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“On the Design of Masonry Dams.”

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PRELIMINARY.

THE determination of the proper cross-section of a masonry dam has hitherto only been effected by a series of trials accompanied by tedious calculations. The Author proposes to show how, by dividing the problem into several distinct parts, and by introducing certain limitations and suppositions, trials may be dispensed with, and a mathematical solution may be gradually arrived at as exact as may be desired.

The calculations in question, moreover, are in no way more complicated or difficult than those in actual use for determining the stresses in other large engineering structures, such as metal bridges.

I. ORIGIN AND PRESENT STATE OF THE THEORY OF MASONRY DAMS.

The idea of giving to the cross-sections of masonry dams a shape of uniform stability and strength is of French origin, and was first published by Mr. de Sazilly in the “*Annales des ponts et chaussées*,” 1852.

The first solution of the problem, at once scientific and practical, was given by the French engineers Messrs. Graeff and Delocre, who built the Furens dam, the largest masonry dam in existence until recent times.¹

Delocre, starting from de Sazilly's principles, was led to a standard shape of cross-section which has ever since been used as

¹ Graeff, “Rapport sur la forme et le mode de construction du barrage du Gouffre d'Enfer,” &c., “*Annales des ponts et chaussées*,” 1866, p. 184.

Delocre, “Sur la Forme du profil à adopter pour les grands Barrages en Maçonnerie des Reservoirs,” *ibid.*, p. 212.

a model, and has been usefully developed by another eminent French engineer, Mr. Krantz, who, in 1870, published a selection of types of reservoir-banks, designed according to the principles established by Sazilly and Delocre.

An interesting and valuable report on the design and construction of masonry dams, by Professor Rankine, was published shortly before his death,¹ and was probably one of the last Papers issued by that unrivalled investigator. Rankine added two very essential conditions to those established by de Sazilly and accepted by Delocre, so that the basis for any sound theory of these structures may now be stated as follows:—

1. That, at any horizontal section, the intensity of pressure at the faces of the wall shall never exceed a certain value fixed upon as the safe crushing load of the material of the dam.

2. That at no horizontal layer of the masonry shall there be any danger of sliding.

3. That at those parts of the profile where the wall has a batter, the intensity of pressure at the faces shall be diminished below the limits answering to vertical faces.

4. (A principle not referred to by the French authors.) That there ought to be no practically appreciable tension at any point of the masonry, whether at the outer face, when the reservoir is empty, or at the inner face, when it is filled. The lines of resistance therefore should not deviate from the middle of the thickness of the wall to an extent exceeding one-sixth of the thickness.

Lastly, mention should be made of an excellent Paper of the late Professor Harlacher, of Prague,² Austria, containing full information for calculating cross-sections of reservoir-walls by means of graphic statics.

Now, until recently, whenever it has been attempted to design a reservoir-wall of an economic cross-section, the French types have served as models, modifications being made to meet the special cases. Papers on the subject published lately have scarcely contributed any new points of view or appreciably forwarded the theory of masonry dams, so that, in comparison with the progress made in the theory of structures in general during the last forty years, the results arrived at by the process now in use are sensibly inferior in exactness to those obtained in other departments of modern statics. By the present communication the Author hopes to make a modest step in advance.

¹ *The Engineer*, 1872, p. 1.

² Published in "*Technische Blätter*," 1875, pp. 89 and 169.

II. PRINCIPLES OF THE NEW THEORY.

A. *The following assumptions are made:—*1. The water-level is supposed to reach to the top of wall.

2. The vertical component of the water-pressure upon the battered part of inner face of the wall is provisionally neglected.

3. The shearing-stresses acting parallel to the layers of the wall are not allowed for.

For the rest, the principles as established by de Sazilly and Rankine, and mentioned in the preceding paragraph, will be adhered to.

The first and second of the above assumptions at once serve to simplify the calculations and to favour safety. The third is of no importance when the wall is properly executed according to the rules first established by Graeff and accepted by Krantz and Rankine.

The first supposition generally corresponds with the most unfavourable case that can occur, when the pressure of water is greatest. A quantity of mud occasionally collected at the bottom of the reservoir, indeed is apt to exert a pressure exceeding that of pure water; but this quantity, of course, never should attain a considerable depth, and moreover it must not be forgotten that the working water-level is always kept considerably below the top of the wall.

The margin of safety given by adopting the first assumption may be calculated as follows:

Let x be the vertical depth of any point in the face of the wall below the top of the wall, and x' the depth of the same point below the actual water-level; let w' be the weight of unit of volume of water. Then the actual moment of the water-pressure on the wall above any horizontal section is $\frac{w' x'^3}{6}$, whereas the moment deduced from the assumption of the reservoir being full is $\frac{w' x^3}{6}$. This assumption gives to the wall, therefore, at the cross-section in question, a factor of safety of $\left(\frac{x}{x'}\right)^3$.

As x and x' increase, this factor diminishes and tends to the limit unity. On the other hand, its value increases in approaching the upper part of the wall, where such a factor is more valuable, owing to the smaller mass to resist impacts from waves or ice.

When the cross-section of the wall has been determined on this

assumption, and when its line of resistance is known, it is an easy matter to find the line of resistance of the same wall for any given lower water-level.

Thus, let P and P' be the centres of pressure in the two cases respectively for any horizontal plane, y and y' the distances of P and P' from the vertical through the centre of gravity of the part of the wall above that plane (Fig. 1), W the weight of the same part of the wall.

Then the moment about P of the weight W is $W y$, and this is equal to the moment of the water-pressure, which, as stated above, is $\frac{w' x^3}{6}$.

Similarly

$$W y' = \frac{w' x'^3}{6}.$$

Therefore

$$\frac{y}{y'} = \left(\frac{x}{x'}\right)^3,$$

and y being known for every cross-section, y' can be calculated. The second limitation, consisting in at first neglecting the vertical component of water-pressure acting on those parts of the inner face of wall which are not vertical, besides simplifying the calculations, offers the advantage that in the lower parts of the wall, where the influence of the ratio $\frac{x}{x'}$, considerably diminishes, the line of pressure, corresponding to the reservoir filled, will be displaced towards the middle, and consequently the intensity of the vertical pressure at the outer slope of the wall will diminish as the outer slope of the walls flattens towards the bottom.

It will be shown, however, that the vertical component of the water-pressure may be allowed for in cases where it may be found to be desirable.

Fig. 2.

B. *Primary Shape of Cross-section.*—The simplest among appropriate cross-sections, namely, those of uniform stability throughout, is a right-angled triangle, as in Fig. 2, with vertical inner face.

Then the thickness of the wall at any horizontal cross-section is given by the equation¹

$$\frac{t}{x} = \sqrt{\frac{w'}{w}}$$

¹ Rankine "Applied Mechanics," 4th ed., p. 247.

where t is the thickness of the wall at the cross-section whose depth is x , and w is the weight of unit volume of masonry.

The lines of resistance for the reservoir full and empty are both straight lines, trisecting each horizontal cross-section of the wall, and the pressures at the faces of the wall in the two cases increase proportionally to the depth. This form of cross-section, however, cannot be realized in practice owing to the necessity of giving a certain width to the top of the wall, and often a sufficient width to form a road. The uppermost part of the wall is frequently a rectangular block of masonry, and where it is not so, it may be represented for the purposes of design by such a block, equivalent to the actual structure; *i.e.*, the equivalent block will be of the same weight as the actual structure superposed on the dam, and will have its centre of gravity in the same vertical line.

If such a superstructure be placed upon the triangular dam above considered, the lines of resistance will be displaced throughout the whole body of the dam. If, then, it is desired that the condition of uniform stability should prevail, that is to say, that the conditions laid down above (p. 64) should be fulfilled by every horizontal cross-section to the same extent, it becomes necessary to depart from the triangular cross-section, and among the elements determining the cross-section to be sought will be the mass and form of the superstructure. Delocre recognized this necessity and divided a reservoir-wall into three portions, to which different methods were applied; namely (1) the rectangular superstructure, (2) the body of the wall with vertical inner face and curved outer face carried down to a level where the pressure at the inner face with the reservoir empty becomes equal to the maximum pressure permissible, and (3) the lowest portion or base of the wall with both faces curved.

It is well known that if the weight of the masonry alone were to be considered, the faces of this third or lowest portion of the cross-section would be logarithmic curves, and it will be shown hereafter that for the case under consideration the curves will be of the same class.

This division of the subject as given by Delocre is incomplete, because if the same conditions of stability are made to apply throughout, there will be a discontinuity at the plane of junction of the portions Nos. (1) and (2), as will be seen at once by considering what those conditions are.

In the portion No. (1), which is a rectangular block either forming the actual superstructure or equivalent to it, the only condition to be fulfilled is that the line of pressure with full

reservoir must be at a distance from the outer face of the wall of at least one-third of the thickness. With the reservoir empty the line of pressure is vertical, bisecting the thickness. In the portion No. (2) the lines of pressure for the reservoir full and empty must trisect each horizontal cross-section. It is therefore necessary to introduce between the portions Nos. (1) and (2) another portion, to which the condition of No. (1) will apply at its upper limit and those of No. (2) at its lower limit.

The following investigation is therefore divided into four parts treating respectively of the four portions of the wall to which different conditions have to be applied. These portions will be hereafter referred to as Nos. (1), (2), (3) and (4), of which Nos. (3) and (4), correspond to Delocre's divisions Nos. (2) and (3) above mentioned, with this difference, that the inner face of the wall is not assumed to be vertical in portions Nos. (2) and (3), although it is nearly so.

III. CALCULATION OF CROSS-SECTION.

The process of determining the cross-section may be divided into three stages, namely: (A) to determine the thickness of the wall throughout from top to bottom, but without regard to the form of the inner or outer faces: (B) knowing the thickness of the wall at every horizontal section, to determine the right profile for the faces: (C) to make allowance in the figure of the cross-section, if need be, for the vertical component of the water-pressure on the inner face.

A. To determine the thickness of the wall throughout.—*Portion No. (1).* Let the width of the top of the wall be given and be t_o . Then the height, a , of a rectangular block, such that at its base the line of pressure with the reservoir full falls at one-third of the thickness of the wall from the outside, is given by the equation¹—

$$\frac{a}{t_o} = \sqrt{\frac{w}{w'}} \quad . \quad . \quad . \quad . \quad (1)$$

If the wall were carried down with vertical faces to a greater depth than this, the centre of pressure would be within the outer third of the thickness, which (on the assumption of uniformly varying pressure) means that there would be a tension in the masonry at the inner face.

¹ Rankine, "Applied Mechanics," 4th ed., p. 247.

Portion No. (2). This portion has only to fulfil the conditions of portion No. (1) at its upper limit and of portion No. (3) at its lower limit. In other respects the thickness is arbitrary so long as the lines of resistance do not pass outside the middle third. The simplest form to take satisfying these conditions is the trapezoidal. Let $A B D C$ (Fig. 3) be the superstructure and $C D F E$ a trapezoidal block forming portion No. (2). The slope of $C E$ and $D F$ is at present quite arbitrary. The height of the trapezoid is also arbitrary, since by choosing the slopes suitably the conditions at the upper and lower limits can be satisfied with any height. The height must not, however, be taken too small, as this would require a considerable batter to be given to the inner face, which is by hypothesis nearly vertical.

Fig. 3.

A convenient height to take, and one having the further advantage of simplifying the calculations, is that equal to the height of the superstructure or a . Let t be the thickness at the base $E F$. The weight of masonry, then, resting on $E F$ for unit of length of the wall is

$$w\{a t_0 + \frac{a}{2}(t_0 + t)\} \text{ or } \frac{w a}{2} (3 t_0 + t).$$

Let P_1 and P_2 be the points of trisection of $E F$.

The centre of gravity of this weight is by hypothesis vertically above P_2 , and therefore its moment about P_1 is $\frac{w a}{2} (3 t_0 + t) \times \frac{t}{3}$,

while the moment of the water-pressure is $\frac{w' (2a)^3}{6}$.

Since P_1 is the centre of pressure for the reservoir full, these moments are equal; therefore

$$w t (3 t_0 + t) = 8 w' a^2.$$

But since, by equation (1), $w' a^2 = w t_0^2$, it follows that

$$t^2 + 3 t_0 t - 8 t_0^2 = 0,$$

the solution of which equation gives

$$t = 1.7 t_0 \quad \dots \dots \dots (2)$$

Portion No. (3.) Let t be the thickness of the wall at a depth x below the top, A the total area of the cross-section to that depth.

Taking moments exactly as was done for portion No. (2) we have for any horizontal cross-section at depth x —

$$w A \times \frac{t}{3} = \frac{w' x^3}{6}.$$

Let $\frac{w}{w'} = \rho.$

Then $2 A t = \frac{x^3}{\rho} \quad . \quad . \quad . \quad . \quad (3)$

If x increase by a small increment Δx , $\Delta A = t \Delta x$, and in the limit when Δx is infinitely small,

$t = \frac{dA}{dx}$, whence the preceding equation becomes

$$2 A \frac{dA}{dx} = \frac{x^3}{\rho}.$$

Integrating, $A^2 = \frac{x^4}{4\rho} + C$, where C is the constant of integration.

Let A_0 be the area of portions Nos. (1) and (2) which the preceding work enables us to calculate.

$$A = A_0 \text{ when } x = 2a.$$

$$\text{Therefore } A_0^2 = \frac{16 a^4}{4\rho} + C,$$

$$\text{and } A^2 = A_0^2 + \frac{x^4 - 16 a^4}{4\rho}.$$

Substituting this value of A in equation (3) we obtain:

$$t = \frac{x^3}{\sqrt{\rho(x^4 - 16 a^4 + 4\rho A_0^2)}} \quad . \quad . \quad (4)$$

As x increases, t tends to the value $\frac{x}{\sqrt{\rho}}$, which is that of the primary triangular form. The curves for the faces of the wall given by (4) have the sides of the triangle as asymptotes.

The limit of depth to which equation (4) can be used is reached when the pressure at the face of the wall becomes equal to the maximum permissible pressure. Let p be this pressure. Under

Fig. 4.

the conditions assumed to prevail, the maximum pressure is twice the mean pressure,

$$\text{or } p = \frac{2 A w}{t} \text{ at the limiting depth,}$$

$$\text{or, by (3) } p t^2 = \frac{x^3 w}{\rho} = w' x^3.$$

This equation may be solved by trial, commencing with the value $x = \frac{p}{w}$ which applies to the primary triangle.

If the slope of the outer face is allowed for in the limiting pressure (as is done by Rankine), $p \cos \delta$ must be substituted for p , where approximately

$$\tan \delta = \frac{x}{t} = \sqrt{\rho}.$$

Portion No. (4). The fresh condition introduced here is that the pressure at the face—i.e., either at the inner face when the reservoir is empty or at the outer face when it is full—must not exceed the value p . As before, let t be the thickness of the wall at any depth, x , and A the entire area of cross-section above that depth; A_0 the area of cross-section determined according to the foregoing methods for portions Nos. (1), (2) and (3). The mean intensity of vertical pressure is $\frac{w A}{t}$, and, on the theory of uniformly varying pressure,

the pressures at the faces are $\frac{w A}{t} + \frac{k t}{2}$ and $\frac{w A}{t} - \frac{k t}{2}$, where k is a constant differing for each manner of loading. An investigation which it is needless to reproduce here, shows that the best practical result is arrived at by making the minimum pressure zero, in which case the maximum pressure is double the mean, or $p = \frac{2 w A}{t}$.

$$\text{As before } t = \frac{d A}{d x}.$$

$$\text{Whence } \frac{1}{A} \frac{d A}{d x} = \frac{2 w}{p}.$$

Integrating, $\log \frac{A}{C} = \frac{2 w x}{p}$, where C is a constant of integration.

$$\text{Therefore } A = C e^{\frac{2 w x}{p}}.$$

If x_0 be the depth at which portion No. (3) ends and No. (4) begins, $A_0 = C e^{\frac{2 w x_0}{p}}$.

Therefore
$$A = A_o \epsilon^{\frac{2w}{p}(x-x_o)},$$

and
$$t = \frac{2w}{p} A = \frac{2w}{p} A_o \epsilon^{\frac{2w}{p}(x-x_o)}. \quad (5)$$

If the thickness of the wall be determined according to equation (5), one line of resistance (say that for the reservoir empty) will be everywhere at a distance from the face of one-third the thickness, but in that case the same cannot be true of the other line of resistance, because the equation of moments has to be considered. Let r be the distance between the two centres of pressure.

$$r \times w A = \frac{w' x^3}{6},$$

or
$$r = \frac{x^3}{6 \rho A}.$$

This equation is of a similar form to (3), which was found for portion No. (3). As, by equation (5) above, the thickness increases very rapidly with the depth, A increases much more rapidly than x^3 , and r , starting from the value $\frac{t}{3}$, constantly decreases. When r becomes less than $\frac{t}{6}$, the two centres of pressure are in the same half of the thickness.

Since the cross-section increases so rapidly with the depth after portion No. (4) is reached, it is advisable to extend the limit of depth to which portion No. (3) is carried as far down as possible, and this can only be done either by using lighter materials or by admitting higher pressures. There is not much scope for choice in the weight of the materials, but it is certainly possible to allow a greater stress than 85 lbs. per square inch, which has hitherto been taken as a limit, through the influence of French practice, although recent experiments in Germany have demonstrated the great strength of masonry laid in cement. A factor of safety of 20 to 30 seems an unnecessary precaution against crushing when the wall has a factor many times less against overturning.

B. *To determine the Configuration of the Wall, the thickness being already known throughout.*—The preceding investigation enables us to draw a provisional cross-section of the wall, having everywhere the thickness necessary to comply with the conditions of stability. The configuration is yet to be determined by certain conditions of equilibrium, which have not yet been considered. This may be

done either by analytical or by graphic methods, the former of which alone will be here dealt with.

Portion No. (1). This is already completely determined, both faces being, by hypothesis, vertical.

Portion No. (2). The position of this trapezoid, with respect to the rectangle above it, is fixed by the condition that the centre of gravity of the two together must be vertically above the point of trisection of the base nearer to the inner face. Let (*Fig. 3*) y be the distance between the middle point of EF, and the vertical through the middle point of AB, y' the distance of the vertical through the centre of gravity of the area CDEF from the same point. The area of ABCD is $t_0 a$, and that of CDEF is (equation (2)) $a \times \frac{t_0 + 1.7 t_0}{2}$, or $1.35 t_0 a$. The area of both together is $2.35 t_0 a$.

The above condition gives therefore:—

$$t_0 a \times y + 1.35 t_0 a \times y' = 2.35 t_0 a \times \frac{1.7 t_0}{6};$$

that is,

$$y + 1.35 y' = \frac{2.35 \times 1.7}{6} t_0,$$

and y' is given by the formula for the centre of gravity of a trapezoid, viz.:—

$$\frac{y'}{y} = \frac{2CD + EF}{3(CD + EF)} = \frac{3.7}{8.1} = 0.457;$$

therefore

$$y(1 + 1.35 \times 0.457) = \frac{2.35 \times 1.7}{6} t_0$$

$$y = 0.4 t_0.$$

The vertical BD overhangs the point F by the amount—

$$y + \frac{t_0}{2} - \frac{EF}{2} = (0.4 + 0.5 - 0.85) t_0 = 0.05 t_0,$$

or $\frac{1}{20}$ part of the width of the wall at the top.

This trifling amount of overhang is unobjectionable from the point of view of construction, but, in order to remove any doubt, the effect has been calculated of the upward water-pressure on the overhanging face DF. It is found that the tensional stress on the masonry from this cause amounts to $90 \cdot t_0 \sqrt{\rho}$ in kilograms per square metre. The value of ρ will lie between 2 and 3, and if the width of the wall on top, t_0 , be even so unusually large as 5 metres, the factor of safety for good rubble masonry laid in hydraulic

mortar will be between 250 and 300, to resist this tension. It is also to be remembered that the water-level in practice does not reach the top of the wall, and that any parapet along the inner side of the wall tends to counteract the upward pressure.

Portion No. (3).—The condition determining the outline of the faces of this portion is similar to that applying to No. (2), viz., that the material must be so distributed as to conform to the rule that the line of resistance with the reservoir empty will everywhere pass at a distance from the inner face of one-third of the thickness. This line of resistance is the locus of the projections on successive horizontal cross-sections of the wall, of the centres of gravity of the whole mass above those cross-sections. Consider a lamina of thickness dx , bounded by horizontal planes.

If y be the distance from any vertical axis of the centre of gravity of the mass of the wall above the lamina, x and y are the Cartesian co-ordinates of the point where the line of resistance, reservoir empty, cuts the plane x . The thickness t being already known as a function of x , static conditions determine dy in terms of dx . Thus, if A denote, as before, the area of the whole wall above the level x ,

$$A \, dy = c \times t \, dx$$

c being the distance of the vertical through the centre of gravity of the area A from the middle point of the thickness t , and equal to $\frac{t}{6}$.

Therefore the equation to the line of resistance in question is—

$$y = \int \frac{c \, t}{A} \, dx.$$

This integration cannot be made directly, but the line of

resistance can be plotted by calculating the elements of the quantity to be integrated step by step.

The starting point of the curve at the base of portion No. (2) is known. For every element of depth dx , the numerator and denominator of the fraction under the sign of integration admit of calculation from the work previously done, and the course of the curve can in that way be accurately laid down.

Portion No. (4).—The condition which determines the outline, and also the method to be pursued are identical with those already stated for portion No. (3).

C. To Make Allowance for the Vertical Component of the Water-Pressure.—It is only in portion No. (4) that the water-pressure on the inner face has a vertical component of any appreciable magnitude. Where, however, the wall is of such height that portion No. (4) enters into it, a valuable economy results from taking this vertical component into account. This may be approximately performed by a combined graphic and analytical process, which moreover appears to be very simple, since the two limiting cases between which the solution required is to be found may be represented with fully sufficient accuracy according to the preceding investigation.

These limiting cases are (*Fig. 6.*):

(1) Intensity of pressure at one edge, p , continuously increasing downwards, and nil at the opposite edge; that is to say, the portion No. (4) being left out and the whole cross-section, from the base of portion No. (2), EF, downwards, being calculated by equation (4), as portion No. (3). Here it is certainly allowable to neglect the vertical component of the water-pressure. This case is represented by VLECBDFMW in *Fig. 6.*

(2) The calculation of the cross-section, with portion No. (4) according to equation (5), without allowing for the vertical component of the water-pressure, and represented by QLMR in *Fig. 6.*, evidently represents the second, or opposite limiting case.

The intensities of the pressures at the edges QR of the base may be obtained by calculation.

Let p_o denote the utmost pressure consistent with safety to be admitted in portion No. (4), and p_1 and p_2 pressures obtained in the limiting cases (1) and (2) respectively, then it results:—

Case.	Intensity of Pressure at the edge of Base. Reservoir filled.	
	Outer Face.	Inner Face.
(1) Limit	$p_1 > p_o$	$p_1' = 0$;
(2) Opposite limit	$p_2 < p_o$	$p_2' > 0$;
(3) Problem to be solved	p	$p' < p_2' > 0$.

Now let the two profiles (1) and (2) be laid upon each other so that the upper parts, beginning from the limit LM , will coincide, as has been done in *Fig. 6*. The profiles, as well as the bases z_1 and z_2 , will then have a definite relative position. In the *Fig.* $VW = z_1$ is the base of the first, $QR = z_2$ that of the second profile.

If at the edges QR , VW , the respective intensities of pressure be produced as perpendiculars, namely, $\overline{QQ'} = p_2$, $\overline{VV'} = p_1$, $\overline{RR'} = p'_2$; by joining their proper ends, the corresponding "surfaces of load" may be obtained, the areas of which evidently must

Fig. 6.

be proportionate to the respective total loads of masonry and water.

Thus for the first case, and base z_1 , there is as surface of load the triangle $VV'W$; for the second case, and base z_2 , the trapezoid $QQ'R'R$. If now the intensity of pressure at the outer edge of the base shall become p_2 , the edge Q must evidently move towards V or the point Q' towards V' . At the same time, it is obvious that R as well as R' will move towards W .

The curves then described by the points Q' and R' are indicated in the *Fig.*, but will not be needed in respect of the solution of the problem.

Let the whole load of water resting on the curved inner face MR of portion No. (4) be reduced to masonry, by dividing its depths n by the specific gravity ρ of masonry, and let the lengths thus obtained

$$n' = \frac{n}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

be plotted vertically upwards from the curve MR and their ends be joined by a continuous curve op' .

The spandril B D F M W o may be neglected without involving a sensible error. Then the polygon B op' R will represent the inner face of a masonry block, which, when acted upon by the purely horizontal pressure of the water, as already taken into account, would exert at its base z_2 a stress exactly equal to that actually arising if the profile of case (2) were executed and the reservoir were filled.

If now the process be continued from the base QR = z_2 to a base XY = z_1 , lying between the limits z_2 and z_1 , the point p will move horizontally from p' to p , and so will accordingly any other point of the curve op' .

The trapezoid op YW may therefore be assumed as being a parallel projection of op' RW, and, likewise, the triangle LXV as a parallel projection of LQV.

Let the inner surfaces of load op' RW and op YW respectively be denoted by H' and H ; the outer, LQV and LXV, by V' and V ; then, using the symbols marked in the figure, there will be obtained

$$\frac{V}{V'} = \frac{v}{v'} \quad \frac{H}{H'} = \frac{h}{h'} \quad . \quad . \quad . \quad . \quad (7)$$

If, further, r signifies the mean height of the triangle LQV, and s that of the trapezoid op' RW,

$$v'r = V', \quad h's = H' \quad . \quad . \quad . \quad . \quad (8)$$

whence

$$r = \frac{V'}{v'}, \quad s = \frac{H'}{h'} \quad . \quad . \quad . \quad . \quad (8a)$$

By equations (8),

$$vr = V, \quad hs = H \quad . \quad . \quad . \quad . \quad (9)$$

Now there must still be established another relation between V and H , and it is allowable to consider the most simple case answering the purpose, that is—

$$\frac{dv}{dh} = \text{constant} \quad . \quad . \quad . \quad . \quad (10)$$

It is evident that this is sufficient to fulfil the condition, that the intensity of pressure at the outer edge shall pass through the value $xx' = p_0$, when z continuously increases from z_1 towards z_2 .

Therefore there are the further following conditions—

$$\left. \begin{aligned} \frac{dv}{dh} = \frac{v}{h} = \frac{v'}{h'} = \sigma, \text{ a constant} \\ \frac{V}{H} = \frac{V'}{H'} = \frac{rv}{sh} = \frac{r}{s}\sigma, \text{ another constant} \end{aligned} \right\} \quad . \quad . \quad . \quad (11)$$

h and v can now be expressed in terms of z thus—

$$v + h + z_1 = z \quad . \quad . \quad . \quad (12)$$

$$\text{or} \quad \sigma h + h = z - z_1;$$

$$h = \frac{z - z_1}{\sigma + 1} \quad . \quad . \quad . \quad (13)$$

and accordingly

$$v = \frac{\sigma}{\sigma + 1} (z - z_1) \quad . \quad . \quad . \quad (14)$$

Let the centres of gravity of the triangle LVQ and of the trapezoid $op'RW$ be determined by any method; the centres of gravity of LVX and $opYW$ will be parallel-projections of the former.

Let, as an approximation, the lines LV and MW be taken as straight, and moreover MW as vertical.

The distance between the centre of gravity of the area H' and the vertical MW is proportionate to the breadth h' or equal to

$$\phi h',$$

ϕ denoting a constant ratio, also applying for any parallel-projection of $op'PW$, so that the corresponding distance for the area H will be

$$\phi h.$$

Thus will be found the distances of the centres of gravity of the triangle LVQ and its parallel-projections, respectively

$$\psi v' \text{ and } \psi v.$$

These latter distances, however, are to be measured horizontally. The breadth of profile I , as measured at the height of the centres of gravity of the planes V (which centres obviously must be all at the same height above the base) will be denoted by z_s . Let S be the total area of cross-section of profile I ; $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{G}$, the statical

moments of the parts V, S, H of the profile to be deduced, about the middle of the base z ; let the weight of S be assumed to be applied at the outer third of the base z_1 : by so doing, the moment of the horizontal component of water-pressure will be allowed for; since, according to the premisses, the action of that moment consists in shifting the line of action of that weight from the inner to the outer third of the base z_1 .

$\mathfrak{B}$ and $\mathfrak{S}$ obviously are positive, whilst $\mathfrak{H}$ is negative.

Now

$$\begin{aligned}\mathfrak{B} &= V \left(\psi v + z_v + h - \frac{z}{2} \right) w, \\ &= \frac{r\sigma}{\sigma+1} (z - z_1) \left\{ \frac{z - z_1}{\sigma + 1} (\psi\sigma + 1) + z_v - \frac{z}{2} \right\} w. \quad (15)\end{aligned}$$

$$\begin{aligned}\mathfrak{S} &= S \left(h + \frac{2}{3} z_1 - \frac{z}{2} \right) w, \\ &= S \left(\frac{z - z_1}{\sigma + 1} + \frac{2}{3} z_1 - \frac{z}{2} \right) w \quad . \quad . \quad (16)\end{aligned}$$

$$\begin{aligned}\mathfrak{H} &= H \left(\frac{z}{2} - h + \phi h \right) w, \\ &= \frac{s(z - z_1)}{\sigma + 1} \left\{ \frac{z}{2} - \frac{z - z_1}{\sigma + 1} (1 - \phi) \right\} w \quad . \quad . \quad (17)\end{aligned}$$

For determining the intensity of pressure p , at the outer edge of the base, there is the fundamental equation.

$$p = \frac{N}{z} + \frac{6\mathfrak{P}}{z^2} \quad . \quad . \quad . \quad (18)$$

Here N , as before, denotes the total vertical pressure as distributed over the base z , and $\mathfrak{P}$ signifies the resulting moment of the whole water-pressure and weight of masonry about the middle of z . Hence

$$\begin{aligned}N &= w(V + S + H) \} \\ \mathfrak{P} &= \mathfrak{B} + \mathfrak{S} - \mathfrak{H} \} \quad . \quad . \quad . \quad (19)\end{aligned}$$

It is obvious that the most favourable value of z would be obtained, if it were possible so to arrange the cross-section of bottom as to distribute the weight N uniformly over the base, that is to say, if it were possible by solving the equations of conditions

$$\frac{N}{z} = p \text{ and } \mathfrak{P} = 0,$$

to obtain one distinct value of z .

This, however, as may easily be stated, is not possible in every case, when p is fixed; or in other words, these conditions will not lead to real results for any arbitrary value of p , and this is just what is required.

The process must therefore be continued as follows: finding

$$N = w \left(\frac{z - z_1}{\sigma + 1} (r\sigma + s) + S \right) \quad . \quad . \quad . \quad (20)$$

$$\begin{aligned} \mathfrak{P} = w \left\{ \frac{z - z_1}{\sigma + 1} \left[\frac{z - z_1}{\sigma + 1} (r\sigma(\psi\sigma + 1) + s(1 - \phi)) \right] \right. \\ \left. - \frac{z}{2} (r\sigma + s) + r\sigma z_v + S \right] + S \left(\frac{2}{3} z_1 - \frac{z}{2} \right) \Big\} \quad (21) \end{aligned}$$

and putting these values into equation (18) there is obtained

$$\begin{aligned} \frac{z^2 p}{w} = z \left[\frac{z - z_1}{\sigma + 1} (r\sigma + s) + S \right] + 6 \left\{ \frac{z - z_1}{\sigma + 1} \left[\frac{z - z_1}{\sigma + 1} (r\sigma(\psi\sigma + 1) \right. \right. \\ \left. \left. + s(1 - \phi)) - \frac{z}{2} (r\sigma + s) + r\sigma z_v + S \right] + S \left(\frac{2}{3} z_1 - \frac{z}{2} \right) \right\}. \end{aligned}$$

The following quantities are next computed.

$$\frac{p}{w} = a, \frac{r\sigma + s}{\sigma + 1} = b, \frac{r\sigma(\psi\sigma + 1) + s(1 - \phi)}{(\sigma + 1)^2} = c, \frac{r\sigma z_v + S}{\sigma + 1} = B;$$

then, after the required transformations, there results—

$$\begin{aligned} z^2(a + 2b - 6c) - 2z \left[z_1(b - 6c) - S + 3B \right] \\ = 2z_1 \left[2S + 3(cz_1 - B) \right] \quad (22) \end{aligned}$$

Further, computing

$$a + 2b - 6c = e; \quad z_1(b - 6c) - S + 3B = E,$$

$$2z_1 \left[2S + 3(cz_1 - B) \right] = \mathfrak{E};$$

there is finally obtained from (22)

$$z = \frac{E}{e} + \sqrt{\frac{\mathfrak{E}}{e} + \left(\frac{E}{e} \right)^2} \quad . \quad . \quad . \quad (23)$$

By introducing into this equation p_0 instead of p , the length of base required, z'_0 is obtained.

It only remains to plot the latter in the right position. For this purpose equations (13) and (14) are made use of—

$$h_o = \frac{z_o - z_1}{\sigma + 1}, v_o = \frac{\sigma}{\sigma + 1} (z_o - z_1). \quad . \quad . \quad (24)$$

Now, the curve M Y is obtained by reducing in the ratio

$$\overline{WY} : \overline{WR},$$

the several horizontal ordinates of the curve M R, measured from the line M W; and the curve L X is obtained by reducing in the ratio

$$\overline{VX} : \overline{VQ}$$

the several horizontal ordinates of the curve L Q, measured from the line L V.

It is almost unnecessary to mention, that all the calculations required are much facilitated by the use of a slide rule, or some mechanical contrivance of the kind. In Fig. 6, X L C A B M Y represents, on a scale of 1 in 1000, the shape of cross-section which would have been obtained by the foregoing theory for the well-known masonry dam of Furens; that is to say, by employing as fundamental data for the calculations

$$t_o = 5 \text{ metres; } w = 2,000 \text{ kilograms per cubic metre,} \\ \rho = 2.0; p = 6 \text{ kilograms per square centimetre.}$$

The greatest deviation from the above standard value of p (about three per cent.) is arrived at, at a depth of 35 metres from the top, the intensity of pressure at the outer face (reservoir filled) being there 6.2 instead of 6.0 kilograms per square centimetre; whilst at the base it returns to 6.0 exactly. This evidently arises from the neglect of the spandril B F W o.

At the inner face the intensity of pressure diminishes as the slope flattens towards the bottom, becoming 4.8 kilograms per square centimetre at the edge of base, Y, when the reservoir is filled, and 3.5 kilograms when it is empty.

The communication is accompanied by a series of diagrams from which the *Figs.* in the text have been prepared.