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I.—THE METHODS OF MODERN LOGIC AND THE CONCEPTION OF INFINITY.

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IN an address delivered some two years ago and since then published, I made an attempt to estimate the influence on economic ideas of the great advances effected in the methods of modern logicians. Whether I succeeded I do not know. But the moral which I sought to point was that economists of more than one shade of opinion would do well to criticise their categories, and to see to it that they do not confuse what are in reality only valuable working conceptions with concrete realities. My purpose in this paper is to pursue the same method, and to suggest that, in yet another region, abstractions are apt to be hypostatized into realities.

Both in daily parlance and in mathematical science the word "infinite" is freely employed. Yet this expression is, as a rule, either not defined at all, or, if defined, employed in a special sense which excludes what the word means when used in other connections. An infinite series suggests, or ought to suggest, nothing analogous to an infinite God. The former may be limited by a finite quantity which the sum of the members of the series, though themselves increased in number without limit, cannot exceed. Such a numerical series can be treated as collapsing into a whole which is finite, and this whole is determined by the law of the series, through the medium of

the definite relationship between the members which constitutes that law. Finite, in the sense of ended, is in this fashion prevented from coming into contradiction with infinite, in the sense of unending. The two aspects are present in one whole, and are not inconsistent with each other. But there is nothing final about the whole which they constitute. It is one single system alongside of others, and is itself included in an indefinite succession of larger systems, which may or may not be capable of being summed in an analogous finite whole as their limit. In the region of space and time real finality is sought in vain. The progress is endless. And this is entirely true of the more general category of Quantity. Quantity as such is not a concrete thing. It is a relation or category. Even in mathematics the category of quantity is not always used. The new science of Projective Geometry seems expressly to exclude the quantitative relationships of space, and to confine itself to qualitative distinctions of points and lines. In the infinitesimal calculus the better opinion appears to be that from the notion of definite quantities or quanta, as ordinarily understood, we must free our minds and speech if we would escape self-stultification. But before I go further into this topic I should like to examine the notion of Quantity itself a little more closely.

Quantity is sometimes defined as the capacity of being increased or decreased. This definition is tautologous, but it points to a real distinction. The distinctive element in quantity is the capacity of being increased or decreased without alteration of character or quality. In pure quantity, whether we deal with occupation of space or with number, unit passes into unit, and the substratum remains through the change of quantity qualitatively identical. This is so notwithstanding that quantity possesses a discrete as well as a continuous aspect. Quantitative magnitudes are not of two kinds, but inherently possess two aspects, that of unbroken self-identity or continuity, and that of divisibility into units or parts. On whichever side

we take quantity we find ourselves faced by the capacity of indefinite extension. This capacity is sometimes spoken of as extension to infinity. But the word infinite as so used betokens no true infinite. It simply signifies that something greater or less can be set up beyond any given stage. But this something beyond is essentially existence alongside of, and in contrast to, the phase of existence already reached. It is therefore a finite. The true infinite cannot exist by contrast—cannot be a cause, even a first cause, or a substance, or numerically different from anything else; for these, all of them, carry the badge of finitude. A true infinite must be self-contained. It follows that no quantum can be infinite, and that to escape from its inherent finitude we must turn to its qualitative relations. From this side quantity may have an aspect in which it is self-contained—for instance, as the sum of a series, but it is only in one aspect that such a sum has the resemblance of self-containedness. The number 2 is the sum or limit of an arithmetical series that in point of number of members has no limit. But although in its aspect of a limit it contains the whole series, in its aspect as a definite number it has an infinity of numbers outside itself. It is therefore only *sub modo* that it is representative of what is self-contained, and for the series such a summation is possible only in virtue of a law or relationship in which the notion of a limit is inherent. From the failure of mathematicians to notice the ambiguity in the word “infinite” a great amount of apparently unnecessary controversy has arisen. The methods of modern logic, with their stringent insistence on criticism of categories, might, if carefully applied, have delivered them from much.

I am not a mathematician, and I speak with some reluctance on that which I have studied mainly as the *corpus vile* on which to attempt logical investigations. When, however, even an outsider enters on an examination of the principles of the infinitesimal calculus he is impressed with the evidences of confusion. But I will quote the language of men who are not

outsiders, two distinguished mathematicians—both Americans—who have been candid on this subject. I will begin with Dr. A. T. Bledsoe's "Philosophy of Mathematics":—

"The student of mathematics, in passing from the lower branches of the science to the infinitesimal analysis, finds himself in a strange and wholly foreign department of thought. He has not risen by easy and gradual steps, from a lower to a higher, purer, and more beautiful region of scientific truth. On the contrary, he is painfully impressed with the conviction that the continuity of the science has been broken, and its unity destroyed by the influx of principles which are as unintelligible as they are novel. He finds himself surrounded by enigmas and obscurities which only serve to perplex his understanding and darken his aspirations after knowledge."

Commenting on this passage, Professor Buckingham, of Chicago, in his striking book on the "Differential and Integral Calculus," goes further:—

"The student," he declares, "finds himself required to ignore the principles and axioms that have hitherto guided his studies and sustained his convictions, and to receive in their stead a set of notions that are utterly repugnant to all his preconceived ideas of truth. When he is told that one quantity may be added to or subtracted from another, without diminishing it; that one quantity may be infinitely small, another infinitely smaller, and another infinitely smaller still, and so on *ad infinitum*—that a quantity may be so small that it cannot be divided, and yet may contain another an indefinite and even an infinite number of times—that zero is not always nothing, but may not only be something or nothing as occasion may require, and may be *both at the same time*, in the same equation—it is not surprising that he should become bewildered and disheartened. Nevertheless, if he study the text books that are considered orthodox in this country and in Europe, he will find some of these notions set forth in them all; not indeed in their naked deformity, as they are here stated, but softened and made

as palatable as possible by associating them with, or concealing them beneath, propositions that are undoubtedly true. It is indeed strange that a science so exact in its results should have its principles interwoven with so much that is false and absurd in theory, especially as all these absurdities have been so often exposed and charged against the claims of the calculus as a true science. It can be accounted for only by the influence of the great names that first adopted them, and the indisposition of mathematicians to depart from the simple ideas of the ancients in reference to the attributes of quantity. They regard it merely as inert, either fixed in value or subject only to such changes as may be arbitrarily imposed on it. But when they attempt to carry this conception into the operations of the calculus, and to account for the results by some theory consistent with this idea of quantity, they are inevitably entangled in some often absurd notions that have been mentioned. Many efforts have indeed been made to escape such glaring inconsistencies, but they have only resulted in a partial success in concealing them."

I have quoted Professor Buckingham at some length, because, as a mere logician, I prefer to rely on the authority of a trained mathematician, rather than to express views of my own. But there is no need for me to confine myself to Professor Buckingham. In 1684 Leibnitz first published his method in full—the method of Infinitesimals. His fundamental assumptions are thus stated by his disciple, the Marquis de L'Hôpital. First:—"We demand that we may take, indifferently, the one for the other, two quantities which differ from each other by an infinitely small quantity, or what is the same thing, that a quantity which is increased or decreased by another quantity infinitely less than itself, can be considered as remaining the same." Second:—"We demand that a curved line may be considered as the assemblage of an infinity of straight lines, each infinitely small; or, what is the same thing, as a polygon with an infinite number of sides, each infinitely small, which

determine by the angles which they make with each other, the curvature of the lines."

Here we have remarkable confusion. That these infinitesimals are regarded from the side of quantity on which it presents its discrete aspect and are meant to be minute quanta, is plain from the next step which Leibnitz takes. He goes on to treat the infinitesimals of his system as themselves composed of an infinite number of parts, infinitely smaller still, and so on. It is plain that a circle cannot be regarded as really made up of a polygon of an infinite number of sides, as Leibnitz claimed, each infinitely small, and of which the prolongation is the tangent to the circle. No doubt, as Lagrange afterwards explained, the error finds itself corrected in the results of the calculus by the omission which is made of infinitely small quantities. Leibnitz himself likened them to grains of sand in comparison with the sea. But, as Comte remarked, this explanation would, if true, completely change the nature of this analysis, by reducing it to a mere approximative calculus, which, from this point of view, would be radically vicious. As another great critic, himself a mathematician, D'Alembert, said, Leibnitz's explanation "ruined" the geometrical exactness of the calculus.

But we know that the Infinitesimal Calculus is an exact science, and that it cannot rest on an inexact hypothesis. To the logician, who has examined the conception of quantity with the suspicion which a metaphysical training engenders, the source of the confusion is apparent. Quantity, as I reminded you at the beginning of this paper, has a continuous as well as a discrete aspect, and can therefore, in the first of these aspects, be made the subject of a science of rate of change which does not involve the dragging in of the other aspect and the attempt to treat quanta as if they could be infinitesimal, a very contradiction in terms. The infinite of Leibnitz is a false infinite, and imports mere unendingness in increase or decrease of finite quanta. The qualitative infinity which belongs to the con-

tinuous aspect of quantity cannot be reached by addition or subtraction. Leibnitz was too great a genius to be dominated in the result by such a fallacy. But logic, as it has come to be since Kant and Hegel scrutinised its foundations afresh, did not exist in his time, and he fell into uncritical and dogmatic assumptions which led to insoluble antinomies. Newton, whose method of fluxions was founded on larger notions, was able to reach the results of Leibnitz without being subject to the reproach of contradicting himself in first principles.

As far as I can venture to express a view, founded on an examination of a number of treatises which I have endeavoured to study, the broad working conceptions of the Differential Calculus, apart from theories as to their meaning, are usually expressed somewhat as follows:—

(1) A Differential Coefficient expresses the rate of change of a function with respect to its independent variable. The work of the Differential Calculus is to find the derived function when the original function is given. The Integral Calculus seeks, conversely, to find its original function when the derived function is given.

(2) If there be a fixed magnitude to which a variable magnitude can be made as nearly equal as we please, and if it be impossible that the variable magnitude can ever be exactly equal to the fixed magnitude, the fixed magnitude is called the limit of the variable magnitude.

Now, this second proposition bears the mark of the cloven hoof. The idea of negligible difference is present in it. Nevertheless it points to the real underlying principle, which is that we are to take the continuous aspect of quantity as real equally with the discrete aspect with which arithmetic is concerned. It suggests that we may treat quantity, not only as capable of being increased and diminished *ab extra*, but as being actually alive, as it were, and inherently in a state of change. If so the Differential Calculus is a science of rates, and its peculiar subject is quantity regarded as a state of

continuous change. Rate is a complex idea of which the elements are time and change, but where one uniform rate is compared with another, without regard to absolute values, as in dy/dx , the symbol of the differential coefficient, the idea of time may be left out of the question, and the comparison may be made through that of the simultaneous changes of the two rates themselves; so that the ratio of the simultaneous increments of any two variables which increase at constant rates, does in fact perform a double office, representing not only the ratio of their changes, but also that of their rates of change, which will of course be constant. Just because of this characteristic, the rates at which the quantities may be changing are not affected when their values become zero, and may pass from *plus* to *minus* quantities. The equation $0/0 = a$, thus has a meaning: a may be the ratio, not only of quantities or of their increments, but also of their rates of increase, and therefore, when applied to the so-called increments with which the Differential Calculus deals, a must indicate the ratio of the rates of change of their increments as they pass through the zero point of their value. dy/dx is thus a symbol which represents, not actual or separable minute quantities in their relation to each other, but simultaneous rates of change in x and a function of x . Apart from their relation to each other, dy and dx are non-existent; they have reality only as moments in a relationship which belongs to the continuous aspect of quantity. Thus it is only as symbolising rates of change in certain aspects of magnitude that curves may be treated as though made up of lines. The relationship of the differential coefficients of such magnitudes is to be conceived as existing, neither *before* they disappear, nor *after*, but as the relation *with which* they disappear. In virtue of the principle of continuity the vanishing magnitudes still retain the ratio which characterises their relative rates of change. It follows that the transition from the function of a variable to its differential, must be regarded as no difference by so much,

but as a reduction of the finite function to the qualitative relation of its quantitative elements. What is present is in reality only the qualitative relation of quantitative *principia*, which, as being *principia*, are elements but not quanta. Thus we get rid of the "beyond" of the false infinite of quantity taken as discrete. It is by grasping truths of this class that we rid ourselves of the limitations by which the science of quantity, assumed to possess a merely discrete character, is confused, and gain the free use of the great instrument of analysis which the genius of Newton and Leibnitz has placed within our grasp. In the notions themselves which underlie the calculus there is nothing particularly abstruse. It is the confusion of ideas which logically imply different categories that makes this branch of mathematics seem a mystery. As Professor Perry points out in a very useful book which he has written:* "Surely there is no great difficulty in catching the idea of a limiting value. Some people have the notion that we are stating something that is only approximately true; it is often because their teacher will say such things as 'reject $16.1\delta t$ because it is small,' or 'let dt be an infinitely small amount of time' and they proceed to divide something by it, showing that though they may reach the age of Methuselah they will never have the common sense of an engineer."

I have dwelt thus long on the illustration from mathematics of the false infinite of quantity, because I believe that much confusion and many antinomies have arisen in connection with it. The controversy as to the character of this false infinite was raised by Kant, and was pursued still more fully by Hegel in the second section (that which deals with quantity) of the first book of his large *Wissenschaft der Logik*. His "Encyclopædia" Logic does not go into detail on the point. A brilliant exposition of the attack made by Hegel on what he deemed to be the bad metaphysics of the Leibnitzian view is

* *The Calculus for Engineers*, p. 22.

given in the second volume of Dr. Stirling's *Secret of Hegel*, a book which ought to be carefully studied. This exposition produced a keen controversy, in which the late Professors Tait and Robertson Smith took up the cudgels for Leibnitz. Dr. Stirling replied in a pamphlet entitled "Whewell and Hegel," which is bound up with his lectures on the "Philosophy of Law." After that time—nearly 40 years ago—the controversy dropped. It is interesting to find the root question discussed once again in the much later writings of the American mathematicians whom I have quoted. The whole subject is only one more illustration of the truth that there is a borderland between science and metaphysics, the boundaries of which require much more exact study and ascertainment than they have hitherto received. It is the province of modern logic to embrace this region within its survey, and to set up warning posts for men of science and of metaphysicians alike.

One further remark I wish to make here. It is a common delusion that logicians and metaphysicians desire, by some high *a priori* method, to endeavour to do the work of the physicists. This is a delusion, the existence of which is due to the circumstance that the latter have, as a general rule, failed to take the trouble to study the criticisms of the former. The method is the same in both cases, the testing and modification of hypotheses by facts. It is not the necessity for experiment, or the results attained by it, that philosophy claims a right to criticise. The claim is confined to theories about the facts—theories which the history of science itself is constantly showing to have been dogmatic, uncritical, and *a priori*, in the most objectionable sense. It is, in other words, not the physics of the physicist, but his unconscious metaphysics, that are brought under scrutiny.

If, then, we must rule out the notion of mere endlessness as a help to the conception of infinity, and turn away from quantity on its discrete side, it follows that space and time must be rejected as a field of study. For they can yield us no picture

of anything but finiteness endlessly repeated. Nor do these other relationships which are treated in the higher mathematics avail us. They give us, it is true, the notion of infinity in the sense of what is as self-contained, and as expressible in terms of relationships which are free from the characteristic of implying the "*Beyond*" of quantity in its discrete aspect. But they are abstract, and the true infinite cannot be abstract. What is abstract has been divided or wrenched from a context. It has, therefore, something outside it, and is thus infected with the characteristic of finiteness. The true infinite must, it would seem, be not only concrete but completely self-contained, with nothing beyond it. It can be, in fact, only the ultimate reality into which all else is resolvable, and in terms of which everything can be expressed, while it can itself be expressed in terms of and be resolved into nothing beyond. What, then, must be the characteristic of such an infinite? It cannot be a cause, for a cause is really conditioned by its effect. Nor can it be a substance, for then the attribute would be another, and would limit it.

When Bishop Berkeley came to the conclusion that the real world was but a set of ordered ideas in the mind, and when David Hume followed this up by showing that the same analysis disclosed the mind itself as a mere series of ideas, men began to reconsider their position. Kant's doctrine regarding the synthetic Unity of Apperception was simply a way of saying that reality and the substance, both of the external and the internal world, must be sought within the mind itself. Over the significance of this doctrine a great controversy has raged. It has been, on the one hand, insisted that what is not capable of being the object of knowledge of some kind has no meaning and therefore cannot be deemed to exist. On the other hand, it has been rejoined that reality cannot be resolved into universals of knowledge. Now it is obvious that both propositions may be simultaneously true. Moreover, for the second there is warrant in common experience.

When I look at the room full of people opposite me it is plain that I do more than simply reflect ; I perceive. What is my object world is still the content of mind, but content of a different quality. It is true that all perception depends on, and consists in large measure in, the application of universals, but these do not make up the whole nature of perception. The moment of sense or feeling is present throughout, and if thought and feeling are two separate and self-subsisting entities then perception can no more be resolved into mere thought than it can be resolved into mere feeling. But it is a mistake to infer from this that in the process of perception there are two separate elements. The truth seems to be that thought and feeling, the universal and the particular, are both abstractions made within a concrete whole, a self-consciousness that is *individual* and *singular*. Real knowledge appears to be always either concrete and singular or else a process of abstraction from what is concrete and singular. Mind itself appears as a process in which the structure of the universe, including the isolation in knowledge of the mind itself as one of its own objects, is evolved by reflection on a concrete experience which is constantly being made more definite and distinct and thus progressively transformed. The mind is not a thing beside other things, but a process for which the view of other things and of itself with them develops itself within the activity which is the real characteristic of mind. It is only by abstraction that subject and object are separated as if they were independent existences ; whereas, like the universal and the particular, like thought and feeling, they seem to be only abstractions falling within a single entirety.

Much of the controversy over what is called Idealism has arisen from overlooking this circumstance, and from the tacit assumption that the methods and distinctions of psychology can be employed to throw light on the ultimate nature of knowledge. It is of the nature of the subject to manifest itself, not as inert substance, but as activity, the activity which is

knowledge and which embraces within its ambit the universals of thought and the particulars of feeling alike, and distinguishes them, not as independent realities but as aspects arising by abstraction within itself. It is thus that experience develops, and it is thus that it seems to be constituted. The supposed unknowable things in themselves are notions which arise, not in the concrete actuality of this experience, but from abstractions growing out of reflection upon it. They are really limiting notions which are due to another feature of what is actual in knowledge. The subject is real in the form of the self-consciousness within which arises by reflection the object world which it distinguishes from itself, and yet, as part of the process of turning its reflective activity upon itself, regards as containing itself as a part of the whole. It is as belonging to the object world that it constructs the conception of itself, and of necessity constructs it, as that of one among a plurality of subjects. But, regarded as the foundation and essence of the experience which contains the knowledge of itself and the rest of the universe, the activity of the subject has another aspect. In this it is never object. It is the condition of any object world arising, and is describable consequently, in Kantian language, as the synthetic unity of apperception. But this becomes real only as a concrete individual, which is yet, on its universal side, more than individual. In other words, the infinite is no aspect existing independently of the finite, but realises itself only in and through the finite. The finite, on the other hand, is no independent aspect, but is real only for and through the infinite. The real individual of experience is thus an infinite-finite. In the language of Theology, man is as indispensable to God as God is to man. The true infinite is a process of logical development through finite forms, the limiting concept of which is a whole which is never realised as statically complete, but is yet presupposed as the condition and foundation *ab initio* of the entire process.

This result appears not only to get over many of the difficulties which Idealism has to encounter, but to accord with the investigations of modern logicians. These investigations tend to represent the test of truth as the possibility of complete and harmonious inclusion within a system, which again falls within and accords with a larger system. Our intelligence creates and sustains our real world by a continuous judgment which embraces certain forms in their concrete connection within the unity of a single system. It is the fitting into this unity and the accord with the system in question which is the test of truth, and is what compels our assent and our purpose when we reason. As Mr. Bosanquet puts it towards the close of the second volume of his *Logic*: "Necessity is, then, a character attaching to parts or differences interrelated within wholes, universals, or identities. If there were any totality such that it could not be set over against something else as a part or difference within a further system, such a totality could not be known under an aspect of necessity, *i.e.*, as a link in the chain of the process of development of knowledge. The universe, however we may conceive it as including subordinate systems, must ultimately be incapable, *ex hypothesi*, of entering as an element into a system including more than it. Strictly speaking, therefore, its relation to knowledge must be one of reality, not of necessity. But, also, strictly speaking, it is a reality which we have no power to question or to explain, because all our questioning or explanation falls within it. There can be no meaning in talking of what might be the case if the universe were other than it is, or about what has been the case in order to make the universe what it is."

Every judgment assumes a larger whole and is the analysis of it, and affirms a necessity based on the reality of this whole. We can never include the entirety of the universe in the object world of that experience from which we distinguish knowledge as including it within its system. The entirety

and this entirety we must assume, is at once the foundation of reality and at the same time incapable of being completely given in that reality. It is of the nature of the only self-consciousness that we know, that the development of its knowledge should remain incomplete. Yet, such finite self-consciousness presupposes the infinite which becomes real in it. The two are separable, not as concrete individuals of reality, but only in the abstractions of reflection. The one is the other, and neither is reducible to its logical complement. The possible completion of the system is its key note, presupposition, and foundation. Yet this completion is incapable of being realised save in the unending process to which it gives its character and of which it is the summation.

Let us try to gather together results:—

(1) There is no infinity which does not realise itself in finite forms; there is equally no finite form which has not its foundation and signification in what is infinite. In the popular language of a much abused and little understood *Credo* we must neither unduly confound persons nor divide substances. Each aspect is as real as the other.

(2) The infinite does not exist in contrast with the finite—nor as another numerically apart in time or space. It is no more prolongation of the finite to endlessness.

(3) The real infinite must be regarded as a self-contained system which is real under the aspect of a process, a progress of notional development within which time and space and the limited self of experience appear as stages, constituents, or moments.

(4) To ask for the presentation as a completed perception, or even experience, of such a self-contained system is a contradiction in terms. Its presupposition is the foundation on which such perception and experience in general rest, and it contains them as stages of development within itself.

(5) Such an infinite cannot be substance and must be subject. The category of self-consciousness is the highest

category we know. If a completely self-contained self consciousness could present itself to itself, apart from the element of the finite which it requires for its purposive development, there would be no opposition within it between subject and object. This is the meaning of the doctrine of Immanence.

(6) It follows that the supreme controlling power of the universe is to be sought not in causes, but in ends and the systems which progressively realise them. The system of nature is the logical antithesis which reflection sets up between the universality of abstract reflection, and the particulars which are the limit confronting abstract reflection. These are essential as the condition without which finite spirit could have no basis on which to rise into existence. On the other hand, nature is not prior in time to mind, but arises in and through its distinctions. Moreover, the distinction between thought and feeling is not ultimate. It is in so far as they are severally and abstractly brought before consciousness as among its objects that the distinction between them arises. This distinction, therefore, is made and falls within consciousness, and is not prior to it.

(7) The metaphysical principles above indicated, however difficult they may seem, are necessary, and are as old as the teaching of Aristotle. They have been often forgotten and several times rediscovered. They must be borne in mind if the methods of modern logicians are to have their full application, and if confusion is to be eliminated from the treatment of infinity, even in mathematical science.