P cannot be expressed in terms of other forms obeying certain definite laws (*v. Note*, p. 225).

Notation.—The symbol $[a_{r_1} a_{r_2} \dots a_{r_l}]$ denotes any covariant, reducible or otherwise, involving the symbolical letters $a_{r_1}, a_{r_2}, \dots, a_{r_l}$; and the symbol $[a_{r_1} a_{r_2} \dots a_{r_l}]'$ denotes any covariant, reducible or otherwise, involving all the symbolical letters $a_1, a_2, \dots, a_k$ concerned except $a_{r_1}, a_{r_2}, \dots, a_{r_l}$.

If the transvectant $([a_{r_1} \dots a_{r_l}], [a_{r_1} \dots a_{r_l}]')^\lambda$ replace C_k in P , we may assume from our previous knowledge that, unless $\lambda \geq \mu_{r_1 r_2 \dots r_l}$, P is, according to some definition, reducible; here $\mu_{r_1 r_2 \dots r_l}$ is a quantity supposed known from other investigations: its value will depend on our definition of "reducibility."

We shall obtain all such transvectants of covariants involving some of the letters of C_k with covariants involving the remaining letters, if we suppose that $a_{r_1}, a_{r_2}, \dots, a_{r_l}$ are some or all of $a_2, a_3, \dots, a_k$, so that a_1 appears always in $[a_{r_1} a_{r_2} \dots a_{r_l}]'$.

To every choice of the letters $a_{r_1}, a_{r_2}, \dots, a_{r_l}$ corresponds a quantity $\mu_{r_1 r_2 \dots r_l}$, so that the number of such quantities $\mu_{r_1 r_2 \dots r_l}$ is $2^{k-1} - 1$; for

there are $\binom{\kappa-1}{l}$ quantics $\mu_{r_1 r_2 \dots r_l}$ with l suffixes, and l may have any value from 1 to $\kappa-1$.

The order of the suffixes may be conveniently fixed by the rule

$$r_1 < r_2 < r_3 \dots < r_l.$$

We shall use σ_r to denote the sum of all the μ 's whose first suffix is r , $\sigma_{r,s}$ to denote the sum of all the μ 's whose first two suffixes are r, s , and so on.

$$\text{Thus} \quad \sigma_\kappa = \mu_\kappa, \quad \sigma_{\kappa-1} = \mu_{\kappa-1} + \mu_{\kappa-1, \kappa},$$

and so on. We shall establish the following

THEOREM. — The conditions of irreducibility* of the product P are, as far as the factor $C_\kappa \equiv (a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_\kappa)^{\lambda_{\kappa-1}}$ is concerned,

$$\lambda_{\kappa-1} \geq \sigma_\kappa, \quad \lambda_{\kappa-2} \geq \sigma_{\kappa-1}, \quad \dots, \quad \lambda_1 \geq \sigma_2.$$

We have

$$C_\kappa \equiv ([a_\kappa]', [a_\kappa])^{\lambda_{\kappa-1}} + \Sigma \nu ([a_\kappa]', [a_\kappa])^{\lambda_{\kappa-1}-\epsilon},$$

and therefore by hypothesis C_κ is reducible unless $\lambda_{\kappa-1} \geq \mu_\kappa$.

$$\text{Put} \quad (a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_{\kappa-2})^{\lambda_{\kappa-2}} \equiv \alpha \equiv [a_{\kappa-1} a_\kappa]'. \quad \text{Put}$$

Then C_κ differs from $(\alpha a_{\kappa-1})^{\lambda_{\kappa-2}} (a a_\kappa)^{\lambda_{\kappa-1}}$ by terms which we have just defined as reducible.

If $\lambda_{\kappa-2} + \lambda_{\kappa-1}$ is less than either of $\mu_{\kappa-1}$ or $\mu_{\kappa-1, \kappa}$, then obviously C_κ is reducible: if $\lambda_{\kappa-2} + \lambda_{\kappa-1}$ is greater than each of $\mu_{\kappa-1}$ and $\mu_{\kappa-1, \kappa}$, then, by Stroh's series, the covariant

$$(a a_{\kappa-1})^{\lambda_{\kappa-2}} (a a_\kappa)^{\lambda_{\kappa-1}}$$

is linearly expressible in terms of

$$\begin{aligned} & (a_{\kappa-1} a_\kappa)^{\lambda_{\kappa-2} + \lambda_{\kappa-1}}, \quad \left((a_{\kappa-1} a_\kappa)^{\lambda_{\kappa-2} + \lambda_{\kappa-1} - 1}, \alpha \right), \quad \dots, \\ & \left((a_{\kappa-1} a_\kappa)^{\lambda_{\kappa-2} + \lambda_{\kappa-1} - \mu_{\kappa-1, \kappa} + 1}, \alpha \right)^{\mu_{\kappa-1, \kappa} - 1}; \\ & (a a_\kappa)^{\lambda_{\kappa-2} + \lambda_{\kappa-1}}, \quad \left((a a_\kappa)^{\lambda_{\kappa-2} + \lambda_{\kappa-1} - 1}, [a_{\kappa-1}] \right), \quad \dots, \quad \left((a a_\kappa)^{\lambda_{\kappa-2} + \lambda_{\kappa-1} - \mu_{\kappa-1} + 1}, [a_{\kappa-1}] \right)^{\mu_{\kappa-1} - 1}; \\ & (a a_{\kappa-1})^{\lambda_{\kappa-2} + \lambda_{\kappa-1}}, \quad (a a_{\kappa-1})^{\lambda_{\kappa-2} + \lambda_{\kappa-1} - 1} (a a_\kappa), \quad \dots, \quad (a a_{\kappa-1})^{\mu_{\kappa-1} + \mu_{\kappa-1, \kappa}} (a a_\kappa)^{\lambda_{\kappa-2} + \lambda_{\kappa-1} - \sigma_{\kappa-1}}. \end{aligned}$$

Now by hypothesis each form in the first two rows is reducible, and therefore $(a a_{\kappa-1})^{\lambda_{\kappa-2}} (a a_\kappa)^{\lambda_{\kappa-1}}$ is reducible unless $\lambda_{\kappa-2} \geq \mu_{\kappa-1} + \mu_{\kappa-1, \kappa}$.

* We always consider $(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_\kappa)^{\lambda_{\kappa-1}}$ reducible if we can express it in terms of forms $(a_1 a_2)^{\mu_1} (a_1 a_3)^{\mu_2} \dots (a_1 a_\kappa)^{\mu_{\kappa-1}}$ such that the first of the quantities $\mu_1 - \lambda_1, \mu_2 - \lambda_2, \dots, \mu_{\kappa-1} - \lambda_{\kappa-1}$ which is not zero is positive.

Hence a necessary condition for irreducibility is

$$\lambda_{\kappa-2} \geq \sigma_{\kappa-1}.$$

Assume the theorem true for the indices λ_{r-1} where $r > s$; then, as before, if we put $a \equiv [a_1 a_2 \dots a_{s-1}]$, we have to consider the form

$$(aa_s)^{\lambda_{s-1}} (aa_{s+1})^{\lambda_s} \dots (aa_\kappa)^{\lambda_{\kappa-1}}.$$

By Stroh's series this can be expressed in terms of

1. $(a_s a_{s+1})^{\lambda_s + \lambda_{s-1}} [a_s a_{s+1}]', ([a_s a_{s+1}], [a_s a_{s+1}])', \dots, ([a_s a_{s+1}], [a_s a_{s+1}])'^{\sigma_{s, s+1}-1};$
2. $(aa_{s+1})^{\lambda_s + \lambda_{s-1}} \dots, ([a_s]', [a_s]), \dots, ([a_s]', [a_s])^{\sigma_s - \sigma_{s, s+1}-1};$
3. $(aa_s)^{\lambda_s + \lambda_{s-1}} \dots, (aa_s)^{\lambda_s + \lambda_{s-1}-1} (aa_{s+1}) \dots, \dots, (aa_s)^{\sigma_s} (aa_{s+1})^{\lambda_s + \lambda_{s-1} - \sigma_s} (aa_{s+2})^{\lambda_{s+1}} \dots$

We proceed to show that each form in the first two rows is reducible.

(1) Any member of the first row is

$$([a_s a_{s+1}], [a_s a_{s+1}])'^\nu,$$

where

$$[a_s a_{s+1}] \equiv (a_s a_{s+1})^{\lambda_{s-1} + \lambda_s - \nu} \equiv \beta, \text{ say,}$$

and

$$[a_s a_{s+1}]' \equiv (aa_{s+2})^{\lambda_{s+1}} \dots (aa_\kappa)^{\lambda_{\kappa-1}},$$

so that we need only consider the form

$$(a\beta)^\nu (aa_{s+2})^{\lambda_{s+1}} \dots (aa_\kappa)^{\lambda_{\kappa-1}}.$$

In addition to the actual reductions implied by hypothesis, we must also consider those reductions which consist in increasing the earlier indices at the expense of the later ones. (This is a common feature of all inductive proofs, since these reduced indices may be raised to their proper limits by virtue of the inductive proof; *v. Note, p. 225.*)

So, since we have assumed the theorem true for the index λ_s (of a factor followed by not more than $(\kappa - s - 1)$ other factors), we must have $\nu \geq \sigma_{s, s+1}$, if the form last written down is irreducible. Hence every form in the first row is reducible.

(2) If we put

$$\beta \equiv (aa_{s+1})^{\lambda_s + \lambda_{s-1} - \nu},$$

$$[a_s]' \equiv (\beta a_{s+2})^{\lambda_{s+1}} (\beta a_{s+3})^{\lambda_{s+2}} \dots (\beta a_\kappa)^{\lambda_{\kappa-1}};$$

and we have to consider forms

$$(\beta a_s)^\nu (\beta a_{s+2})^{\lambda_{s+1}} \dots (\beta a_\kappa)^{\lambda_{\kappa-1}}.$$

The condition for the irreducibility of this form is by hypothesis

$$\nu \geq \text{sum of } \mu\text{'s whose first suffix is } s, \text{ and which do not contain a suffix } s+1,$$

$$\text{i.e., } \nu \geq \sigma_s - \sigma_{s, s+1}.$$

Hence every form in the second row is reducible: and, since the index of (aa_s) in the last row is in every term $\geq \sigma_s$, we have as the condition, affecting λ_{s-1} , of irreducibility, $\lambda_{s-1} \geq \sigma_s$.

We have demonstrated the truth of the conditions for the indices $\lambda_{\kappa-1}$ and $\lambda_{\kappa-2}$, and therefore, by induction, the theorem is universally true.

2. COR. 1.—The perpetuant type theorem is immediately deducible from the preceding result: for, if C_κ is the only factor of the product P , every quantity μ is unity, since in this case the transvectants are reducible only if they are of zero order.

Hence

$$\begin{aligned}\sigma_s &= \text{number of ways of choosing all, none, or any of the suffixes} \\ &\quad s+1, \dots, \kappa, \\ &= 2^{\kappa-s},\end{aligned}$$

and therefore the conditions for the irreducibility of C_κ are

$$\lambda_{s-1} \geq 2^{\kappa-s}, \quad s = 2, \dots, \kappa.$$

COR. 2.—An especially important case arises when the product P is (for any reason) reducible if C_κ is a Jacobian: in this case every quantity μ is 2, and

$$\sigma_s = 2^{\kappa-s+1},$$

and therefore $\lambda_{s-1} \geq 2^{\kappa-s+1}, \quad s = 2, \dots, \kappa.$

In this case the minimum weight of C_κ for irreducibility is $2^\kappa - 2$.

(Reference may be made to a paper by Grace, "Extension of Two Theorems on Covariants": *Proc. London Math. Soc.*, Ser. 2, Vol. 1.)

II. SYZYGIES OF DEGREES 3 AND 4 RESPECTIVELY.

3. The syzygies of degrees 3 and 4 offer no difficulties, and may be actually written down apart from the consideration of their reducing special forms: those of degree 4 have been fully discussed already.* A discussion for degree 4 is here given from a different point of view in order to exhibit in a simple case the methods by which syzygies of higher degree are treated.

Degree 3.—There are no syzygies except for weight unity, for which there is a single syzygy:—

$$(bc) + (ca) + (ab) \equiv 0.$$

* Wood, "On Perpetuant Syzygies of Degree 4," *Proc. London Math. Soc.*, Ser. 2, Vol. 2.

The generating function for irreducible forms is $\frac{x^3}{(1-x)^2}$; and, since there are three linearly independent product forms $(bc)^\omega$, $(ca)^\omega$, $(ab)^\omega$ for all values of ω , except $\omega = 1, 0$, for which there are only two and one respectively, the generating function for irreducible product forms is

$$3x^3/(1-x) + 2x + 1.$$

Hence the generating function for all forms is

$$\frac{x^3}{(1-x)^2} + \frac{3x^3}{1-x} + 2x + 1 = \frac{1}{(1-x)^2}.$$

4. *Degree 4*.—There are five different kinds of type forms and products of forms of degree 4: they may be conveniently written

$$C_4, C_2^2, C_1 C_3, C_1^2 C_2, C_1^4;$$

where C_r denotes any perpetuant of degree r , C_r^2 denotes the product of two perpetuants of degree r , and so on. This notation will be preserved throughout.

A product is in the first place considered reducible if it is expressible in terms of products of other kinds which follow it in the above sequence. For weight ω the syzygies are nine in number and consist of—

(1) The three syzygies, due to Stroh, of the form

$$\{a_1 a_2 a_3 a_4\}^\omega \equiv \{ (a_1 a_2) + (a_3 a_4) \}^\omega - \{ (a_1 a_4) + (a_3 a_2) \}^\omega = 0.$$

(2) The six syzygies, arising from the Jacobian identity, of the form

$$(a_1 a_2)^{\omega-1} (a_3 a_4) = (a_1 a_2)^{\omega-1} (a_2 a_4) - (a_1 a_2)^{\omega-1} (a_2 a_3).$$

These last syzygies reduce all products C_2^2 , when one of the factors C_2 is a Jacobian, by expressing each such product in terms of products $C_1 C_3$.

In order to fix what products are reduced by Stroh's syzygies, we arrange the products C_2^2 in the following order:—

$$(a_1 a_4)^\lambda (a_2 a_3)^\mu, \quad (a_1 a_3)^\lambda (a_2 a_4)^\mu, \quad (a_1 a_2)^\lambda (a_3 a_4)^\mu,$$

and define—

(i.) $(a_1 a_4)^\lambda (a_2 a_3)^\mu$ as reducible if it is expressible in terms of products $(a_1 a_3)^\lambda (a_2 a_4)^\mu$, $(a_1 a_2)^\lambda (a_3 a_4)^\mu$ and products $C_1 C_3$, &c.

(ii.) $(a_1 a_3)^\lambda (a_2 a_4)^\mu$ as reducible if it is expressible in terms of products $(a_1 a_2)^\lambda (a_3 a_4)^\mu$ and products $C_1 C_3$, &c.

Finally, we shall arrange the products $(a_1 a_r)^\lambda (a_s a_t)^\mu$ in ascending values of λ , and define any such form as reducible if it is expressible in terms of similar forms with greater values of λ .

The syzygy $\{a_1 a_2 a_3 a_4\}_\omega \equiv 0$ will reduce the form $(a_1 a_4)^2 (a_2 a_3)^{\omega-2}$, since $(a_1 a_4)(a_2 a_3)^{\omega-1} = \Sigma C_1 C_3$, by the Jacobian identity; the syzygy $\{a_1 a_2 a_4 a_3\}_\omega \equiv 0$ reduces in the same way the form $(a_1 a_3)^2 (a_2 a_4)^{\omega-2}$, and the syzygy $\{a_1 a_3 a_2 a_4\}_\omega \equiv 0$ reduces

$$\binom{\omega}{2} (a_1 a_4)^2 (a_2 a_3)^{\omega-2} + \binom{\omega}{3} (a_1 a_4)^3 (a_2 a_3)^{\omega-3}.$$

Moreover, from $\{a_1 a_2 a_3 a_4\}_\omega \equiv 0$, we have reduced

$$\binom{\omega}{2} (a_1 a_4)^2 (a_3 a_2)^{\omega-2} + \binom{\omega}{3} (a_1 a_4)^3 (a_3 a_2)^{\omega-3},$$

$$i.e., \quad \binom{\omega}{2} (a_1 a_4)^2 (a_2 a_3)^{\omega-2} - \binom{\omega}{3} (a_1 a_4)^3 (a_2 a_3)^{\omega-3}.$$

Hence both $(a_1 a_4)^2 (a_2 a_3)^{\omega-2}$ and $(a_1 a_4)^3 (a_2 a_3)^{\omega-3}$ are, in accordance with our definition, reducible products.

Hence the irreducible products C_2^2 are

$$(a_1 a_2)^\lambda (a_3 a_4)^\mu, \quad \lambda \geq 2, \quad \mu \geq 2;$$

$$(a_1 a_3)^\lambda (a_2 a_4)^\mu, \quad \lambda \geq 3, \quad \mu \geq 2;$$

$$(a_1 a_4)^\lambda (a_2 a_3)^\mu, \quad \lambda \geq 4, \quad \mu \geq 2;$$

and so the generating function for all products C_2^2 is

$$\frac{x^4}{(1-x)^2} + \frac{x^5}{(1-x)^2} + \frac{x^6}{(1-x)^2} = \frac{x^4 - x^7}{(1-x)^3}.$$

Finally, when the weight is unity we have only forms $C_1^2 C_2$; by the Jacobian transformation, we can compel any one definite letter to appear in the factor C_2 ; so for weight unity there are only three such forms, while for all other weights there are six: the generating function for the products $C_1^2 C_2$ is therefore

$$6x^2 / (1-x) + 3x.$$

The results for degree 4 may be summarized thus :—

Forms.	Generating Functions.
C_4	$\frac{x^7}{(1-x)^3}$
C_2^2	$\frac{x^4-x^7}{(1-x)^3}$
$C_1 C_3$	$\frac{4x^3}{(1-x)^2}$
$C_1^2 C_2$	$\frac{6x^2}{1-x} + 3x$
C_1^4	1

Hence the generating function for the total number of forms of degree 4 is

$$\frac{x^7+x^4-x^7}{(1-x)^3} + \frac{4x^3}{(1-x)^2} + \frac{6x^2}{1-x} + 3x + 1 = \frac{1}{(1-x)^3}.$$

III. DEFINITION OF REDUCIBILITY.—GENERAL METHODS OF REDUCTION BY DIFFERENTIAL OPERATORS.

5. The product forms will be arranged in a definite sequence to be particularized immediately, and a product will be defined as reducible if it is linearly expressible in terms of product forms which follow it in this definite sequence. The sequence of the *letters* involved in a symbolical expression will be that defined by their suffixes, viz., $a_1, a_2, \dots, a_s$.

The criteria for determining the relative positions of any two products A and B in the sequence will be made to depend successively on :—

- (i.) The partial degrees of the factors of each product.
- (ii.) The arrangement of the letters among the factors of each product.
- (iii.) The weights of the factors of each product.
- (iv.) The indices of the symbolical determinants of the factors of each product.

$$(i.) \text{ If } A \equiv C_{m_1} C_{m_2} \dots C_{m_r}, \quad m_1 \leq m_2 \leq m_3 \dots \leq m_r;$$

$$B \equiv C_{n_1} C_{n_2} \dots C_{n_l}, \quad n_1 \leq n_2 \leq n_3 \dots \leq n_l;$$

then A precedes B , if the first of the quantities

$$m_1 - n_1, m_2 - n_2, \dots, m_r - n_r, \dots,$$

which is not zero, is positive.

$$\begin{aligned} \text{(ii.) If } & A \equiv C_{m_1} C_{m_2} \dots C_{m_r} \\ & B \equiv C'_{m_1} C'_{m_2} \dots C'_{m_r} \end{aligned} \left. \vphantom{\begin{aligned} A \equiv C_{m_1} C_{m_2} \dots C_{m_r} \\ B \equiv C'_{m_1} C'_{m_2} \dots C'_{m_r} \end{aligned}} \right\}, \quad m_1 \leq m_2 \leq m_3 \dots \leq m_r,$$

so that the factors of A and B are of the same partial degrees, then the arrangement of the letters is taken into consideration.

Case I.— $m_1 < m_2$.

Suppose C_{m_1} contains the letters $a_{r_1}, a_{r_2}, \dots, a_{r_{m_1}}$,

and C'_{m_2} contains the letters $a_{s_1}, a_{s_2}, \dots, a_{s_{m_2}}$;

where $r_1, r_2, \dots, r_{m_1}$, or $s_1, s_2, \dots, s_{m_2}$ are any m_1 of the suffixes 1, 2, 3, ..., ϕ , such that

$$r_1 < r_2 < r_3 \dots < r_{m_1},$$

and

$$s_1 < s_2 < s_3 \dots < s_{m_2}.$$

Then A precedes B , if the first of the quantities

$$r_1 - s_1, r_2 - s_2, \dots, r_{m_1} - s_{m_1}$$

which is not zero is negative.

This assumes that C_{m_1} and C'_{m_1} do not contain the same set of m_1 letters.

Case II.— $m = m_1 = m_2 = \dots = m_\theta < m_{\theta+1} \leq m_{\theta+2} \dots \leq m_\kappa$.

In the first place the sequence is to be determined as in Case I. by all the letters occurring in the factors $C_{m_1} C_{m_2} \dots C_{m_\theta}$.

When the letters in $C_{m_1} C_{m_2} \dots C_{m_\theta}$ are the same as those in $C'_{m_1} C'_{m_2} \dots C'_{m_\theta}$, it is necessary to distinguish between the factors $C_{m_1}, C_{m_2}, \dots, C_{m_\theta}$, in order to determine whether A precedes B or not. Let a_ϕ be the first of the set of letters which occurs in $C_{m_1} C_{m_2} \dots C_{m_\theta}$; then we define C_{m_1} to be that factor which contains a_ϕ ; similarly C'_{m_1} is that factor of the product $C'_{m_1} C'_{m_2} \dots C'_{m_\theta}$ which contains a_ϕ . Let the letters of C_{m_1}, C'_{m_1} be respectively $a_\phi, a_{r_2}, a_{r_3}, \dots, a_{r_m}$ and $a_\phi, a_{s_2}, a_{s_3}, \dots, a_{s_m}$, where

$$\phi < r_2 < r_3 \dots < r_m,$$

$$\phi < s_2 < s_3 \dots < s_m.$$

Then A precedes B if the first of the differences

$$r_2 - s_2, r_3 - s_3, \dots, r_m - s_m$$

which is not zero is *positive*.

(As an example, see the arrangement of the sequence of products C_2^2 at the foot of p. 228.)

If the letters of C_{m_1} and C'_{m_1} are the same, we apply the same test to the products $C_{m_2} C_{m_3} \dots C_{m_g}$, $C'_{m_2} C'_{m_3} \dots C'_{m_g}$. These two products now contain the same set of letters; we choose C_{m_2} , C'_{m_2} to be the two factors which contain the first letter of this set. Then, unless the letters of C_{m_2} , C'_{m_2} are all the same, the sequence is determined by these letters, in the same way as before. Otherwise we must apply the same test to the products $C_{m_3} \dots C_{m_g}$, $C'_{m_3} \dots C'_{m_g}$.

Finally, when the letters of the various corresponding factors of $C_{m_1} C_{m_2} \dots C_{m_g}$ and $C'_{m_1} C'_{m_2} \dots C_{m_g}$ are in every case the same, the tests which have just been laid down must be applied to the products $C_{m_{g+1}} \dots C_{m_\kappa}$, $C'_{m_{g+1}} \dots C'_{m_\kappa}$ in order to determine whether A precedes B or not.

(iii.) If A and B each have their factors of the same partial degrees in the same sets of letters, let

$$A \equiv C_{m_1} C_{m_2} \dots C_{m_\kappa}, \quad B \equiv C'_{m_1} C'_{m_2} \dots C'_{m_\kappa}, \quad m_1 \leq m_2 \dots \leq m_\kappa,$$

where, if certain of the m 's are equal, the sequence of the factors $C_{m_1}, C_{m_2}, \dots, C_{m_\kappa}$ is determined as in (ii.), and further let

$C_{m_1}, C_{m_2}, \dots, C_{m_\kappa}$ be of total weights $\omega_1, \omega_2, \dots, \omega_\kappa$ respectively,

and $C'_{m_1}, C'_{m_2}, \dots, C'_{m_\kappa}$ be of total weights $\omega'_1, \omega'_2, \dots, \omega'_\kappa$ respectively.

Then A precedes B if the first of the quantities

$$\omega_1 - \omega'_1, \omega_2 - \omega'_2, \dots, \omega_\kappa - \omega'_\kappa$$

which is not zero is *negative*.

(iv.) Finally, we have to consider the case of products whose factors are of the same weights in the same sets of letters. Suppose

$$\left. \begin{aligned} C_{m_r} &\equiv (a_{(r,1)} a_{(r,2)})^{\lambda_{(r,1)}} (a_{(r,1)} a_{(r,3)})^{\lambda_{(r,2)}} \dots (a_{(r,1)} a_{(r,m_r)})^{\lambda_{(r,m_r-1)}} \\ C'_{m_r} &\equiv (a_{(r,1)} a_{(r,2)})^{\lambda'_{(r,1)}} (a_{(r,1)} a_{(r,3)})^{\lambda'_{(r,2)}} \dots (a_{(r,1)} a_{(r,m_r)})^{\lambda'_{(r,m_r-1)}} \end{aligned} \right\},$$

$$r = 1, 2, \dots, \kappa$$

where $(r, 1) < (r, 2) \dots < (r, m_r - 1) < (r, m_r)$.

Then A precedes B , if the first of the quantities

$$\begin{aligned} (1) \quad & \lambda_{(1, 1)} - \lambda'_{(1, 1)}, \quad \lambda_{(1, 2)} - \lambda'_{(1, 2)}, \quad \dots, \quad \lambda_{(1, m_1)} - \lambda'_{(1, m_1)}, \\ (2) \quad & \lambda_{(2, 1)} - \lambda'_{(2, 1)}, \quad \lambda_{(2, 2)} - \lambda'_{(2, 2)}, \quad \dots, \quad \lambda_{(2, m_2)} - \lambda'_{(2, m_2)}, \\ & \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ (r) \quad & \lambda_{(r, 1)} - \lambda'_{(r, 1)}, \quad \lambda_{(r, 2)} - \lambda'_{(r, 2)}, \quad \dots, \quad \lambda_{(r, m_r)} - \lambda'_{(r, m_r)}, \\ & \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ (\kappa) \quad & \lambda_{(\kappa, 1)} - \lambda'_{(\kappa, 1)}, \quad \lambda_{(\kappa, 2)} - \lambda'_{(\kappa, 2)}, \quad \dots, \quad \lambda_{(\kappa, m_\kappa)} - \lambda'_{(\kappa, m_\kappa)} \end{aligned}$$

which is not zero is negative.

A reference to the reductions in the case of degree 5 (in the succeeding section) should go some way towards making the above definition of reducibility quite clear.

General Methods of Reduction by the use of Differential Operators.

6. All the reductions made use of depend ultimately on Stroh's syzygies of degree 4: consider the syzygy

$$\{(ab) + (cd)\}^\lambda - \{(ad) + (cb)\}^\lambda = 0,$$

$$\text{or} \quad \sum_{r=0}^{\lambda} \binom{\lambda}{r} (ab)^r (cd)^{\lambda-r} - \sum_{r=0}^{\lambda} \binom{\lambda}{r} (ad)^r (cb)^{\lambda-r} = 0.$$

Here the letters a, b, c, d denote any of the letters $a_1, a_2, \dots, a_s$ used elsewhere in the present paper. This syzygy may be written

$$e^{(ab)D} (cd)^\lambda - e^{(ad)B} (cb)^\lambda = 0,$$

$$\text{where} \quad B \equiv x_1 \frac{\partial}{\partial b_2} - x_2 \frac{\partial}{\partial b_1}, \quad D \equiv x_1 \frac{\partial}{\partial d_2} - x_2 \frac{\partial}{\partial d_1};$$

$$\text{and therefore} \quad B.b_x^n = 0, \quad D(cd) = c_x, \quad \dots$$

Let Γ_1 be any covariant containing the letters a, b , but not the letters c, d , and let Γ_2 be any covariant containing the letters c, d , but none of the letters of Γ_1 . Then we may write

$$\Gamma_1 = \Sigma P_1(ab)^{\mu_1}, \quad \Gamma_2 = \Sigma P_2(cd)^{\mu_2},$$

where P_1 does not contain the letter b in any determinantal factor, and P_2 does not contain the letter d in any determinantal factor.

Then we have a syzygy

$$P_1 \cdot e^{(ab)D} \{P_2(cd)^\lambda\} = P_1 \cdot e^{(ab)B} \{P_2(cb)^\lambda\}; \quad (I.)$$

and this syzygy is a relation between products of covariants of the form $\Gamma_1 \Gamma_2$ and the same products having the letters b and d interchanged.

Now we have arranged our products in a fixed sequence, we shall suppose that the products on the right-hand side of the relation (I.) come after those on the left in this fixed sequence. The syzygy may then be conveniently written

$$P_1 e^{(ab)D} \{P_2(cd)^\lambda\} = R,$$

where R denotes any products which come in the fixed sequence after the products on the left-hand side: the sum of the products on the left-hand side is, in fact, according to our definition, reducible.

7. In the first place we shall consider the reduction of products $C_2 C_\kappa$: let $C_2 C_\kappa = (ab)^\mu (d_\kappa d_1)^{\lambda_1} (d_\kappa d_2)^{\lambda_2} \dots (d_\kappa d_{r-1})^{\lambda_{r-1}}$: the sequence of such products is primarily determined by the letters contained in the factor C_2 : we suppose that the product in which the letters are ab precedes any of the products in which the letters of C_2 are

$$ad_1, ad_2, \dots, ad_r \text{ respectively.}$$

This is in accordance with our defined sequence of products when all the letters $d_1, d_2, \dots, d_r$ come after the letter b . In this case we have a series of relations of the form

$$e^{(ab)D_1} C_\kappa = R, \quad e^{(ab)D_2} C_\kappa = R, \quad \dots, \quad e^{(ab)D_r} C_\kappa = R,$$

where

$$D_r = x_1 \frac{\partial}{\partial d_{r,2}} - x_2 \frac{\partial}{\partial d_{r,1}}.$$

We proceed to show that *these r relations are all linearly independent*: this result is of fundamental importance.

The operators $D_1, D_2, \dots, D_r$ are obviously all commutative. The first relation may be written

$$C_\kappa + (ab) \cdot D_1 C_\kappa + \frac{(ab)^2}{2!} D_1^2 C_\kappa + \dots = R,$$

or, since the first two terms are obviously reducible, the second by the Jacobian identity, it may be written

$$\frac{(ab)^2}{2!} D_1^2 C_\kappa + \frac{(ab)^3}{3!} D_1^3 C_\kappa + \dots = R, \quad (II.)$$

and this relation is true, whatever covariant C_κ may be involving the letters $d_1, d_2, \dots, d_\kappa$.

8. Consider the equation $D_1 C_\kappa = C'_\kappa$

as an equation to determine C_κ .

If

$$C'_\kappa = (d_\kappa d_1)^{\mu_1} (d_\kappa d_2)^{\mu_2} \dots (d_\kappa d_{\kappa-1})^{\mu_{\kappa-1}},$$

then one solution is obviously

$$C_\kappa = \frac{1}{\mu_1 + 1} (d_\kappa d_1)^{\mu_1 + 1} (d_\kappa d_2)^{\mu_2} \dots (d_\kappa d_{\kappa-1})^{\mu_{\kappa-1}}.$$

Let $C_\kappa^{(1)}$ be any other solution; then we can always write

$$C_\kappa^{(1)} = \sum_{\rho} N_{\rho} (d_\kappa d_1)^{\mu_{\rho,1}} (d_\kappa d_2)^{\mu_{\rho,2}} \dots (d_\kappa d_{\kappa-1})^{\mu_{\rho,\kappa-1}},$$

and hence

$$\sum_{\rho} N_{\rho} \mu_{\rho,1} (d_\kappa d_1)^{\mu_{\rho,1}-1} (d_\kappa d_2)^{\mu_{\rho,2}} \dots (d_\kappa d_{\kappa-1})^{\mu_{\rho,\kappa-1}} \equiv (d_\kappa d_1)^{\mu_1} \dots (d_\kappa d_{\kappa-1})^{\mu_{\kappa-1}}.$$

The covariants in this identity are all independent: it follows, therefore, that C_κ can differ from $\frac{1}{\mu_1 + 1} (d_\kappa d_1)^{\mu_1 + 1} (d_\kappa d_2)^{\mu_2} \dots (d_\kappa d_{\kappa-1})^{\mu_{\kappa-1}}$ only by covariants which are products of the quantic d_1 with covariants of degree $\kappa - 1$. And so, neglecting reducible terms, $D_1^{-1} C_\kappa$ has a unique meaning: in the same way $D_1^{-2} C_\kappa$, $D_1^{-3} C_\kappa$, ... have each a unique meaning. So in (II.) we may replace C_κ by $D_1^{-2} C_\kappa$: the resulting relation is

$$\frac{(ab)^2}{2!} C_\kappa + \frac{(ab)^3}{3!} D_1 C_\kappa + \dots = R,$$

and, in the same way, from the relation $e^{(ab)D_2} C_\kappa = R$ we obtain

$$\frac{(ab)^2}{2!} C_\kappa + \frac{(ab)^3}{3!} D_2 C_\kappa + \dots = R.$$

Therefore, on subtraction,

$$\frac{(ab)^3}{3!} (D_1 - D_2) C_\kappa + \frac{(ab)^4}{4!} (D_1^2 - D_2^2) C_\kappa + \dots = R,$$

$$\text{or } \left[\frac{(ab)^3}{3!} + \frac{(ab)^4}{4!} (D_1 + D_2) + \frac{(ab)^5}{5!} (D_1^2 + D_1 D_2 + D_2^2) \dots \right] (D_1 - D_2) C_\kappa = R.$$

Writing this

$$\chi (D_1 - D_2) C_\kappa = R,$$

and replacing C_κ by $D_1^{-1} C_\kappa$, we have

$$\chi (1 - D_2 D_1^{-1}) C_\kappa = R.$$

In this replace C_κ by $D_2 D_1^{-1} C_\kappa$: we have

$$\chi (D_2 D_1^{-1} - D_2^2 D_1^{-2}) C_\kappa = R.$$

So, adding, we get $\chi(1 - D_2^2 D_1^{-2}) C_\kappa = R$.

Proceeding in this way, we obtain

$$\chi(1 - D_2^\rho D_1^{-\rho}) C_\kappa = R.$$

Since $D_1^{-\rho}$ has no effect on the index of $(d_\kappa d_\rho)$ in the covariant C_κ , ρ may be taken so large that $D_2^\rho D_1^{-\rho} C_\kappa = 0$: so the last relation is

$$\chi C_\kappa = R,$$

or
$$\left[\frac{(ab)^3}{3!} + \frac{(ab)^4}{4!} (D_1 + D_2) + \dots \right] C_\kappa = R.$$

9. Now let us assume that, in general, we can deduce from the σ relations

$$e^{(ab)D_1} C_\kappa = R, \quad e^{(ab)D_2} C_\kappa = R, \quad \dots, \quad e^{(ab)D_\sigma} C_\kappa = R$$

the relation:—

$$\begin{aligned} & \left\{ \frac{(ab)^{\sigma+1}}{(\sigma+1)!} + \frac{(ab)^{\sigma+2}}{(\sigma+2)!} (D_1 + D_2 + \dots + D_\sigma) + \dots \right. \\ & \quad \left. + \frac{(ab)^{\sigma+\tau}}{(\sigma+\tau)!} \left(\sum_{\varpi_1 + \dots + \varpi_\sigma = \tau-1} D_1^{\varpi_1} D_2^{\varpi_2} \dots D_\sigma^{\varpi_\sigma} \right) + \dots \right\} C_\kappa = R. \quad (\text{III.}) \end{aligned}$$

We shall show that, if we take another relation $e^{(ab)D_{\sigma+1}} C_\kappa = R$, then the resulting relation will be of the same form as (III.) with σ changed to $\sigma+1$.

On our assumption we can deduce from the σ relations

$$e^{(ab)D_1} C_\kappa = R, \quad \dots, \quad e^{(ab)D_{\sigma-1}} C_\kappa = R, \quad e^{(ab)D_{\sigma+1}} C_\kappa = R$$

the relation:—

$$\begin{aligned} & \left\{ \frac{(ab)^{\sigma+1}}{(\sigma+1)!} + \frac{(ab)^{\sigma+2}}{(\sigma+2)!} (D_1 + D_2 + \dots + D_{\sigma-1} + D_{\sigma+1}) + \dots \right. \\ & \quad \left. + \frac{(ab)^{\sigma+\tau}}{(\sigma+\tau)!} \left(\sum_{\varpi_1 + \dots + \varpi_{\sigma-1} + \varpi_{\sigma+1} = \tau-1} D_1^{\varpi_1} \dots D_{\sigma-1}^{\varpi_{\sigma-1}} D_{\sigma+1}^{\varpi_{\sigma+1}} \right) + \dots \right\} C_\kappa = R. \end{aligned}$$

Subtracting this last relation from (III.), we obtain

$$\begin{aligned} & \left\{ \frac{(ab)^{\sigma+2}}{(\sigma+2)!} + \frac{(ab)^{\sigma+3}}{(\sigma+3)!} \left(\sum_{\rho=1}^{\rho=\sigma+1} D_\rho \right) + \dots \right. \\ & \quad \left. + \frac{(ab)^{\sigma+\tau}}{(\sigma+\tau)!} \left(\sum_{\varpi_1 + \dots + \varpi_\sigma + \varpi_{\sigma+1} = \tau-2} D_1^{\varpi_1} \dots D_{\sigma+1}^{\varpi_{\sigma+1}} \right) + \dots \right\} (D_{\sigma-1} - D_{\sigma+1}) C_\kappa = R, \end{aligned}$$

and by the same argument as before we can show that $(D_{\sigma-1} - D_{\sigma+1}) C_\kappa$

may be replaced by C_κ . Now we have seen that (III.) is true when $\sigma = 1, 2$: it is therefore true for all values of σ .

An immediate consequence of this is the

THEOREM.—The r relations

$$e^{(ab)D_1} C_\kappa = R, \quad e^{(ab)D_2} C_\kappa = R, \quad \dots, \quad e^{(ab)D_r} C_\kappa = R$$

are in general linearly independent, and together reduce the product $(ab)^\lambda C_\kappa$, when $\lambda \leq r+1$.

10. Certain special cases remain for consideration.

(i.) The relation $e^{(ab)D_\sigma} C_\kappa = R$ is true if the product $C_2 C_\kappa$, in which the letters of C_2 are (ab) , precedes the product $C_2 C_\kappa$, in which the letters of C_2 are (ad_κ) . We have previously written C_κ in the form

$$(d_\kappa d_1)^{\lambda_1} (d_\kappa d_2)^{\lambda_2} \dots (d_\kappa d_{\kappa-1})^{\lambda_{\kappa-1}},$$

but it may be equally well written in the form

$$\Sigma N (d_1 d_\kappa)^{\mu_1} (d_1 d_2)^{\mu_2} \dots (d_1 d_{\kappa-1})^{\mu_{\kappa-1}}.$$

Now the result of operating on each of the terms of this sum is R , and therefore the result of operating on the whole sum, or C_κ , is also R : in other words, our preceding results are not affected by the form of C_κ .

(ii.) If the product $(ab)^\lambda C_\kappa$ comes before the product $(d_\sigma b)^\lambda C_\kappa$, we have in the same way another syzygy

$$e^{(ab)D_\sigma} C_\kappa = R$$

or

$$e^{(ab)(-D_\sigma)} C_\kappa = R;$$

all such syzygies are independent of those already considered: we may, in fact, write $D_{\kappa+1} \equiv -D_\sigma$, and then the extension of the result embodied in (III.) to the case where the operators are $D_1, D_2, \dots, D_\sigma, D_{\kappa+1}$ is almost identical with the previous investigation. Hence we deduce that, if r_1 letters of C_κ come after a , and r_2 letters of C_κ come after b , then

$$(ab)^\lambda C_\kappa = R, \quad \text{if } \lambda \leq r_1 + r_2 + 1.$$

(iii.) We may suppose that the letters a and b refer to covariants of the original quantics and not to the quantics themselves, and that one of the letters—say, d_1 —refers to a covariant involving the letters, say, $e_1, e_2, \dots, e_i$. Then, if the product $(ab)^\lambda C_\kappa$ precedes the product $(ad)^\lambda C'_\kappa$, we have a syzygy

$$e^{(ab)D_1} C_\kappa = R,$$

or, from the identity $(D_1 + D_2 + \dots + D_r) C_\kappa = 0$, which is easily verified,

$$e^{(ab)(-D_2-D_3-\dots-D_r)} C_\kappa = R.$$

Let P be the covariant obtained from C_κ by replacing d_1 by the letters $e_1, e_2, \dots, e_l$; then

$$(D_2 + D_3 + \dots + D_r + E_1 + E_2 + \dots + E_l) P = 0,$$

and hence the syzygy may be written

$$e^{(ab)(E_1+E_2+\dots+E_l)} C_\kappa = R.$$

Now C_κ may be represented as a sum of transvectants of covariants of a definite set of r of its letters with covariants of the remainder, and so we may also obtain syzygies of the form

$$e^{(ab)(D_1+D_2+\dots+D_r)} C_\kappa = R,$$

where $d_1, d_2, \dots, d_r$ are any of the letters of C_κ (the existence of any such relation will, of course, depend on the sequence of the products). We have to show that all such syzygies are linearly independent. We write $D_{\kappa+1}$ for any such composite operator as $(D_1 + D_2 + \dots + D_r)$, and observe that all operators, simple or composite, are commutative. The proof of the relation (III.) then proceeds on the same lines as before for both simple and composite operators: only one point arises:—Has the operation

$$D_{\kappa+1}^{-1} C_\kappa \equiv (D_1 + D_2 + \dots + D_r)^{-1} C_\kappa$$

a perfectly definite meaning?

To prove that it has, we write C_κ in the form $\Sigma N(\alpha, \beta)^\lambda$, where β is a covariant of the letters $d_1, d_2, \dots, d_r$, and α is a covariant of the remaining letters: then, in the same way as before, we see that

$$(D_1 + D_2 + \dots + D_r)^{-1} (\alpha\beta)^\lambda$$

differs from $(\lambda+1)^{-1}(\alpha\beta)^{\lambda+1}$ only by products of covariants of $d_1, d_2, \dots, d_r$ with covariants of the remaining letters, and so the meaning is for our purpose definite and unique.

IV. SYZYGIES OF DEGREE 5.

11. We shall treat the syzygies of degree 5 at some length in explanation of the general principles set forth in Section III.

The product forms to be considered are of the following classes:—

$$C_2 C_3, C_1 C_4, C_1 C_2^2, C_1^2 C_3, C_1^3 C_2, C_1^5.$$

The generating functions for all the irreducible products, except $C_2 C_3$ and $C_1^3 C_2$, are known from the results for degrees 3 and 4.

(i.) *Generating Function for Products $C_2 C_3$.*

The $C_2 C_3$ products are to be arranged in a sequence to be determined thus :

(a) by virtue of the letters involved in the factor C_2 : the following is the sequence :—

$$\begin{array}{ccccc} \text{(i.)} & \text{(ii.)} & \text{(iii.)} & \text{(iv.)} & \text{(v.)} \\ (a_1 a_2)^r C_3, & (a_1 a_3)^r C_3, & (a_1 a_4)^r C_3, & (a_1 a_5)^r C_3, & (a_2 a_3)^r C_3, \\ \text{(vi.)} & \text{(vii.)} & \text{(viii.)} & \text{(ix.)} & \text{(x.)} \\ (a_2 a_4)^r C_3, & (a_2 a_5)^r C_3, & (a_3 a_4)^r C_3, & (a_3 a_5)^r C_3, & (a_4 a_5)^r C_3. \end{array}$$

(b) If two products $C_2 C_3$ have the same letters in the C_2 factor, that product whose C_2 factor is of smaller weight precedes the other.

Any $C_2 C_3$ product is then defined as reducible, if it is expressible linearly in terms of products of other classes and of $C_2 C_3$ products which follow it in the sequence just determined.

First consider the products $(a_1 a_2)^r C_3$: using the notation of the preceding section, we have six linearly independent relations (§ 9)

$$e^{\pm(a_1 a_2) D_3} C_3 = R, \quad e^{\pm(a_1 a_2) D_1} C_3 = R, \quad e^{\pm(a_1 a_2) D_5} C_3 = R,$$

where, for instance, in the relation $e^{(a_1 a_2) D_3} C_3 = R$, R represents $C_2 C_3$ product forms such as $(a_1 a_3)^r C_3$ together with product forms $C_1 C_2^2$, $C_1^2 C_3$, and, in the relation $e^{-(a_1 a_2) D_1} C_3 = R$, R represents $C_2 C_3$ product forms such as $(a_2 a_3)^r C_3$ with product forms $C_1 C_2^2$, $C_1^2 C_3$. So, for irreducibility, we must have $\nu > 6+1$, that is, $\nu \geq 8$.

Moreover, C_3 must not be a Jacobian of the form

$$\left((a_r a_s)^\lambda, a_t \right), \text{ where } r, s, t \text{ are } 3, 4, 5 \text{ in any order ;}$$

$$\begin{aligned} \text{for } (a_1 a_2)^r \left((a_r a_s)^\lambda, a_t \right) &= (a_r a_s)^\lambda \left((a_1 a_2)^r, a_t \right) - a_t \left((a_1 a_2)^r, (a_r a_s)^\lambda \right) \\ &= R, \text{ whatever } r \text{ and } s \text{ may be.} \end{aligned}$$

Hence, by the second Corollary of Section I. (§ 2), the weight of C_3 is at least 6, and therefore the minimum weight of an irreducible product $(a_1 a_2)^r C_3$ is 14.

The treatment of the products $(a_1 a_3)^r C_3$ is similar: here (by § 9) there are reductions corresponding to the interchange of a_3 with a_4 , a_5 , and the interchange of a_1 with a_2 , a_4 , a_5 . The five linearly independent relations are

$$e^{(a_3 a_1) D_2} C_3 = R, \quad e^{\pm(a_1 a_3) D_4} C = R, \quad e^{\pm(a_1 a_3) D_5} C_3 = R.$$

Hence $\nu \geq 7$, and, as before, C_3 must not be a Jacobian of the form $\left((a_r a_s)^\lambda, a_t \right)$, r, s, t being 2, 4, 5 in any order: therefore the minimum

weight of C_3 is 6, and the minimum weight of an irreducible product $(a_1 a_3)^\nu C_3$ is 18.

Similarly for an irreducible product $(a_1 a_4)^\nu C_3$, $\nu \geq 6$, and the minimum weight of C_3 is 6; while, for an irreducible product $(a_1 a_5)^\nu C_3$, $\nu \geq 5$, and the minimum weight of C_3 is 6.

Next consider the products $(a_2 a_3)^\nu C_3$: here there are reductions corresponding to the interchanges of a_2 with a_4 , a_5 and the interchanges of a_3 with a_4 , a_5 . The four linearly independent relations are

$$e^{\pm(a_2 a_3) D_4} C_3 = R, \quad e^{\pm(a_2 a_3) D_5} C_3 = R; \quad \text{so that } \nu \geq 6.$$

Moreover, C_3 must not be a Jacobian of the form $((a_4 a_5)^\lambda, a_1)$. Hence, by the second Corollary of Section I. (§ 2), the weight of C_3 is at least 4: therefore the minimum weight of an irreducible product $(a_2 a_3)^\nu C_3$ is 10. In the same way, for an irreducible product $(a_2 a_4)^\nu C_3$, $\nu \geq 5$, and the weight of $C_3 \geq 4$, while, for an irreducible product $(a_2 a_5)^\nu C_3$, $\nu \geq 4$, and the weight of $C_3 \geq 4$.

Now consider the products $(a_3 a_4)^\nu C_3$: we have for the reduction of such forms two linearly independent relations

$$e^{\pm(a_3 a_4) D_5} C_3 = R;$$

so that $\nu \geq 4$, while, by the perpetuant type theorem, C_3 is of weight 3 at least. So, for $(a_3 a_5)^\nu C_3$, $\nu \geq 3$, and C_3 is of weight 3 at least.

Finally, for the products $(a_4 a_5)^\nu C_3$, we have $\nu \geq 2$, for, if $\nu = 1$, there is a Jacobian identity; also the weight of C_3 is 3 at least.

12. So the irreducible products $C_2 C_3$ may be tabulated thus:—

$$\begin{array}{ll} (a_1 a_2)^\nu (a_3 a_4)^\lambda (a_3 a_5)^\mu, & \nu \geq 8, \quad \lambda \geq 4, \quad \mu \geq 2; \\ (a_1 a_3)^\nu (a_2 a_4)^\lambda (a_2 a_5)^\mu, & \nu \geq 7, \quad \lambda \geq 4, \quad \mu \geq 2; \\ (a_1 a_4)^\nu (a_2 a_3)^\lambda (a_2 a_5)^\mu, & \nu \geq 6, \quad \lambda \geq 4, \quad \mu \geq 2; \\ (a_1 a_5)^\nu (a_2 a_3)^\lambda (a_2 a_4)^\mu, & \nu \geq 5, \quad \lambda \geq 4, \quad \mu \geq 2; \\ (a_2 a_3)^\nu (a_1 a_4)^\lambda (a_1 a_5)^\mu, & \nu \geq 6, \quad \lambda \geq 3, \quad \mu \geq 1; \\ (a_2 a_4)^\nu (a_1 a_3)^\lambda (a_1 a_5)^\mu, & \nu \geq 5, \quad \lambda \geq 3, \quad \mu \geq 1; \\ (a_2 a_5)^\nu (a_1 a_3)^\lambda (a_1 a_4)^\mu, & \nu \geq 4, \quad \lambda \geq 3, \quad \mu \geq 1; \\ (a_3 a_4)^\nu (a_1 a_2)^\lambda (a_1 a_5)^\mu, & \nu \geq 4, \quad \lambda \geq 2, \quad \mu \geq 1; \\ (a_3 a_5)^\nu (a_1 a_2)^\lambda (a_1 a_4)^\mu, & \nu \geq 3, \quad \lambda \geq 2, \quad \mu \geq 1; \\ (a_4 a_5)^\nu (a_1 a_2)^\lambda (a_1 a_3)^\mu, & \nu \geq 2, \quad \lambda \geq 2, \quad \mu \geq 1. \end{array}$$

Here the minimum value of ν is determined by the reductions of

Section III., while the minimum values of the indices λ, μ are determined by the theorem of Section I. (§§ 1, 2).

The generating functions for the products $C_2 C_3$ are, therefore, as follows :—

Product.	G. F.	Product.	G. F.
$(a_1 a_2)^\nu C_3$	$\frac{x^{14}}{(1-x)^3}$	$(a_2 a_4)^\nu C_3$	$\frac{x^9}{(1-x)^3}$
$(a_1 a_3)^\nu C_3$	$\frac{x^{13}}{(1-x)^3}$	$(a_2 a_5)^\nu C_3$	$\frac{x^8}{(1-x)^3}$
$(a_1 a_4)^\nu C_3$	$\frac{x^{12}}{(1-x)^3}$	$(a_3 a_4)^\nu C_3$	$\frac{x^7}{(1-x)^3}$
$(a_1 a_5)^\nu C_3$	$\frac{x^{11}}{(1-x)^3}$	$(a_3 a_5)^\nu C_3$	$\frac{x^6}{(1-x)^3}$
$(a_2 a_3)^\nu C_3$	$\frac{x^{10}}{(1-x)^3}$	$(a_4 a_5)^\nu C_3$	$\frac{x^5}{(1-x)^3}$

Hence the generating function for all products $C_2 C_3$ is

$$\frac{x^5 + x^6 + \dots + x^{13} + x^{14}}{(1-x)^3} = \frac{x^5 - x^{15}}{(1-x)^4}.$$

(ii.) *Generating Function for Products $C_1^3 C_2$.*

Here the only possible reductions arise from the Jacobian identity, when the weight is unity. In this case we can insist on a_1 appearing in the symbolical determinant factor; so that there are only four linearly independent products of unit weight, while there are ten of any other weight.

Hence the generating function is $10x^2/(1-x) + 4x$.

The generating functions for types and products of degree 5 may be tabulated thus :—

Forms.	G. F.	Forms.	G. F.
C_5	$\frac{x^{15}}{(1-x)^4}$	$C_1^2 C_3$	$\frac{10x^3}{(1-x)^2}$
$C_2 C_3$	$\frac{x^5 - x^{15}}{(1-x)^4}$	$C_1^3 C_2$	$\frac{10x^2}{1-x} + 4x$
$C_1 C_4$	$\frac{5x^7}{(1-x)^3}$	C_1^5	1
$C_1 C_2^2$	$\frac{5(x^4 - x^7)}{(1-x)^3}$		

The sum of all these generating functions is $\frac{1}{(1-x)^4}$, which is known to be the generating function for the enumeration of the linearly independent covariants of degree 5: from this it follows that the perpetuant type of theorem is exact, and that we have found all the syzygies of degree 5.

V. (i.) PRODUCTS HAVING NO FACTOR OF DEGREE LESS THAN 2.—(ii.) PRODUCTS HAVING NO FACTOR OF DEGREE LESS THAN 3.

13. In the present section we shall show that—

- (i.) The generating function for types and irreducible products of degree δ having no factor C_1 is

$$\frac{x^\delta}{(1-x)^{\delta-1}}.$$

- (ii.) The generating function for types and irreducible products of degree δ having no factor C_1 or C_2 is

$$\frac{x^{\binom{\delta}{1} + \binom{\delta}{2}}}{(1-x)^{\delta-1}}.$$

It seems probable that, in general, the generating function for types and irreducible products of degree δ having no factor of degree κ or less is

$$\frac{x^{\binom{\delta}{1} + \binom{\delta}{2} + \dots + \binom{\delta}{\kappa}}}{(1-x)^{\delta-1}}.$$

This general result is verified in Section VI. in the case of

$$3\kappa + 2 \geq \delta \geq 2\kappa.$$

- (i.) *Products having no Factor of Degree less than 2.*

14. Those products which do contain a factor C_1 are all of the form

$$C_1^r . P_{\delta-r}, \quad r = 1, 2, \dots, \delta-2, \delta.$$

Let V_δ be the generating function for all types and products not having a factor C_1 ; then the expression

$$V_\delta + \binom{\delta}{1} V_{\delta-1} + \binom{\delta}{2} V_{\delta-2} + \dots + \binom{\delta}{r} V_{\delta-r} + \dots + \binom{\delta}{\delta-2} V_2 + V_0$$

is clearly the generating function for all types and irreducible products of degree δ , and so we must have

$$\frac{1}{(1-x)^{\delta-1}} = V_{\delta} + \binom{\delta}{1} V_{\delta-1} + \dots + \binom{\delta}{r} V_{\delta-r} + \dots + \binom{\delta}{\delta-2} V_2 + V_0.$$

In the first place $V_0 = 1$: consider $\binom{\delta}{\delta-2} V_2$, the generating function for the products $C_1^{\delta-2} C_2$; if this product is of unit weight, so that C_2 is a Jacobian, we can insist on a definite letter being present in the symbolical determinant; and therefore the coefficient of x in $\binom{\delta}{\delta-2} V_2$ is $(\delta-1)$; for any other weight the products $C_2 C_1^{\delta-2}$ are irreducible, and so

$$\binom{\delta}{\delta-2} V_2 = (\delta-1)x + \binom{\delta}{\delta-2} \frac{x^2}{1-x}.$$

By reference to our previous results, we find that

$$V_3 = \frac{x^3}{(1-x)^3}, \quad V_4 = \frac{x^4}{(1-x)^3}, \quad V_5 = \frac{x^5}{(1-x)^4}.$$

Let us assume that

$$V_{\delta'} = \frac{x^{\delta'}}{(1-x)^{\delta'-1}} \quad \text{for the values } 3, 4, 5, \dots, \delta-1 \text{ of } \delta';$$

then, from the above relation, we have

$$\begin{aligned} V_{\delta} &= \frac{\{x + (1-x)\}^{\delta}}{(1-x)^{\delta-1}} - \binom{\delta}{1} \frac{x^{\delta-1}}{(1-x)^{\delta-2}} + \dots \\ &\quad + \binom{\delta}{r} \frac{x^{\delta-r}}{(1-x)^{\delta-r-1}} + \dots + (\delta-1)x + 1, \\ &= \frac{x^{\delta}}{(1-x)^{\delta-1}}, \text{ on expanding the first term.} \end{aligned}$$

Hence by induction the result is true in general.

Therefore:—

The generating function for all types and irreducible products of degree δ which contain no factor of unit degree is $\frac{x^{\delta}}{(1-x)^{\delta-1}}$.

(ii.) *Products having no Factor of Degree less than 3.*

15. We proceed to find the generating functions for the following products:—

$$(1) C_2^m, \quad (2) C_2^m P_{\delta-2m}, \quad (3) C_2^m C_3, \quad (4) C_2^m C_4;$$

R 2

where $P_{\delta-2m}$ is any product of degree $\delta-2m$ which contains no factor of degree < 3 and is neither C_3 nor C_4 .

It will be seen that the peculiarity in the treatment of the products $C_2^m C_3$ and $C_2^m C_4$ arises from the Jacobian identity; it may be easily verified that this special identity yields no exceptional reductions of other products $C_2^m P_r$, even if P_r is one of the products C_3^3 , C_4^2 , $C_3 C_4$, $C_3 C_5$, $C_4 C_5$ (see § 17, Note).

(1) *Generating Function for C_2^m .*

We know that the generating function for C_2^2 is

$$\frac{x^4 - x^7}{(1-x)^8} = \frac{x^4(1-x)(1-x^3)}{(1-x)^4}.$$

Consider first the product C_2^3 : we have the following products:—

$$(a_1 a_6)^\wedge C_2^2, \quad (a_1 a_5)^\wedge C_2^2, \quad (a_1 a_4)^\wedge C_2^2, \quad (a_1 a_3)^\wedge C_2^2, \quad (a_1 a_2)^\wedge C_2^2.$$

Now any of these products is reducible if it is expressible in terms of products following it: thus for $(a_1 a_2)^\wedge C_2^2$ there is no such reduction, so that $\lambda \geq 2$, and the generating function for such products is $\frac{x^2 V_2}{1-x}$, where V_2 is the generating function of C_2^2 ; for $(a_1 a_3)^\wedge C_2^2$ there is a reduction corresponding to the interchange of a_2 and a_3 , so that $\lambda \geq 3$, and so on; finally, for $(a_1 a_6)^\wedge C_2^2$ there are reductions corresponding to the interchanges of a_6 with a_2 , a_3 , a_4 , a_5 respectively, and therefore $\lambda \geq 6$; hence the generating function for all these products is

$$\frac{x^2 + x^3 + x^4 + x^5 + x^6}{1-x} V_2 = \frac{x^2 - x^7}{(1-x)^2} V_2 = \frac{x^6(1-x)(1-x^3)(1-x^5)}{(1-x)^6}.$$

Assume that the generating function for C_2^{m-1} is

$$V_{m-1} = x^{2m-2} \frac{(1-x)(1-x^3)\dots(1-x^{2m-3})}{(1-x)^{2m-2}}.$$

Then the products C_2^m are

$$(a_1 a_{2m})^\wedge C_2^{m-1}, \quad (a_1 a_{2m-1})^\wedge C_2^{m-1}, \quad \dots, \quad (a_1 a_r)^\wedge C_2^{m-1}, \quad \dots, \quad (a_1 a_3)^\wedge C_2^{m-1}.$$

Here, as before, any product is reducible if it is expressible in terms of products following it: thus for $(a_1 a_r)^\wedge C_2^{m-1}$ there are $(r-2)$ reductions corresponding to the interchanges of a_r with a_2 , a_3 , ..., a_{r-1} respectively, and so $\lambda \geq r$; for, by Section III., these reductions are all independent; the

generating function for this set is $\frac{x^r}{1-x} V_{m-1}$, and therefore the generating function for all products C_2^m is

$$\begin{aligned} V_m &= \frac{x^3 + x^8 + \dots + x^{2m}}{1-x} V_{m-1} = \frac{(x^3 - x^{2m+1})}{(1-x)^2} V_{m-1} \\ &= \frac{x^{2m}(1-x)(1-x^8) \dots (1-x^{2m-1})}{(1-x)^{2m}}. \end{aligned}$$

Therefore, by induction, the generating function for products C_2^m is

$$\frac{x^{2m}(1-x)(1-x^8) \dots (1-x^{2m-1})}{(1-x)^{2m}}.$$

(2) *Generating Function for $C_2^m P_{\delta-2m}$ (where $P_{\delta-2m}$ is any form or product, other than C_3 or C_4 , which contains neither C_1 nor C_2).*

16. *Lemma.*—If the roots of $y^n - p_1 y^{n-1} + p_2 y^{n-2} - \dots + (-)^n p_n = 0$ are $1, x, x^3, \dots, x^{n-1}$, then

$$(1) \quad p_\kappa = x^{\binom{\kappa}{2}} \frac{(1-x^n)(1-x^{n-1}) \dots (1-x^{n-\kappa+1})}{(1-x)(1-x^3) \dots (1-x^\kappa)}.$$

$$\begin{aligned} (2) \quad p_n + (1-x)p_{n-2} + (1-x)(1-x^8)p_{n-4} + \dots \\ + (1-x)(1-x^8) \dots (1-x^{n-1}) &= 1, \quad n \text{ even}; \\ p_n + (1-x)p_{n-2} + (1-x)(1-x^8)p_{n-4} + \dots \\ + (1-x)(1-x^8) \dots (1-x^{n-2})p_1 &= 1, \quad n \text{ odd}. \end{aligned}$$

(1) Assuming the theorem true for all values up to and including n , we shall show that it is also true for the value $(n+1)$; it is obviously true when $n = 1, 2$.

$$\begin{aligned} \text{If } y^{n+1} - q_1 y^n + q_2 y^{n-1} + \dots + (-)^{n-1} q_{n+1} \\ &= (y^n - p_1 y^{n-1} + p_2 y^{n-2} + \dots + (-)^n p_n)(y - x^n), \\ q_\kappa &= p_\kappa + x^n p_{\kappa-1}, \quad \kappa = 1, 2, \dots, n, \\ q_{n+1} &= p_n x^n; \end{aligned}$$

and therefore, by hypothesis, if $\kappa = 1, 2, \dots, n$,

$$\begin{aligned} q_\kappa &= x^{\binom{\kappa}{2}} \frac{(1-x^n) \dots (1-x^{n-\kappa+1})}{(1-x) \dots (1-x^\kappa)} + x^{n+\binom{\kappa-1}{2}} \frac{(1-x^n) \dots (1-x^{n-\kappa+2})}{(1-x) \dots (1-x^{\kappa-1})} \\ &= x^{\binom{\kappa}{2}} \frac{(1-x^n) \dots (1-x^{n-\kappa+2})}{(1-x) \dots (1-x^\kappa)} \{1-x^{n-\kappa+1}+x^{n-\kappa+1}(1-x^\kappa)\} \\ &= x^{\binom{\kappa}{2}} \frac{(1-x^{n+1})(1-x^n) \dots (1-x^{n-\kappa+2})}{(1-x)(1-x^2) \dots (1-x^\kappa)}, \end{aligned}$$

and $q_{n+1} = x^n \cdot x^{\binom{n}{2}} = x^{\binom{n+1}{2}}.$

Therefore the result is true universally.

(2) Let the roots of

$$y^{n+1} - q_1 y^n + \dots + (-)^{n+1} q_{n+1} = 0 \quad \text{be } 1, x, x^2, \dots, x^n,$$

and those of

$$y^{n-1} - r_1 y^{n-2} + \dots + (-)^{n-1} r_{n-1} = 0 \quad \text{be } 1, x, x^2, \dots, x^{n-2}.$$

Assuming the result for the p 's and the r 's, it will be sufficient to prove it for the q 's; for brevity we consider only the case of n being even. Then

$$\begin{aligned} q_{n+1} + (1-x)q_{n-1} + (1-x)(1-x^3)q_{n-3} + \dots + (1-x)(1-x^3) \dots (1-x^{n-1})q_1 \\ = x^n p_n + (1-x) \{p_{n-1} + x^n p_{n-2}\} + \dots + (1-x)(1-x^3) \dots (1-x^{n-1})(p_1 + x^n), \end{aligned}$$

and
$$p_\kappa = x^{\binom{\kappa}{2}} \frac{(1-x^n) \dots (1-x^{n-\kappa+1})}{(1-x) \dots (1-x^\kappa)} = \frac{1-x^n}{1-x^{n-\kappa}} r_\kappa$$

by the first part of the Lemma. Therefore

$$\begin{aligned} q_{n+1} + (1-x)q_{n-1} + \dots + (1-x)(1-x^3) \dots (1-x^{n-1})q_1 \\ = x^n \{p_n + (1-x)p_{n-2} + \dots + (1-x)(1-x^3) \dots (1-x^{n-1})\} \\ + (1-x)p_{n-1} + (1-x)(1-x^3)p_{n-3} + \dots + (1-x)(1-x^3) \dots (1-x^{n-1})p_1 \\ = x^n + (1-x^n) \{r_{n-1} + (1-x)r_{n-3} + \dots + (1-x)(1-x^3) \dots (1-x^{n-3})r_1\} \\ = x^n + (1-x^n) \quad (\text{by hypothesis}) \\ = 1. \end{aligned}$$

17. Let V be the generating function of $P_{\delta-2m}$, $\delta = 2m + r$ being the total degree, and let the letters in C_2^m be

$$a_{\kappa_1}, a_{\kappa_2}, \dots, a_{\kappa_{2m}}, \quad \text{where } \kappa_1 < \kappa_2 < \kappa_3 \dots < \kappa_{2m}.$$

Then, by definition, the interchange of the letter $a_{\kappa_{2m}}$ with any one of the letters $a_{\kappa_{2m}+1}, a_{\kappa_{2m}+2}, \dots, a_{\kappa_s}$ implies a reduction, and, by Section III., corre-

sponding to each of these interchanges the weight of C_2^m is increased by unity. So in general, by interchanging $a_{\kappa_{2m-\epsilon}}$ with succeeding letters occurring in $P_{\delta-2m}$, the minimum weight for irreducibility is increased by $(\delta - \kappa_{2m-\epsilon} - \epsilon)$, since this is the possible number of interchanges. Here ϵ may be any one of $0, 1, 2, \dots, 2m-1$, and therefore, by interchanging all the letters of C_2^m with the letters of $P_{\delta-2m}$ succeeding, the weight is increased by

$$(\delta - \kappa_1) + (\delta - \kappa_2) + \dots + (\delta - \kappa_{2m}) - \binom{2m}{2}.$$

If the $2m$ letters of C_2^m are fixed, the generating function is

$$\frac{x^{2m}(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m-1}} V;$$

and therefore, if these $2m$ letters are chosen in all possible ways and interchanged with the letters of $P_{\delta-2m}$, the generating function required is

$$V \cdot x^{-\binom{2m}{2}} \frac{x^{2m}(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m-1}} \sum x^{(\delta-\kappa_1)+(\delta-\kappa_2)+\dots+(\delta-\kappa_{2m})},$$

where the Σ applies to all possible choices of the suffixes $\kappa_1, \kappa_2, \dots, \kappa_{2m}$ from the suffixes $1, 2, 3, \dots, \delta$.

The numbers $\delta - \kappa_1, \delta - \kappa_2, \dots, \delta - \kappa_{2m}$ are all different and each is one of the numbers $0, 1, 2, \dots, \delta - 1$; so, by the Lemma,

$$\sum x^{\delta-\kappa_1+\delta-\kappa_2+\dots+\delta-\kappa_{2m}} = p_{2m} = x^{\binom{2m}{2}} \frac{(1-x^\delta)\dots(1-x^{\delta-2m+1})}{(1-x)\dots(1-x^{2m})}.$$

Therefore the generating function for $C_2^n P_{\delta-2m}$ is

$$V \frac{x^{2m}(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m}} \frac{(1-x^\delta)(1-x^{\delta-1})\dots(1-x^{\delta-2m+1})}{(1-x)(1-x^2)\dots(1-x^{2m})},$$

where V is the generating function for $P_{\delta-2m}$ and $\delta = 2m + r$ is the total degree.

Note.—If $P_{\delta-2m}$ contains a factor C_3 or C_4 which is a Jacobian, any extra reductions due to a Jacobian transformation will be included in V , the generating function for $P_{\delta-2m}$.

(8) *Generating Function for $C_2^m C_3$.*

18. A different kind of reduction is introduced only in the case where C_3 is a Jacobian: three cases must be separately considered—

(a) C_3 is a Jacobian and does not contain a_1 : here there is a new reduction.

(β) C_3 is a Jacobian and contains a_1 , but not a_2 : here there is a new reduction if, and only if, C_3 is of the form (C_3, a_1) .

(γ) C_3 contains both a_1 and a_2 : here there is never any new reduction.

(a) If the product is irreducible, C_3 can be written in the form $(a_{r_1} a_{r_2})^\lambda (a_{r_1} a_{r_3})^\mu$, $\lambda \geq 4$, $\mu \geq 2$, by Section I., Cor. 2.

The number of reductions due to replacing the letters of C_3 by earlier letters contained in C_2^m is

$$(r_1 - 1) + (r_2 - 2) + (r_3 - 3) = (r_1 - 2) + (r_3 - 2) + (r_3 - 2),$$

and therefore the generating function for such products is

$$\begin{aligned} & \frac{x^6}{(1-x)^2} \frac{x^{2m}(1-x) \dots (1-x^{2m-1})}{(1-x)^{2m}} \sum_{r_1, r_3, r_3=2}^{2m+3} x^{(r_1-2)+(r_3-2)+(r_3-2)} \\ &= x^{2m+6} \frac{(1-x)(1-x^3) \dots (1-x^{2m-1})}{(1-x)^{2m+2}} \frac{x^3(1-x^{2m+2})(1-x^{2m+1})(1-x^{2m})}{(1-x)(1-x^3)(1-x^3)}, \end{aligned}$$

by the Lemma (§ 16).

(β) If the product is irreducible, C_3 can be written in the form

$$(a_1 a_{r_1})^\lambda (a_1 a_{r_2})^\mu, \quad \lambda \geq 3, \quad \mu \geq 1,$$

by Section I., § 1, and the number of reductions due to replacing a_{r_1} , a_{r_2} by earlier letters contained in C_2^m is

$$(r_1 - 2) + (r_2 - 3) = (r_1 - 3) + (r_2 - 3) + 1;$$

therefore the generating function for such products is

$$\begin{aligned} & \frac{x^5}{(1-x)^2} \frac{x^{2m}(1-x)(1-x^3) \dots (1-x^{2m-1})}{(1-x)^{2m}} \sum_{r_1, r_2=3}^{2m+3} x^{(r_1-3)+(r_2-3)} \\ &= x^{2m+5} \frac{(1-x)(1-x^3) \dots (1-x^{2m-1})}{(1-x)^{2m+2}} \frac{x(1-x^{2m+1})(1-x^{2m})}{(1-x)(1-x^3)}. \end{aligned}$$

(γ) Here it makes no difference to the irreducibility of the product whether C_3 is a Jacobian or not; so we may write

$$C_3 = (a_1 a_2)^\lambda (a_1 a_r)^\mu, \quad \lambda \geq 2, \quad \mu \geq 1;$$

and the number of reductions due to replacing a_r by earlier letters contained in C_2^m is $(r-3)$; and therefore the generating function for such products is

$$\begin{aligned} & \frac{x^3}{(1-x)^2} \frac{x^{2m}(1-x)(1-x^3) \dots (1-x^{2m-1})}{(1-x)^{2m}} \sum_{r=3}^{2m+3} x^{r-3} \\ &= x^{2m+3} \frac{(1-x)(1-x^3) \dots (1-x^{2m+1})}{(1-x)^{2m+3}}. \end{aligned}$$

So, finally, the generating function for all products $C_2^m C_3$ is

$$\frac{x^{2m+3}(1-x)(1-x^3)\dots(1-x^{2m+1})}{(1-x)^{2m+3}} \left\{ 1 + \frac{x^3(1-x^{2m})}{1-x^3} + x^6 \frac{(1-x^{2m})(1-x^{2m+2})}{(1-x^3)(1-x^6)} \right\}$$

$$= \frac{x^3}{(1-x)^{\delta-1}} \{ (1-x)(1-x^3)\dots(1-x^{\delta-4})p_3 + (1-x)(1-x^3)\dots(1-x^{\delta-2})p_1 \},$$

where $\delta = 2m+3$ is the total degree, and (as in the Lemma of § 16)

$$p_\kappa = x^{\binom{\kappa}{2}} \frac{(1-x^\delta)(1-x^{\delta-1})\dots(1-x^{\delta-\kappa+1})}{(1-x)(1-x^3)\dots(1-x^\kappa)}.$$

(4) *Generating Function for $C_2^m C_4$.*

19. A different kind of reduction from those considered in § 17 is introduced when C_4 is a Jacobian of the form (C_2, C_3) , and does not contain a_1 , and in no other case.

If C_4 does not contain a_1 , then, by Section I., if the product is irreducible, it can be written in the form

$$C_4 \equiv (a_{r_1} a_{r_2})^{\lambda_1} (a_{r_1} a_{r_3})^{\lambda_2} (a_{r_1} a_{r_4})^{\lambda_3}, \quad \text{where } \lambda_1 \geq 6, \lambda_2 \geq 3, \lambda_3 \geq 1,$$

and, in the same way as before, we find that the generating function for such products is

$$\frac{x^{10}}{(1-x)^8} \frac{x^{2m}(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m}} \sum x^{(r_1-1)+(r_2-2)+(r_3-3)+(r_4-4)}$$

$$= x^{2m+10} \frac{(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m+3}} \frac{x^4(1-x^{2m+3})\dots(1-x^{2m})}{(1-x)(1-x^3)(1-x^6)(1-x^4)}.$$

If C_4 does contain the letter a_1 , we write

$$C_4 \equiv (a_1 a_{r_1})^{\lambda_1} (a_1 a_{r_2})^{\lambda_2} (a_1 a_{r_3})^{\lambda_3}, \quad \text{where } \lambda_1 \geq 4, \lambda_2 \geq 2, \lambda_3 \geq 1;$$

and the generating function for such products is

$$\frac{x^7}{(1-x)^8} \frac{x^{2m}(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m}} \sum x^{(r_1-2)+(r_2-3)+(r_3-4)}$$

$$= x^{2m+7} \frac{(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m+3}} \frac{(1-x^{2m+3})(1-x^{2m+2})(1-x^{2m+1})}{(1-x)(1-x^3)(1-x^6)}.$$

Hence, adding and simplifying these two results, we find for the complete

generating function of products $C_2^m C_4$,

$$\frac{x^\delta}{(1-x)^{\delta-1}} \left\{ (1-x)(1-x^3) \dots (1-x^{\delta-5}) p_4 + (1-x)(1-x^3) \dots (1-x^{\delta-3}) p_2 \right. \\ \left. + (1-x)(1-x^3) \dots (1-x^{\delta-1}) \left(1 - \frac{1}{1-x} \right) \right\},$$

where $\delta = 2m + 4$ is the total degree, and p_2, p_4 are defined above (§ 16).

20. We are now in a position to find V_δ , the generating function for the types and those irreducible products which contain no factor of degree less than 3; for we know that $\frac{x^\delta}{(1-x)^{\delta-1}}$ is the generating function for the irreducible forms with no factor C_1 , and every such product is of the form $C_2^n P_{\delta-2n}$. We must consider two cases according as δ is even or odd.

δ even.—

$$\frac{x^\delta}{(1-x)^{\delta-1}} = V_\delta + \frac{x^2(1-x)}{(1-x)^2} \frac{(1-x^\delta)(1-x^{\delta-1})}{(1-x)(1-x^3)} V_{\delta-2} + \dots \\ + \frac{x^{2m}(1-x)(1-x^3) \dots (1-x^{2m-1})}{(1-x)^{2m}} \frac{(1-x^\delta)(1-x^{\delta-1}) \dots (1-x^{\delta-2m+1})}{(1-x)(1-x^3) \dots (1-x^{2m})} V_{\delta-2m} + \dots \\ + \frac{x^\delta}{(1-x)^{\delta-1}} \left\{ (1-x)(1-x^3) \dots (1-x^{\delta-5}) p_4 + (1-x)(1-x^3) \dots (1-x^{\delta-3}) p_2 \right. \\ \left. + (1-x)(1-x^3) \dots (1-x^{\delta-1}) \left(1 - \frac{1}{1-x} \right) \right\} \\ + x^\delta \frac{(1-x)(1-x^3) \dots (1-x^{\delta-1})}{(1-x)^\delta};$$

the last two terms being the generating functions for $C_2^n C_4$ and C_2^{n+2} respectively, where $\delta = 2n + 4$.

δ odd.—

$$\frac{x^\delta}{(1-x)^{\delta-1}} = V_\delta + \frac{x^3(1-x)}{(1-x)^2} \frac{(1-x^\delta)(1-x^{\delta-1})}{(1-x)(1-x^3)} V_{\delta-2} + \dots \\ + \frac{x^{2m}(1-x)(1-x^3) \dots (1-x^{2m-1})}{(1-x)^{2m}} \frac{(1-x^\delta)(1-x^{\delta-1}) \dots (1-x^{\delta-2m+1})}{(1-x)(1-x^3) \dots (1-x^{2m})} V_{\delta-2m} + \dots \\ + \frac{x^\delta}{(1-x)^{\delta-1}} \left\{ (1-x)(1-x^3) \dots (1-x^{\delta-4}) p_3 + (1-x)(1-x^3) \dots (1-x^{\delta-2}) p_1 \right\};$$

the last term being the generating function for $C_2^n C_3$, where $\delta = 2n + 3$.

Now assume that, for $r < \delta$ and > 4 ,

$$V_r = \frac{x^{\binom{\delta}{1} + \binom{r}{2}}}{(1-x)^{r-1}},$$

and it is easily verified, from the two results of the preceding Lemma (§ 16), that, whether δ is even or odd,

$$V_\delta = \frac{x^{\binom{\delta}{1} + \binom{\delta}{2}}}{(1-x)^{\delta-1}}.$$

Hence the result is established in general.

VI. PRODUCTS OF THE KINDS (1) C_κ^2 ; (2) $C_\kappa C_m$, $\kappa < m < 2\kappa$; (3) $C_\kappa C_{2\kappa}$.

21. The general principle involved by the present application of Stroh's syzygies of degree 4 may be explained thus: Suppose we are seeking for the syzygies in which a product $C_\kappa C_m$ occurs. Let us divide up the letters of C_κ into two sets,

$$a_1, a_2, \dots, a_{r_1}, \quad b_1, b_2, \dots, b_{r_2};$$

and those of C_m into two sets,

$$c_1, c_2, \dots, c_{s_1}, \quad d_1, d_2, \dots, d_{s_2}.$$

Let $\alpha, \beta, \gamma, \delta$ be covariants, whose symbolical characters are comprised in the sets $a_1, a_2, \dots, a_{r_1} : b_1, b_2, \dots, b_{r_2} : c_1, c_2, \dots, c_{s_1} : d_1, d_2, \dots, d_{s_2}$ respectively. Then C_κ can be expressed in terms of transvectants of the form $(\alpha, \beta)^\lambda$, and C_m can be expressed in terms of transvectants of the form $(\gamma, \delta)^\mu$.

Consider the Stroh syzygy

$$\{(\alpha\beta) + (\gamma\delta)\}^\omega - \{(\alpha\delta) + (\gamma\beta)\}^\omega = 0,$$

which may be written $e^{(\alpha\beta)D_\delta}(\gamma\delta)^\omega = e^{(\alpha\delta)D_\beta}(\gamma\beta)^\omega$,

where

$$D_\beta = x_1 \frac{\partial}{\partial \beta_2} - x_2 \frac{\partial}{\partial \beta_1},$$

$$D_\delta = x_1 \frac{\partial}{\partial \delta_2} - x_2 \frac{\partial}{\partial \delta_1}.$$

Now let us suppose $(\alpha\delta)^\rho(\gamma\beta)^\sigma$ is a product which, by the definition of Section III., succeeds the product $(\alpha\beta)^\rho(\gamma\delta)^\sigma$ in the sequence of products: then we have the relation

$$e^{(\alpha\beta)D_\delta}(\gamma\delta)^\omega = \text{reducible terms}$$

or

$$e^{(\alpha\beta)D_\delta}(c_1, c_2, \dots, c_{s_1}, \delta) = \text{reducible terms},$$

where $(c_1, c_2, \dots, c_{s_1}, \delta)$ is the symbolical product obtained by replacing γ by the letters of which it is composed. Moreover, since $c_1, c_2, \dots, c_{s_1}, \delta$ are the only letters involved in this symbolical product, we know that

$$(D_{c_1} + D_{c_2} + \dots + D_{c_{s_1}} + D_\delta)(c_1, c_2, \dots, c_{s_1}, \delta) = 0.$$

Therefore we may write the relation

$$\left. \begin{aligned} e^{-(\alpha\beta)\{D_{c_1} + D_{c_2} + \dots + D_{c_{s_1}}\}} C_m &= \text{reducible terms} \\ \text{or} \quad e^{(\alpha\beta)\{D_{d_1} + D_{d_2} + \dots + D_{d_{s_2}}\}} C_m &= \text{reducible terms} \end{aligned} \right\} \quad (1)$$

This relation gives us a reduction for the product $C_\kappa C_m$ by virtue of the fact that the product

$$(a_1, a_2, \dots, a_{r_1}, b_1, b_2, \dots, b_{r_2})(c_1, c_2, \dots, c_{s_1}, d_1, d_2, \dots, d_{s_2})$$

precedes, in our fixed arrangement, the product

$$(a_1, a_2, \dots, a_{r_1}, d_1, d_2, \dots, d_{s_2})(b_1, b_2, \dots, b_{r_2}, c_1, c_2, \dots, c_{s_1}).$$

In this way we obtain a series of relations such as (1): such relations have been considered in Section III. [§ 10, (iii.)], and have there been shewn to be all independent. It follows, therefore, that we obtain one reduction for $C_\kappa C_m$ corresponding to every product $C_{\kappa'} C_{m'}$ following $C_\kappa C_m$ in the sequence of products already determined, provided that each of $C_{\kappa'}$, $C_{m'}$ contains letters from each of the factors C_κ , C_m ; and for every such reduction the minimum weight of C_κ for irreducibility is increased by unity.

(i.) *Generating Function for C_κ^2 .*

22. Let

$$\begin{aligned} C_\kappa^2 &\equiv (a_1 a_{r_1})^{\lambda_1} (a_1 a_{r_2})^{\lambda_2} \dots (a_1 a_{r_{\kappa-1}})^{\lambda_{\kappa-1}} (b_1 b_2)^{\mu_1} (b_1 b_3)^{\mu_2} \dots (b_1 b_\kappa)^{\mu_{\kappa-1}}, \\ &\equiv C_\kappa^{(1)} C_\kappa^{(2)}, \end{aligned}$$

where $r_1 < r_2 < \dots < r_{\kappa-1}$, and the letters involved are $a_1, a_2, \dots, a_{2\kappa}$ in some order. If either $C_\kappa^{(1)}$ or $C_\kappa^{(2)}$ is a Jacobian, the product is reducible, since it is expressible in terms of products $C_{\kappa-e} C_{\kappa+e}$: hence, by the second Corollary of Section I., each of $C_\kappa^{(1)}$ or $C_\kappa^{(2)}$ is of weight $(2\kappa - 2)$ at least.

In the first place we suppose the letters of $C_\kappa^{(1)}$ and of $C_\kappa^{(2)}$ to be fixed and we disregard their arrangement: we then find how many reductions are possible by virtue of expressing $C_\kappa^{(1)} C_\kappa^{(2)}$ in the form $\Sigma C_{\kappa-e} C_{\kappa+e}$. To this end we consider all possible products $C_r C_{2\kappa-r}$, such that r takes any value from 2 to $\kappa - 1$, where each of C_r , $C_{2\kappa-r}$ contains letters from each of the factors $C_\kappa^{(1)}$, $C_\kappa^{(2)}$: then we know that, corresponding to every such product, there is a reduction for $C_\kappa^{(1)} C_\kappa^{(2)}$, and that these reductions are all

independent; so that, corresponding to each such product, the minimum weight of $C_\kappa^{(1)}$ for the irreducibility of $C_\kappa^{(1)} C_\kappa^{(2)}$ is increased by unity.

The letters of C_r may be chosen in $\binom{2\kappa}{r}$ ways, but we must exclude every case where these r letters are all included among the letters of $C_\kappa^{(1)}$ or among those of $C_\kappa^{(2)}$, for in such cases there is no corresponding reduction of the product $C_\kappa^{(1)} C_\kappa^{(2)}$; hence the number of choices for $C_r C_{2\kappa-r}$ is

$$\binom{2\kappa}{r} - \binom{\kappa}{r} - \binom{\kappa}{r}.$$

So the total number of reductions of this kind for $C_\kappa^{(1)} C_\kappa^{(2)}$ is

$$\begin{aligned} \sum_{r=2}^{r=\kappa-1} \left\{ \binom{2\kappa}{r} - 2 \binom{\kappa}{r} \right\} &= \frac{1}{2} \left[2^{2\kappa} - 4\kappa - 2 - \binom{2\kappa}{\kappa} \right] - 2[2^\kappa - \kappa - 2] \\ &= 2^{2\kappa-1} - 2^{\kappa+1} - \frac{1}{2} \binom{2\kappa}{\kappa} + 3. \end{aligned}$$

Further, by the Jacobian identities, which are independent of these reductions, the minimum weight of $C_\kappa^{(1)}$ is $2^\kappa - 2$. So, whatever are the letters of $C_\kappa^{(1)}$, the minimum weight of $C_\kappa^{(1)}$ to ensure the irreducibility of $C_\kappa^{(1)} C_\kappa^{(2)}$ is

$$2^{2\kappa-1} - 2^\kappa - \frac{1}{2} \binom{2\kappa}{\kappa} + 1.$$

It remains to discover how many products $P_1 P_2$ follow $C_\kappa^{(1)} C_\kappa^{(2)}$ in the sequence of products already determined, where P_1 and P_2 are each of degree κ , and P_1 is taken as that factor which contains a_1 .

If $r_1 > 2$, there is a reduction corresponding to every product $P_1 P_2$, where P_1 contains a_2 , and the number of such products is $\binom{2\kappa-2}{\kappa-2}$.

If $r_1 > 3$, there is a reduction corresponding to every product $P_1 P_2$, where P_1 contains a_3 and not a_2 , and the number of such products is $\binom{2\kappa-3}{\kappa-2}$.

Continuing in this way, we obtain

$$\binom{2\kappa-2}{\kappa-2} + \binom{2\kappa-3}{\kappa-2} + \dots + \binom{2\kappa-r_1+1}{\kappa-2} = \binom{2\kappa-1}{\kappa-1} - \binom{2\kappa-r_1+1}{\kappa-1}$$

reductions.

If we have any further reductions, P_1 must contain both a_1 and a_{r_1} , and we get a reduction if P_1 contains also a_{r_1+1} where $r_1+1 < r_2$; in

this way we have

$$\binom{2\kappa-r_1-1}{\kappa-3} + \binom{2\kappa-r_1-2}{\kappa-3} + \dots + \binom{2\kappa-r_2+1}{\kappa-3} = \binom{2\kappa-r_1}{\kappa-2} - \binom{2\kappa-r_2+1}{\kappa-2}$$

reductions.

The same argument is pursued for each of the letters of $C_\kappa^{(1)}$ in turn: hence the number of reductions, corresponding to the products $C_\kappa^2 = P_1 P_2$, which follow $C_\kappa^{(1)} C_\kappa^{(2)}$ in the determined sequence is

$$\begin{aligned} & \binom{2\kappa-1}{\kappa-1} - \binom{2\kappa-r_1+1}{\kappa-1} + \binom{2\kappa-r_1}{\kappa-2} - \binom{2\kappa-r_2+1}{\kappa-2} + \dots - \binom{2\kappa-r_{\kappa-1}+1}{1} \\ &= \binom{2\kappa-1}{\kappa-1} - \binom{2\kappa-r_1}{\kappa-1} - \binom{2\kappa-r_2}{\kappa-2} - \binom{2\kappa-r_3}{\kappa-3} - \dots - \binom{2\kappa-r_{\kappa-1}}{1} - 1. \end{aligned}$$

Hence, if the letters of $C_\kappa^{(1)}$ are $a_1, a_{r_1}, \dots, a_{r_{\kappa-1}}$, its minimum weight is

$$2^{2\kappa-1} - 2^\kappa - \left[\binom{2\kappa-r_1}{\kappa-1} + \binom{2\kappa-r_2}{\kappa-2} + \dots + \binom{2\kappa-r_{\kappa-1}}{1} \right],$$

and, since the minimum weight of $C_\kappa^{(2)}$ is $(2^\kappa - 2)$, the generating function for the irreducible products C_κ^2 is

$$\frac{x^{2^{2\kappa-1}-2}}{(1-x)^{2^\kappa-2}} \sum_{r_1, r_2, \dots, r_{\kappa-1}} x^{-\binom{2\kappa-r_1}{\kappa-1} - \binom{2\kappa-r_2}{\kappa-2} - \dots - \binom{2\kappa-r_{\kappa-1}}{1}},$$

where $r_1, r_2, \dots, r_{\kappa-1}$ take all values satisfying

$$1 < r_1 < r_2 < \dots < r_{\kappa-1} < 2\kappa + 1.$$

We proceed to show that

$$S \equiv \sum_{r_1, r_2, \dots, r_{\kappa-1}} x^{-\binom{2\kappa-r_1}{\kappa-1} - \binom{2\kappa-r_2}{\kappa-2} - \dots - \binom{2\kappa-r_{\kappa-1}}{1}} = \frac{1-x^{-\binom{2\kappa-1}{\kappa-1}}}{1-x^{-1}}.$$

First find the sum for the possible values $r_{\kappa-2}+1, r_{\kappa-2}+2, \dots, 2\kappa$ of $r_{\kappa-1}$; we have

$$\sum_{r_{\kappa-1}} x^{-\binom{2\kappa-r_{\kappa-1}}{1}} = \frac{1-x^{-\binom{2\kappa-r_{\kappa-2}}{1}}}{1-x^{-1}};$$

and therefore

$$S \equiv \frac{1}{1-x^{-1}} \left\{ \sum_{r_1, r_2, \dots, r_{\kappa-2}} \left(x^{-\binom{2\kappa-r_1}{\kappa-1} - \dots - \binom{2\kappa-r_{\kappa-2}}{2}} - x^{-\binom{2\kappa-r_1}{\kappa-1} - \dots - \binom{2\kappa-r_{\kappa-2}}{2} - \binom{2\kappa-r_{\kappa-2}+1}{2}} \right) \right\};$$

$$\text{also } \sum_{r_{\kappa-2}=r_{\kappa-3}+1}^{2\kappa-1} \left\{ x^{-\binom{2\kappa-r_{\kappa-2}}{2}} - x^{-\binom{2\kappa-r_{\kappa-2}+1}{2}} \right\} = 1 - x^{-\binom{2\kappa-r_{\kappa-3}}{2}};$$

and therefore

$$S \equiv \frac{1}{1-x^{-1}} \left\{ \sum_{r_1, r_2, \dots, r_{\kappa-3}} \left(x^{-(\frac{2\kappa-r_1}{\kappa-1}) - \dots - (\frac{2\kappa-r_{\kappa-3}}{3})} - x^{-(\frac{2\kappa-r_1}{\kappa-1}) - \dots - (\frac{2\kappa-r_{\kappa-3}+1}{3})} \right) \right\} \\ \equiv \frac{1}{1-x^{-1}} \sum_{r_1, r_2, \dots, r_{\kappa-2}} \left\{ x^{-(\frac{2\kappa-r_1}{\kappa-1}) - \dots - (\frac{2\kappa-r_{\kappa-2}}{\epsilon})} - x^{-(\frac{2\kappa-r_1}{\kappa-1}) - \dots - (\frac{2\kappa-r_{\kappa-2}+1}{\epsilon})} \right\},$$

by repeated application of the same method.

Hence, finally,

$$S \equiv \frac{1}{1-x^{-1}} \sum_{r_1=2}^{r_1=\kappa+2} \left\{ x^{-(\frac{2\kappa-r_1}{\kappa-1})} - x^{-(\frac{2\kappa-r_1+1}{\kappa-1})} \right\} = \frac{1}{1-x^{-1}} \left\{ 1 - x^{-(\frac{2\kappa-1}{\kappa-1})} \right\}.$$

So the generating function for C_κ^2 is

$$\frac{x^{2^{2\kappa-1}-2}}{(1-x)^{2^{2\kappa-2}}} \frac{1-x^{-(\frac{2\kappa-1}{\kappa-1})}}{1-x^{-1}} = \frac{x^{2^{2\kappa-1}-1-\frac{1}{2}(2\kappa)}}{(1-x)^{2^{2\kappa-1}}} \left\{ 1 - x^{\frac{1}{2}(2\kappa)} \right\},$$

since

$$\binom{2\kappa-1}{\kappa-1} = \binom{2\kappa-1}{\kappa} = \frac{1}{2} \binom{2\kappa}{\kappa}.$$

(ii.) *Generating Function for $C_\kappa C_m$, where $\kappa < m < 2\kappa$.*

23. In the first place we suppose the letters of C_κ and C_m to be fixed, and we disregard their arrangement; we seek for products $C_r C_s$ ($r+s=m+\kappa$), such that r takes all values from 2 to $\kappa-1$, where each of $C_r C_s$ contains letters from each of $C_\kappa C_m$; then, as before, for every such product the minimum weight of C_κ , for the irreducibility of the product $C_\kappa C_m$, is increased by unity.

The letters of C_r may be chosen in $\binom{\kappa+m}{r}$ ways; but we must exclude the cases where these r letters are all included among the letters of C_κ or among those of C_m ; hence the total number of such products and therefore of reductions is

$$\sum_{r=2}^{r=\kappa-1} \left\{ \binom{\kappa+m}{r} - \binom{\kappa}{r} - \binom{m}{r} \right\} = \sum_{r=1}^{r=\kappa-1} \left\{ \binom{m+\kappa}{r} - \binom{m}{r} \right\} - (2^\kappa - 2).$$

Independently of these reductions the minimum weight of C_κ must be $2^\kappa - 2$, for, if C_κ is a Jacobian, the product $C_\kappa C_m$ is certainly reducible; hence the minimum weight of C_κ for irreducibility is

$$\sum_{r=1}^{r=\kappa-1} \left\{ \binom{m+\kappa}{r} - \binom{m}{r} \right\}.$$

Again, if C_m is a Jacobian of the form (C_l, C_{m-l}) , then

$$C_\kappa C_m = (C_l, C_\kappa) C_{m-l} - (C_{m-l}, C_\kappa) C_l,$$

and this is a reduction if both l and $m-l$ are less than κ ($m < 2\kappa$). So l may have the values $\kappa-1, \kappa-2, \dots, m-\kappa+1$ (it is to be remarked that the values $\theta, m-\theta$ of l lead to the same result); now the letters of C_m can be chosen for C_l in $\binom{m}{l}$ ways, so that the number of ways in which C_m can be chosen to give rise to a Jacobian reduction, whatever be the letters of C_κ , is

$$\begin{aligned} \frac{1}{2} \left\{ \binom{m}{\kappa-1} + \binom{m}{\kappa-2} + \dots + \binom{m}{m-\kappa+1} \right\} \\ = \frac{1}{2} \{2^m - 2\} - \left\{ \binom{m}{\kappa} + \binom{m}{\kappa+1} + \dots + \binom{m}{m-1} \right\}. \end{aligned}$$

Moreover, by the perpetuant type theorem, the minimum weight of C_m is $2^{m-1} - 1$, and all these reductions are independent. Hence, without regarding the arrangement of the letters, we see that the minimum weight of the product $C_\kappa C_m$ for irreducibility is

$$\begin{aligned} \sum_{r=1}^{r=\kappa-1} \left\{ \binom{m+\kappa}{r} - \binom{m}{r} \right\} + 2^m - 2 - \sum_{r=\kappa}^{r=m-1} \binom{m}{r} \\ = \sum_{r=1}^{r=\kappa-1} \binom{m+\kappa}{r} + 2^m - 2 - (2^m - 2) \\ = \sum_{r=1}^{r=\kappa-1} \binom{m+\kappa}{r}. \end{aligned}$$

In considering the arrangement of the letters, there is one reduction corresponding to each product $C'_\kappa C'_m$ which follows in the sequence of products the product $C_\kappa C_m$ which we are considering: this is obvious if C'_κ contain letters from C_κ and C_m . If, however, C'_κ contain only letters from C_m , there is still one reduction; for, if the product $C_\kappa C_m$ is irreducible, C_m must not be expressible in terms of a Jacobian of which C'_κ is one of the forms.

Now the total number of forms $C_\kappa C_m$ is $\binom{m+\kappa}{\kappa}$, and the extra reductions due to the arrangements of the letters number in the various cases

$$0, 1, 2, \dots, \binom{m+\kappa}{\kappa} - 1, \text{ respectively;}$$

hence the generating function for $C_\kappa C_m$ is

$$\frac{x^{\binom{m+\kappa}{\kappa-1} + \binom{m+\kappa}{\kappa-2} + \dots + \binom{m+\kappa}{1}} \{1 + x + x^2 + \dots + x^{\binom{m+\kappa}{\kappa}-1}\}}{(1-x)^{m+\kappa-2}} \\ = \frac{x^{\binom{m+\kappa}{\kappa-1} + \binom{m+\kappa}{\kappa-2} + \dots + \binom{m+\kappa}{1}} \{1 - x^{\binom{m+\kappa}{\kappa}}\}}{(1-x)^{m+\kappa-1}}.$$

This result proves that the generating function for products of degree δ having no factor of degree less than $(\kappa+1)$, provided $\kappa \geq \frac{1}{3}\delta$ (and of course $\leq \frac{1}{2}\delta$), is

$$\frac{x^{\binom{\delta}{1} + \binom{\delta}{2} + \dots + \binom{\delta}{\kappa}}}{(1-x)^{\delta-1}}$$

(see also Section V., § 13).

The case of $\kappa = \frac{1}{2}\delta$ requires a word of explanation: the irreducible forms considered in this case are the perpetuant types of degree δ ; their generating function is known to be

$$\frac{x^{2^{\delta-1}-1}}{(1-x)^{\delta-1}},$$

and this has been proved to be exact;* when δ is odd and $= 2n+1$, the generating function as given by the preceding result is

$$\frac{x^{\binom{2n+1}{1} + \dots + \binom{2n+1}{n-1} + \binom{2n+1}{n}}}{(1-x)^{2n}} = \frac{x^{2^{\delta-1}-1}}{(1-x)^{\delta-1}}.$$

A slight exception occurs in the case where δ is even and $= 2n$; here the generating function is

$$\frac{x^{2^{\delta-1}-1}}{(1-x)^{\delta-1}} = \frac{x^{\binom{2n}{1} + \binom{2n}{2} + \dots + \binom{2n}{n-1} + \frac{1}{2}\binom{2n}{n}}}{(1-x)^{2n-1}}.$$

(iii.) *Generating Function for $C_\kappa C_{2\kappa}$.*

24. Similar methods are applied in the investigation of products $C_\kappa C_{2\kappa}$, but the resulting generating function does not in general admit of ex-

* Wood, "On the Irreducibility of Perpetuant Types," *Proc. London Math. Soc.*, Ser. 2, Vol. 2.

pression in a simple form; in the special case of products $C_8 C_6$ we shall evaluate the generating function for the purpose of the subsequent discussion of syzygies of degree 9.

The number of reductions possible, where the letters of $C_\kappa C_{2\kappa}$ are fixed, is

$$\sum_{r=2}^{r=\kappa-1} \left\{ \binom{3\kappa}{r} - \binom{2\kappa}{r} - \binom{\kappa}{r} \right\} + 2^\kappa - 2 + 2^{2\kappa-1} - 1;$$

for C_κ must not be a Jacobian, and the minimum weight of $C_{2\kappa}$ is $2^{2\kappa-1} - 1$. Hence the minimum weight is, for a fixed choice of letters,

$$\begin{aligned} & \sum_{r=1}^{r=\kappa-1} \binom{3\kappa}{r} + \frac{1}{2} \binom{2\kappa}{\kappa} - \frac{1}{2} (2^{2\kappa} - 2) - (2^\kappa - 2) + 2^\kappa - 2 + 2^{2\kappa-1} - 1 \\ &= \sum_{r=1}^{r=\kappa-1} \binom{3\kappa}{r} + \frac{1}{2} \binom{2\kappa}{\kappa}. \end{aligned}$$

Now suppose that the letters of C_κ are $a_{r_1}, a_{r_2}, \dots, a_{r_\kappa}$, where

$$1 < r_1 < r_2 \dots < r_\kappa;$$

then the number of products $C'_\kappa C'_{2\kappa}$, such that C'_κ contains letters of C_κ and of $C_{2\kappa}$, which follow the product $C_\kappa C_{2\kappa}$ in the sequence already determined is easily seen to be

$$N \equiv \binom{3\kappa-r_1}{\kappa} - \binom{2\kappa-r_1+1}{\kappa} + \binom{3\kappa-r_2}{\kappa-1} + \dots + \binom{3\kappa-r_\kappa}{1};$$

and, as before, we can show that

$$\sum_{r_1, r_2, \dots, r_\kappa} x^N = \sum_{r_1} x^{\binom{3\kappa-r_1}{\kappa} - \binom{2\kappa-r_1+1}{\kappa}} \frac{1 - x^{\binom{3\kappa-r_1}{\kappa-1}}}{1-x};$$

here r_1 may have any value from 2 to $2\kappa+1$.

If C_κ contains a_1 (corresponding to $r_1 = 1$), we get a term

$$x^{\binom{3\kappa-1}{\kappa} - \frac{1}{2} \binom{2\kappa}{\kappa}} \frac{1 - x^{\binom{3\kappa-1}{\kappa-1}}}{1-x};$$

where the extra index $\frac{1}{2} \binom{2\kappa}{\kappa}$ arises from the fact that, if $C_{2\kappa}$ does not contain a_1 and can be expressed as a Jacobian (C_κ, C'_κ), there is a reduction,

Hence the generating function for irreducible products $C_\kappa C_{2\kappa}$ is

$$\frac{x^{\binom{3\kappa}{1} + \binom{3\kappa}{2} + \dots + \binom{3\kappa}{\kappa-1} + \frac{1}{2} \binom{2\kappa}{\kappa}}{(1-x)^{3\kappa-1}} \left\{ x^{\binom{3\kappa-1}{\kappa} - \frac{1}{2} \binom{2\kappa}{\kappa}} \left(1 - x^{\binom{3\kappa-1}{\kappa-1}} \right) \right. \\ \left. + \sum_{r=2}^{\kappa-2\kappa+1} x^{\binom{3\kappa-r}{\kappa} - \binom{2\kappa-r+1}{\kappa}} \left(1 - x^{\binom{3\kappa-r}{\kappa-1}} \right) \right\}.$$

Putting $\kappa = 3$, we find for the irreducible products $C_3 C_6$ of degree 9 the generating function

$$\frac{x^{55} (1 - x^{10} + x^9 - x^{19} + x^{16} - x^{31} + x^{25} - x^{74})}{(1-x)^9}.$$

Summary of Results giving Generating Functions for the Enumeration of certain Classes of Irreducible Products of total Degree δ .

Products.	Generating Functions.
All products containing no factor C_1 (§ 14)	$\frac{x^\delta}{(1-x)^{\delta-1}}$
C_2^n (where $\delta = 2n$) (§ 15)	$\frac{x^\delta (1-x^3)(1-x^5) \dots (1-x^{\delta-1})}{(1-x)^{\delta-1}}$
$C_2^m C_3$ (where $\delta = 2m+3$) (§ 18)	$\frac{x^\delta (1-x^3)(1-x^5) \dots (1-x^{\delta-2})}{(1-x)^{\delta-1}} \left\{ 1 + \frac{x^3(1-x^{\delta-3})}{1-x^2} + x^6 \frac{(1-x^{\delta-3})(1-x^{\delta-1})}{(1-x^2)(1-x^3)} \right\}$
$C_2^m C_4$ (where $\delta = 2m+4$) (§ 19)	$\frac{x^{\delta+3} (1-x^3)(1-x^5) \dots (1-x^{\delta-5})(1-x^{\delta-3})(1-x^{\delta-2})(1-x^{\delta-1})}{(1-x)^{\delta-1} (1-x^2)(1-x^3)} \times \left\{ 1 + \frac{x^7(1-x^{\delta-4})}{(1-x^4)} \right\}$
$C_2^m P_{\delta-2m}$ ($P_{\delta-2m}$ contains no factor of degree less than 3 and is neither C_3 nor C_4) (§ 17)	$\frac{x^{2m} (1-x^3)(1-x^5) \dots (1-x^{2m-1})}{(1-x)^{2m-1}} \frac{(1-x^3)(1-x^5) \dots (1-x^{\delta-2m-1})}{(1-x)(1-x^2) \dots (1-x^{2m})} V$ (where V is the generating function of $P_{\delta-2m}$)
All products containing no factor of degree less than 3 (§ 20)	$\frac{x^{\binom{\delta}{1} + \binom{\delta}{2}}}{(1-x)^{\delta-1}}$
C_κ^2 (where $\delta = 2\kappa$) (§ 22)	$\frac{x^{\binom{\delta-1}{1} - 1 - \frac{1}{2} \binom{\delta}{2}} \left(1 - x^{\binom{\delta}{2}} \right)}{(1-x)^{\delta-1}}$
$C_\kappa C_m$ (where $\delta = \kappa + m$, $\kappa < m < 2\kappa$) (§ 23)	$\frac{x^{\binom{\delta}{1} + \binom{\delta}{2} + \dots + \binom{\delta}{\kappa-1}} \left(1 - x^{\binom{\delta}{\kappa}} \right)}{(1-x)^{\delta-1}}$

VII. SYZYGIES OF DEGREES 6, 7, AND 8.—NOTE ON SYZYGIES OF DEGREE 9.

25. We are now in a position to write down from the preceding general results the generating functions for the enumeration of the irreducible products of degrees 6, 7, and 8 respectively.

Degree 6.

	Products.	Generating Functions.	How obtained.
Type forms :—	C_6	$\frac{x^{31}}{(1-x)^5}$	Perpetuant type theorem
Products :—	C_3^2	$\frac{x^{21}-x^{31}}{(1-x)^5}$	C_{κ}^2 , $\kappa = 3$ (Section VI., § 22)
	$C_2 C_4$	$\frac{x^9+x^{11}-x^{14}-x^{21}}{(1-x)^5}$	$C_2^m C_4$, $m = 1$ (Section V., § 19)
	C_2^3	$\frac{x^6-x^9-x^{11}+x^{14}}{(1-x)^5}$	C_2^m , $m = 3$ (Section V., § 15)
	$C_1 C_5$	$\frac{6x^{15}}{(1-x)^4}$	Perpetuant type theorem
	$C_1 C_2 C_3$	$\frac{6(x^5-x^{15})}{(1-x)^4}$	$C_2 C_3$, $m = 1$ (Section V., § 18)
	$C_1^2 C_4$	$\frac{15x^7}{(1-x)^3}$	Perpetuant type theorem
	$C_1^2 C_2^2$	$\frac{15(x^4-x^7)}{(1-x)^3}$	C_2^m , $m = 2$ (Section V., § 15)
	$C_1^3 C_3$	$\frac{20x^3}{(1-x)^3}$	Perpetuant type theorem
	$C_1^4 C_2$	$\frac{15x}{1-x} - 10x$	$C_1^{n-2} C_2$, $n = 6$ (Section V., § 14)
	C_1^6	1	

Also—

(i.) Sum of generating functions for forms C_6 , C_3^2 is $x^{21}/(1-x)^5$ (see Section V., § 13).

(ii.) Sum of generating functions for forms C_6 , C_3^2 , $C_2 C_4$, C_2^3 is $x^6/(1-x)^5$ (see Section V., § 13).

(iii.) Sum of all generating functions is $1/(1-x)^6$, so that all the enumeration of the irreducible forms is exact and all the syzygies for degree 6 have been identified.

One point calls for comment: in reducing the products $C_2 C_4$, if $C_2 = (a_5 a_6)^\nu$, the only restriction on ν is (Section V., § 19) $\nu \geq 2$. Now consider the form

$$C_2 C_4 = (a_5 a_6)^2 \{ (a_1 a_2)^3, (a_3 a_4)^3 \},$$

which arises in products of weight 9. By the ordinary Jacobian identity we have

$$(a_5 a_6)^2 \{ (a_1 a_2)^3, (a_3 a_4)^3 \} \equiv (a_1 a_2)^3 \{ (a_5 a_6)^2, (a_3 a_4)^3 \} - (a_3 a_4)^3 \{ (a_5 a_6)^2, (a_1 a_2)^3 \}.$$

Now each of the Jacobians $\{ (a_5 a_6)^2, (a_3 a_4)^3 \}$, $\{ (a_5 a_6)^2, (a_1 a_2)^3 \}$ is of degree 4 and weight 6, and therefore is expressible as a sum of product forms. Hence the product $(a_5 a_6)^2 \{ (a_1 a_2)^3, (a_3 a_4)^3 \}$ is reducible: this is not in opposition to the general result, as we can shew that the Jacobian $\{ (a_1 a_2)^3, (a_3 a_4)^3 \}$, although of degree 4 and weight 7, is itself reducible. For, by the perpetuant type theorem,

$$\begin{aligned} & \{ (a_1 a_2)^3, (a_3 a_4)^3 \} \\ & \equiv \kappa (a_1 a_2)^4 (a_2 a_3)^3 (a_3 a_4) + \text{product forms} \\ & \quad \kappa \{ (a_1 a_2)^4, (a_3 a_4) \}^2 + \lambda \{ (a_1 a_2)^5, (a_3 a_4) \} + \mu \{ (a_1 a_2)^4, (a_3 a_4)^2 \} \\ & \quad + \text{product forms,} \end{aligned}$$

and every term of $\{ (a_1 a_2)^5, (a_3 a_4) \}$ is a product form. Hence

$$\{ (a_1 a_2)^3, (a_3 a_4)^3 \} \equiv \kappa \{ (a_1 a_2)^4, (a_3 a_4) \}^2 + \mu \{ (a_1 a_2)^4, (a_3 a_4)^2 \} + \text{product forms.}$$

In this result interchange a_1 and a_2 and subtract; then

$$2 \{ (a_1 a_2)^3, (a_3 a_4)^3 \} = \text{product forms};$$

and therefore the Jacobian is reducible.

In the treatment of product forms of degree 3κ ($\kappa \geq 3$) no such difficulty can arise, for, although we have the analogous result

$$C'_\kappa(C_\kappa, \bar{C}_\kappa) = \Sigma (\text{product of three factors}),$$

where C'_κ is of weight $2^{2\kappa-2}-2$, and each of $C_\kappa, \bar{C}_\kappa$ is of weight $2^{2\kappa-2}-1$, it will be seen that the minimum weight, as determined in Section V., § 24, of the factor $C'_\kappa C_{2\kappa}$ is in all cases greater than $2^{2\kappa-2}-2$; so that there is no occasion to investigate the reducibility of the Jacobian form $(C_\kappa, \bar{C}_\kappa)$ of weight $2^{2\kappa-1}-1$ and degree 2κ (see § 24).

Degree 7.

26.

	Products.	Generating Functions.	How obtained.
Type forms:—	C_7	$\frac{x^{68}}{(1-x)^6}$	Perpetuant type theorem
Products:—	$C_3 C_4$	$\frac{x^{28} - x^{68}}{(1-x)^6}$	$C_\kappa C_m$, $\kappa = 3$, $m = 4$ (Section VI., § 29)
	$C_2 C_5$	$\frac{x^{17}(1-x^7)(1+x^2+x^4)}{(1-x)^6}$	$C_2^m C_r$, $m = 1$, $r = 5$ (Section V., § 17)
	$C_2^2 C_3$	$\frac{x^7(1-x^5)(1-x^7)(1+x^5+x^7)}{(1-x)^6}$	$C_2^m C_3$, $m = 2$ (Section V., § 18)
	$C_1 C_6$	$\frac{7x^{31}}{(1-x)^5}$	Results for degree 6
	$C_1 C_3^2$	$\frac{7(x^{21} - x^{31})}{(1-x)^5}$	
	$C_1 C_2 C_4$	$\frac{7(x^9 + x^{11} - x^{14} - x^{21})}{(1-x)^5}$	
	$C_1 C_2^3$	$\frac{7(x^6 - x^9 - x^{11} + x^{14})}{(1-x)^5}$	
	$C_1^2 C_5$	$\frac{21x^{15}}{(1-x)^4}$	
	$C_1^2 C_2 C_3$	$\frac{21(x^5 - x^{15})}{(1-x)^4}$	
	$C_1^3 C_4$	$\frac{35x^7}{(1-x)^3}$	
	$C_1^3 C_2^2$	$\frac{35(x^4 - x^7)}{(1-x)^3}$	
	$C_1^4 C_3$	$\frac{35x^3}{(1-x)^2}$	
	$C_1^5 C_2$	$\frac{21x}{1-x} - 15x$	$C_1^{n-2} C_3$, $n = 7$ (Section V., § 14)
	C_1^7	1	

Also—

(i.) Sum of generating functions for forms C_7 , $C_3 C_4$ is $x^{28}/(1-x)^6$
(see Section V., § 13).

(ii.) Sum of generating functions for forms $C_7, C_3C_4, C_2C_5, C_2^2C_3$ is $x^7/(1-x)^6$ (see Section V., § 13).

(iii.) Sum of all generating functions is $1/(1-x)^6$; so that all the syzygies are enumerated above.

Degree 8.

It is clear that we need only determine the generating functions for the products $C_8, C_4^2, C_3C_5, C_2C_6, C_2C_3^2, C_2^2C_4, C_2^4$; since all other products have a factor C_1 , and the corresponding generating functions may be determined from the results for degree 7 (see Section V., § 14).

Products.	Generating Functions.	How obtained.
Type forms:— C_8	$\frac{x^{127}}{(1-x)^7}$	Perpetuant type theorem
Products:— C_4^2	$\frac{x^{92}-x^{127}}{(1-x)^7}$	$C_\kappa^2, \kappa = 4$ (Section VI., § 22)
C_3C_5	$\frac{x^{36}-x^{92}}{(1-x)^7}$	$C_\kappa C_m, \kappa = 3, m = 5$ (Section VI., § 23)
C_2C_6	$\frac{x^2(1-x^7)(1-x^8)}{(1-x^2)(1-x)^2} \frac{x^{31}}{(1-x)^5}$	$C_2^m C_r, m = 1, r = 6$ (Section V., § 17)
$C_2C_3^2$	$\frac{x^2(1-x^7)(1-x^8)}{(1-x^2)(1-x)^2} \frac{x^{21}-x^{31}}{(1-x)^5}$	$C_2^m C_r, m = 1, r = 6$ (Section V., § 17)
$C_2^2C_4$	$\frac{x^{11}(1-x^5)(1-x^6)(1-x^{14})}{(1-x^2)(1-x)^7}$	$C_2^m C_4, m = 2$ (Section V., § 19)
C_2^4	$\frac{x^8(1-x^3)(1-x^5)(1-x^7)}{(1-x)^7}$	$C_2^m, m = 4$ (Section V., § 15)

Also—

(i.) Sum of generating functions for forms C_8, C_4^2 (containing no factor of degree 3 or less) is

$$\frac{x^{92}}{(1-x)^7} = \frac{x^{\binom{8}{1} + \binom{8}{2} + \binom{8}{3}}}{(1-x)^7}.$$

(ii.) Sum of generating functions for forms C_8, C_4^2, C_3C_5 is $x^{36}/(1-x)^7$ (see Section V., § 13).

(iii.) Sum of all generating functions above is $x^8/(1-x)^7$ (see Section V., § 13).

Note on Syzygies of Degree 9.

27. The only types and products of degree 9 which have no factor of degree less than 3 are C_9 , $C_4 C_5$, $C_3 C_6$, C_3^3 ; and therefore the generating function for all such irreducible forms is $x^{45}/(1-x)^8$. Also the generating function for C_9 is $x^{255}/(1-x)^8$: the generating function for $C_4 C_5$ is $\frac{x^{129}(1-x^{126})}{(1-x)^8}$, while the generating function for $C_3 C_6$ is (§ 24)

$$\frac{x^{55}(1-x^{10}+x^9-x^{19}+x^{16}-x^{31}+x^{25}-x^{74})}{(1-x)^8}.$$

Hence the generating function for the products C_3^3 should be

$$\begin{aligned} & \frac{x^{45}-x^{255}-x^{129}(1-x^{126})-x^{55}(1-x^{10}+x^9-x^{19}+x^{16}-x^{31}+x^{25}-x^{74})}{(1-x)^8} \\ &= \frac{x^{45}(1-x^{10}+x^{20}-x^{19}+x^{29}-x^{26}+x^{41}-x^{35})}{(1-x)^8}. \end{aligned}$$

This generating function may be written in the form

$$\begin{aligned} & \frac{x^{45}(1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+8x^7+9x^8+10x^9+10x^{10}+10x^{11} \\ & +10x^{12}+10x^{13}+10x^{14}+10x^{15}+10x^{16}+10x^{17}+10x^{18}+9x^{19}+9x^{20} \\ & +9x^{21}+9x^{22}+9x^{23}+9x^{24}+9x^{25}+8x^{26}+7x^{27}+6x^{28}+6x^{29}+6x^{30}+6x^{31} \\ & +6x^{32}+6x^{33}+6x^{34}+5x^{35}+4x^{36}+3x^{37}+2x^{38}+x^{39})}{(1-x)^8}. \end{aligned}$$

The sum of the coefficients in the numerator is 280, the same as the number of possible different arrangements of the letters in the factors of C_3^3 .

The first few terms may be accounted for by the methods already used, thus

$(a_1 a_2 a_3)(a_4 a_5 a_6)(a_7 a_8 a_9)$ is of minimum weight 45,

$(a_1 a_2 a_4)(a_3 a_5 a_6)(a_7 a_8 a_9)$ } " " 46,
 $(a_1 a_2 a_5)(a_4 a_6 a_7)(a_3 a_8 a_9)$ }

$(a_1 a_2 a_6)(a_3 a_4 a_5)(a_7 a_8 a_9)$ }
 $(a_1 a_2 a_7)(a_3 a_5 a_6)(a_4 a_8 a_9)$ } " " 47,
 $(a_1 a_2 a_8)(a_4 a_5 a_6)(a_3 a_7 a_9)$ }

$(a_1 a_2 a_9)(a_3 a_4 a_5)(a_7 a_8 a_6)$ }
 $(a_1 a_2 a_6)(a_3 a_4 a_7)(a_5 a_8 a_9)$ }
 $(a_1 a_2 a_7)(a_3 a_5 a_8)(a_4 a_6 a_9)$ } " " 48.
 $(a_1 a_2 a_8)(a_4 a_5 a_9)(a_3 a_6 a_7)$ }

But our known reductions give six products of minimum weight 49 (instead of five products): they are

$$\begin{aligned}
 &(a_1 a_2 a_7) (a_3 a_4 a_5) (a_6 a_8 a_9), \\
 &(a_1 a_2 a_6) (a_3 a_4 a_7) (a_5 a_8 a_9), \\
 &(a_1 a_2 a_5) (a_3 a_4 a_6) (a_6 a_7 a_9), \\
 &(a_1 a_2 a_4) (a_3 a_5 a_6) (a_6 a_7 a_8), \\
 &(a_1 a_2 a_3) (a_4 a_6 a_7) (a_5 a_8 a_9), \\
 &(a_1 a_3 a_4) (a_2 a_5 a_6) (a_7 a_8 a_9),
 \end{aligned}$$

and one of these forms should be of minimum weight 50 at least.

This would seem to imply that Stroh's syzygies of degree 4 as used hitherto are insufficient for the investigation of the products C_3^3 : it seems probable that similar difficulties will arise successively in the treatment of products $C_4^4, C_5^5, \dots, C_m^m, \dots$, that is, whenever we arrive at a degree m^2 .

[*Note*.—The symbol $(a_1 a_2 a_3)$ is used simply to denote any covariant linear in the coefficients of each of the quantics represented by the letters a_1, a_2, a_3 .]