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the purity of the nitric acid is a matter of indifference, strong commercial acid answering every purpose.

I remain, dear Sir, very truly yours,

JOHN THOMAS COOPER.

82, Blackfriars Road, London, Dec. 10, 1839.

P.S. I have found within these few days that some kinds of common coke, such as is very brilliant, close-grained, and has a columnar fracture, is equally good with the other varieties of carbon, and admits of being selected of almost any variety of form and size.

J. T. C.

VI. *On Derivation of Coexistence: Part I. Being the Theory of simultaneous simple homogeneous Equations.* By J. J. SYLVESTER, F.R.S. & R.A.S., Professor of Natural Philosophy in University College, London*.

Art. (1.) WE shall have constant occasion in this paper to denote different quantities by the same letter affected with different subscribed numerical indices.

Such a letter is to be termed a "Base."

Every character consisting of a base and an inferior index I call an argument of the base, viz. the first, second, or n th argument, according as 1, 2, or in general (n), be the number subscribed.

Art. (2.) I use the symbol PD to denote the product of the differences of the quantities to which it is prefixed (each being to be subtracted *from* each that follows); thus

PD ($a\ b\ c$) indicates $(b-a)\ (c-a)\ (c-b)$.

PD ($o\ a\ b\ c$) indicates $a\ b\ c\ (b-a)\ (c-a)\ (c-b)$.

PD ($o\ a\ b\ c\ \dots\ l$) indicates $a\ b\ c\ \dots\ l \times$ PD ($a\ b\ c\ \dots\ l$).

Art. (3.) For want of a better symbol I use the Greek letter ζ to denote that the product of factors to which it is prefixed is to be effected after a certain symbolical mannner. This I shall distinguish as the zeta-ic product.

The symbol ζ will never be prefixed except to factors, each of which is made up of one or more terms, consisting solely of linear arguments of different bases, i.e. characters bearing indices below but none above.

I am thereby enabled to give this short rule for zeta-ic multiplication: "Imagine all the inferior indices to become superior, so that each argument is transformed into a *power* of its base; multiply according to the rules of ordinary algebra; after the multiplication has been *done fully out* depress all the

* Communicated by the Author. Part I. appeared in L. & E. Phil. Mag. Dec. 1839, vol. xv. p. 428.

indices into their original position; the result is the zeta-ic product*."

Thus for example $\zeta(a_r \cdot b_s)$ is the same as simply $a_r \cdot b_s$, but $\zeta(a_r \cdot a_s)$ represents not $a_r \cdot a_s$ but a_{r+s} .

So in like manner

$$\zeta((a_h - b_k)(a_l - b_m))$$

$$= a_{h+l} - a_h \cdot b_m - b_k \cdot a_l + b_{m+k}$$

$$\zeta((a_l - b_l)(a_l - c_l)(b_l - c_l))$$

$$= \text{the depressed product of } (a-b)(a-c)(b-c)$$

$$= \text{the depressed value of } a^2(b'-c') + b^2(c'-a') + c^2(a'-b')$$

$$\text{i.e.} = a_2 \cdot b_1 - a_2 \cdot c_1 + b_2 \cdot c_1 - b_2 \cdot a_1 + c_2 \cdot a_1 - c_2 \cdot b_1.$$

Art. (4.) We shall have occasion in this part to combine the two symbols ζ , PD: thus we shall use

$$\zeta \text{ PD } (a_l b_l) \text{ to denote } \zeta(b_l - a_l)$$

$$\zeta \text{ PD } (a_l b_l c_l) \text{ to denote } \zeta((b_l - a_l)(c_l - a_l)(c_l - b_l)).$$

Art. (5.) For the sake of elegance of diction I shall in future sometimes omit to insert the inferior index when it is unity; but the reader must always bear in mind that it is to be *understood* though not expressed.

I shall thus be able to speak of the zeta-ic product of such and such bases mentioned by name.

Art. (6.) We are not yet come to the limit of the powers of our notation. The zeta-ic product of the sum of arguments will consist of the sum of products of arguments, each argument being (as I have defined) made up of a base and an inferior index. Now we may imagine each index of every term of the zeta-ic product *after it is fully expanded* to be increased or diminished by unity, or each at the same time to be increased or diminished by 2, or each in general to be increased or diminished by r . I shall denote this alteration by affixing an (r) with the positive or negative sign to the ζ . Thus

$$\zeta(a_l - b_l)(a_l - c_l) \text{ being equal to } a_2 - a_1 \cdot c_1 + b_1 \cdot c_1 - b_1 \cdot a_1$$

$$\zeta_{+1}(a_l - b_l)(a_l - c_l) \text{ is equal to } a_3 - a_2 \cdot c_2 + b_2 \cdot c_2 - b_2 \cdot a_2$$

$$\zeta_{-1}(a_l - b_l)(a_l - c_l) \text{ is equal to } a_1 - a_0 \cdot c_0 + b_0 \cdot c_0 - b_0 \cdot a_0.$$

In like manner $\zeta \text{ PD } (a b c)$ indicating

$$b_2 \cdot a_1 - b_2 \cdot c_1 + c_2 \cdot b_1 - c_2 \cdot a_1 + a_2 \cdot c_1 - a_2 \cdot b_1$$

* It is scarcely necessary to add that an analogous interpretation may be extended to any zeta-ic function whatever. Thus

$$\zeta(a_1 + b_1)^2 = a_2 + 2a_1 b_1 + b_2$$

$$\zeta \cos(a_1) = 1 - \frac{a_2}{1 \cdot 2} + \frac{a_4}{1 \cdot 2 \cdot 3 \cdot 4}, \text{ \&c.}$$

$$\zeta_{\pm r} \text{ PD } (a \ b \ c) \text{ indicates}$$

$$b_{2\pm r} \cdot a_{1\pm r} - b_{2\pm r} \cdot c_{1\pm r} + c_{2\pm r} \cdot b_{2\pm r} - c_{2\pm r} \cdot c_{1\pm r}$$

$$+ a_{2\pm r} \cdot c_{1\pm r} - a_{2\pm r} \cdot b_{1\pm r}.$$

I shall in general denote $\zeta_{\pm r} \text{ PD } (a \ b \ c \dots l)$ *actually expanded* as the zeta-ic product of $a_i b_i c_i \dots l$ in its r th phase.

Art. (7.) General Properties of Zeta-ic Products of Differences.

If there be made one interchange in the order of the bases to which ζ is prefixed, the zeta-ic product, in whatever phase it be taken, remains unaltered in magnitude, but changes its sign.

Art. (8.) If in any phase of a zeta-ic product two of the bases be made to coincide, the expansion vanishes.

Art. (9.) Let $\int_1$ be used, agreeably to the ordinary notation, to denote the sum of the quantities to which it is prefixed, $\int_2$ to denote the sum of the binary products, $\int_3$ of the ternary ones, and so on.

$$\text{Thus let } \int_1 (a_i \ b_i \ c_i) \text{ or } \int_1 (a \ b \ c) \text{ indicate } a_i + b_i + c_i$$

$$\text{and } \int_2 (a_i \ b_i \ c_i) \text{ or } \int_2 (a \ b \ c) \text{ indicate } a_i b_i + a_i c_i + b_i c_i$$

$$\text{and } \int_3 (a_i \ b_i \ c_i) \text{ or } \int_3 (a \ b \ c) \text{ indicate } a_i b_i c_i$$

we shall be able now to state the following remarkable proposition connecting the several phases of certain the same zeta-ic products.

Art. (8.) Let $a, b, c, \dots l$, denote any number of independent bases, say $(n-1)$; but let the arguments of each base be periodic, and the number of terms in each period the same for every base, namely (n) , so that

$$\begin{array}{ll} a_r = a_{r+n} = a_{r-n} & a_n = a_o = a_{-n} \\ b_r = b_{r+n} = b_{r-n} & b_n = b_o = b_{-n} \\ c_r = c_{r+n} = c_{r-n} & c_n = c_o = c_{-n} \\ \cdot & \cdot \\ l_r = l_{r+n} = l_{r-n} & l_n = l_o = l_{-n} \end{array}$$

r being any number whatever. Then

$$\begin{array}{l} \zeta_{-1} \text{ PD } (o \ a \ b \ c \dots l) = \zeta(\int_1 (a \ b \ c \dots l) \cdot \zeta \text{ PD } (o \ a \ b \ c \dots l)) \\ \zeta_{-2} \text{ PD } (o \ a \ b \ c \dots l) = \zeta(\int_2 (a \ b \ c \dots l) \cdot \zeta \text{ PD } (o \ a \ b \ c \dots l)) \\ \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ \zeta_{-r} \text{ PD } (o \ a \ b \ c \dots l) = \zeta(\int_r (a \ b \ c \dots l) \cdot \zeta \text{ PD } (o \ a \ b \ c \dots l)) \end{array}$$

This proposition admits of a great generalization*, but we have now all that is requisite for enabling us to arrive at a proposition exhibiting under one *coup d'œil* every combination and every effect of every combination that can possibly be made with any number of coexisting equations of the first degree, containing any number of *repeated*, or to use the ordinary language of analysts, (variable or) unknown quantities.

Art. (9.) For the sake of symmetry I make every equation homogeneous; so that to eliminate n repeated terms, no more than n equations will be required.

In like manner the problem of determining n quantities from n equations will be here represented by the case in which we have to determine the *ratios* of $(n + 1)$ quantities from n equations.

Art. (9.) *Statement of the Equations of Coexistence.*

Let there be any number of bases ($a \ b \ c \dots l$), and as many repeated terms ($x \ y \ z \dots t$), and let the number of equations be any whatever, say (n). The system may be represented by the *type* equation

$$a_r \cdot x + b_r \cdot y + c_r \cdot z + \dots + l_r \cdot t = 0.$$

In which r can take up all integer values from $-\infty$ to $+\infty$. The specific number of equations given will be represented by making the arguments of each base *periodic*, so that

$$a_r = a_{\mu n + r} \quad b_r = b_{\mu n + r} \quad c_r = c_{\mu n + r} \quad \dots \quad l_r = l_{\mu n + r},$$

μ being any integer whatever.

Art. (10.) *Combination of the given Equations.—
Leading Theorem.*

Take $f, g, \dots k$ as the *arbitrary* bases of new and absolutely independent but periodic arguments, having the same index of periodicity (n) as $a \ b \ c \dots l$, and being in number $(n - 1)$, i. e. one fewer than there are units in that index.

The number of *differing* arbitrary constants thus *manufactured* is $n \cdot (n - 1)$.

Let $Ax + By + Cz + \dots + Lt = 0$ be the general *prime* derivative from the given equations, then we may make

$$A = \zeta \text{ PD } (o \ a \ f \ g \dots k)$$

$$B = \zeta \text{ PD } (o \ b \ f \ g \dots k)$$

$$C = \zeta \text{ PD } (o \ c \ f \ g \dots k)$$

$$\dot{L} = \zeta \text{ PD } (o \ l \ f \ g \dots k).$$

Art. (11.) Cor. (1.) *Inferences from the Leading Theorem.*

Let the number of equations, or, which is the same thing,

* See the Postscript to this paper for *one* specimen.

the index of periodicity (n), be the same as the number of repeated terms ($x y z \dots t$), then one relation exists between the coefficients: this is found by making the $(n - 1)$ new bases coincide with $(n - 1)$ out of the old bases. We get accordingly, as the result of elimination,

$$\zeta \text{ PD } (o a b c \dots l) = 0.$$

Art. (11.) Cor. (2.) Let the number of equations be one more than that of the given bases, there will then be two equations of condition. These are represented by preserving one new arbitrary base, as λ . The result of elimination being in this case

$$\zeta \text{ PD } (o a b c \dots l \lambda) = 0.$$

Ex. The result of eliminating between

$$a_1 \cdot x + b_1 \cdot y = 0$$

$$a_2 \cdot x + b_2 \cdot y = 0$$

$$a_3 \cdot x + b_3 \cdot y = 0$$

is $\zeta \text{ PD } (o a b \lambda) = 0$

$$\text{i. e. } \lambda_3 \cdot b_2 a_1 - \lambda_3 \cdot b_1 a_2 + \lambda_1 \cdot b_3 a_2 - \lambda_1 \cdot b_2 a_3 + \lambda_2 \cdot b_1 a_3 - \lambda_2 \cdot b_3 a_1 = 0,$$

from which we infer, seeing that $\lambda_3 \lambda_2 \lambda_1$ are independent,

$$b_2 \cdot a_1 - b_1 \cdot a_2 = 0$$

$$b_3 \cdot a_2 - b_2 \cdot a_3 = 0$$

$$b_1 \cdot a_3 - b_3 \cdot a_1 = 0,$$

any two of which imply the third.

In like manner, in general, if the number of equations exceed in any manner the number of bases or repeated terms, the rule is to introduce so many *new* and *arbitrary* bases as together with the old bases shall make up the number of equations, and then equate the zeta-ic product of the differences of zero, the old bases and the new bases, to nothing.

Art. (12.) Cor. (3.) Let the number of equations be *one* fewer than the number (n) of bases or repeated terms; the number of introduced bases in the general theorem is here $(n - 2)$. Make these $(n - 2)$ bases equal severally to the bases which in the type equation are affixed to $z, u \dots t$, then

$$C = 0$$

$$D = 0$$

$$\dots\dots\dots$$

$$\dots\dots\dots$$

$$L = 0,$$

and we have left simply

$$\zeta \text{ PD } (o a c d \dots k l) x + \zeta \text{ PD } (o b c d \dots k l) y = 0.$$

In like manner we may make to vanish all but A and C, and thus get

$$\zeta \text{ PD } (o a b d \dots k l) x + \zeta \text{ PD } (o c b d \dots k l) z = 0,$$

and similarly

$$\zeta \text{ PD } (o a b \dots k) x + \zeta \text{ PD } (o b c \dots l) t = 0.$$

$$\text{Hence } \left. \begin{array}{c} x \\ y \\ z \\ . \\ . \\ . \\ t \end{array} \right\} \text{ are severally as } \left\{ \begin{array}{l} \zeta \text{ PD } (o b c \dots l) \\ \zeta \text{ PD } (a o c \dots l) \\ \zeta \text{ PD } (a b o \dots l) \\ \dots\dots\dots \\ \dots\dots\dots \\ \dots\dots\dots \\ \zeta \text{ PD } (a b c \dots o) \end{array} \right.$$

This is the symbolical representation as a *formula* of the remarkable *method* discovered by Cramer, perfected by Bezout and demonstrated by Laplace for the solution of simultaneous simple equations.

Art. (13.) Cor. (4.) In like manner if the number of repeated terms be two greater than the number of equations, we have for the relation between any *three* of them, taken at pleasure, for instance, x, y, z ,

$$\zeta \text{ PD } (o a d \dots l) x + \zeta \text{ PD } (o b d \dots l) y + \zeta \text{ PD } (o c d \dots l) z = 0.$$

And in like manner we may proceed, however much in excess the number of repeated terms (unknown quantities) is over the number of equations.

Art. (14.) *Subcorollary to Corollary (3.)*

If there be any number of bases $(a b c \dots l)$, and any other two fewer in number $(f g \dots k)$

$$\left. \begin{array}{l} \zeta \text{ PD } (a f g \dots k) \times \zeta \text{ PD } (b c \dots l) \\ + \zeta \text{ PD } (b f g \dots k) \times \zeta \text{ PD } (a c \dots l) \\ + \zeta \text{ PD } (a f g \dots k) \times \zeta \text{ PD } (b c \dots l) \\ \dots\dots\dots \\ + \zeta \text{ PD } (l f g \dots k) \times \zeta \text{ PD } (a b c \dots) = 0, \end{array} \right\} \begin{array}{l} \text{The cross is used} \\ \text{to denote ordinary} \\ \text{algebraical multipli-} \\ \text{cation.} \end{array}$$

a formula that from its very nature suggests and *proves* a wide extension of itself.

In conclusion I feel myself bound to state that the principal substance of *corollaries* (1), (2) and (3) may be found in Garnier's *Analyse Algébrique*, in the chapter headed "Développement de la Théorie donnée par M. Laplace, &c." But I am not aware of having been anticipated either in the fertile notation which serves to express them nor in the general theorems to which it has given birth.

End of Part (2).

[The subject to be continued.]

University College, London, Dec. 9, 1839.

P.S. I shall content myself for the present with barely enunciating a theorem, one of a class destined it seems to the

author to play no secondary part in the development of some of the most curious and interesting points of analysis.

Let there be $(n-1)$ bases $a, b, c \dots l$, and let the arguments of each be "recurrents of the n th order*," that is to say let

$$\begin{aligned} a_i &= \phi \left(\cos \cdot \frac{2\pi i}{n} \right) \quad b_i = \psi \left(\cos \cdot \frac{2\pi i}{n} \right) \quad c_i = \chi \left(\cos \cdot \frac{2\pi i}{n} \right) \\ &\dots\dots l_i = \omega \left(\cos \cdot \frac{2\pi i}{n} \right). \end{aligned}$$

Let R_i denote that any symmetrical function of the r th degree is to be taken of the quantities in a parenthesis which come after it, and let $\mathfrak{S}$ indicate any function whatever. Then the zeta-ic product

$$\zeta \left(\zeta R_i (a \ b \ c \ \dots \ l) \times \zeta_{\mathfrak{S}} \mathfrak{S} \text{PD} (o \ a \ b \ c \ \dots \ l) \right)$$

is equal to the product of the *number*

$$\begin{aligned} R_i \left(\left(\cos \frac{2\pi}{n} + \sqrt{-1} \cdot \sin \frac{2\pi}{n} \right) \left(\cos \frac{4\pi}{n} + \sqrt{-1} \cdot \sin \frac{4\pi}{n} \right) \right. \\ \left. \left(\cos \frac{6\pi}{n} + \sqrt{-1} \cdot \sin \frac{6\pi}{n} \right) \dots\dots \right. \\ \left. \cos \left(\frac{(2n-1)\pi}{n} + \sqrt{-1} \cdot \sin \frac{2(n-1)\pi}{n} \right) \right) \end{aligned}$$

multiplied by the zeta-ic phase

$$\zeta_{\mathfrak{S}-i} \mathfrak{S} \text{PD} (o \ a \ b \ c \ \dots \ l) !!$$

* I am indebted for this term to Professor De Morgan, whose pupil I may boast to have been. I have the sanction also of his authority, and that of another profound analyst, my colleague Mr. Graves, for the use of the arbitrary terms zeta-ic, zeta-ically. I take this opportunity of retracting the symbol S P D used in my last paper, the letter S having no meaning except for English readers. I substitute for it Q D P , where Q represents the Latin word *Quadratus*. On some future occasion I shall enlarge upon a new method of notation, whereby the language of analysis may be rendered much more expressive, depending essentially upon the use of similar figures inserted within one another, and containing numbers or letters, according as quantities or operations are to be denoted. This system to be carried out would require special but very simple printing types to be founded for the purpose.

In the next part of this paper an easy and *symmetrical* mode will be given of representing any polynomial either in its developable or expanded form.