The Determination of Coma from a Central Ray

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XXII. *The Determination of Coma from a Central Ray.* By
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THE optical sine condition, due to Clausius, but more generally associated with the name of Abbe, embodies a result of great importance to the lens designer, since it enables him to form an estimate of the degree to which a system is corrected for coma * without having to calculate any rays other than those necessary to determine the spherical aberration. The extent of the departure from the sine condition is not in itself a measure of the amount of coma. Investigations of the meaning which should be given to a departure from the sine condition when spherical aberration is present have been made by Conrady † and Chalmers. ‡ The conclusion reached by Conrady is that departure from the sine condition produces a proportionate zonal change of magnification. That this is true for rays lying in a principal plane is indeed suggested by the well-known law that $\mu h \sin \theta$ is invariant on refraction, where μ is the refractive index, θ the inclination of the ray to the axis, and h a short length perpendicular to the axis at the point where it is met by the ray. Such a statement of the meaning of the condition may suffice to give an indication of the extent to which it is permissible to depart from the condition, but it is evidently not a quite exact statement. For consider a system which is quite free from aberration on the axis, but which does not satisfy the sine condition. If this departure from the sine condition involved only zonal differences of magnification, the image of a point off the axis would be a straight radial line, an effect inconsistent with the essential character of coma, which in this case is the only aberration the departure from the sine condition involves.

* Coma is to be understood to include all those aberrations, the presence of which influences the fulfilment or non-fulfilment of the sine condition, excepting those which do not disappear when the object point is situated on the axis of the instrument; the latter are referred to as spherical aberrations. Generally speaking, rays from a given object point meeting the stop in an arc of a circular zone will intersect the image plane in a circular arc of equal angular extent when spherical aberration alone is present. When coma alone is present the same rays will meet the image plane in an arc of twice that angular extent.

† "The Optical Sine-Condition," *Monthly Notices of R.A.S.*, Vol. LXV., p. 501.

‡ "The Sine Condition in Relation to the Coma of Optical Systems," *Proc. Phys. Soc.*, Vol. XXII., p. 1.

Chalmers expresses his results in the form :—

Departure from sine condition

= $a \times$ longitudinal spherical aberration

+ $b(1/3 \text{ coma of first order} + 1/5 \text{ coma of second order}$
 $+ \dots)$,

where a and b are constants.

He assumes that the effect of coma in the image plane may be represented by

$$\frac{\delta x}{x} = i_1(2 \cos^2 \psi + 1) \sin^2 A + i_2(4 \cos^2 \psi + 1) \sin^4 A + \dots,$$

$$\frac{\delta y}{x} = i_1(2 \sin \psi \cos \psi) \sin^2 A + i_2(4 \sin \psi \cos \psi) \sin^4 A + \dots,$$

where x is the distance from the axis at which a perfect image would be formed, δx and δy are the displacements in the image plane due to coma of the intersection point of a ray from a zone subtending an angle $2A$ at the axial point of the image plane, and ψ is the angle made by the axial plane through the point in which the ray meets the zone with the axial plane through the object point—i.e., with the plane $y=0$. It will be shown that these assumptions involve a special interpretation of what is to be understood by the term zone.

The problem may be simply investigated by the aid of a potential function. Let the origins of co-ordinates in the object and image spaces be the point in which the axis meets the object plane, and its image formed by paraxial rays respectively. The co-ordinates and corresponding direction cosines in the object space will be denoted by x, y, z and l, m, n , and in the image space by the same letters accented. The axes of x and x' coincide with the axis of symmetry of the system. The length of the path between a given point of the object plane and the foot of the perpendicular to the refracted ray from the origin in the image space being denoted by U ,* the potential function to be used will be B , where $U+B=\text{constant}$. The variables being y, z, m', n' , it is not difficult to show that the direction cosines of the incident ray and the point of intersection of the refracted ray with the paraxial image plane are given by

$$m = \frac{\partial B}{\partial y}, \quad n = \frac{\partial B}{\partial z}, \quad y' = \frac{\partial B}{\partial m'}, \quad z' = \frac{\partial B}{\partial n'},$$

* U is called by Bromwich the "modified characteristic function."

where the initial and final refractive indices are taken to be unity. The extension to the more general case will be easily obtained later.

Since the system is symmetrical about the x axis B must evidently be a function of y^2+z^2 , $m'y+n'z$, and $m'^2+n'^2$ only. Taking the focal length to be unity, B may be written in the form—

$$\begin{aligned} B = & g(m'y+n'z) + \frac{1}{2}g(y^2+z^2) \\ & + \frac{1}{4}\alpha_1(m'^2+n'^2)^2 + \frac{1}{6}\alpha_2(m'^2+n'^2)^3 + \frac{1}{8}\alpha_3(m'^2+n'^2)^4 + \dots \\ & + (m'y+n'z)\{\beta_1(m'^2+n'^2) + \beta_2(m'^2+n'^2)^2 \\ & \qquad \qquad \qquad + \beta_3(m'^2+n'^2)^3 + \dots\} \\ & + (y^2+z^2)\{\gamma_1(m'^2+n'^2) + \gamma_2(m'^2+n'^2)^2 + \gamma_3(m'^2+n'^2)^3 + \dots\} \\ & + (m'y+n'z)^2\{\delta_1 + \delta_2(m'^2+n'^2) + \delta_3(m'^2+n'^2)^2 + \dots\} \\ & + \text{terms involving higher powers of } y \text{ and } z. \end{aligned}$$

The magnification of the object produced by paraxial rays is evidently g . The image point for the ray defined by $(y, 0, m', n')$ is—

$$\begin{aligned} y' = & gy + m'\{\alpha_1(m'^2+n'^2) + \alpha_2(m'^2+n'^2)^2 + \alpha_3(m'^2+n'^2)^3 + \dots\} \\ & + y\{\beta_1(3m'^2+n'^2) + \beta_2(5m'^2+n'^2)(m'^2+n'^2) \\ & \qquad \qquad \qquad + \beta_3(7m'^2+n'^2)(m'^2+n'^2)^2 + \dots\} \\ & + \text{terms involving higher powers of } y. \\ z' = & n'\{\alpha_1(m'^2+n'^2) + \alpha_2(m'^2+n'^2)^2 + \alpha_3(m'^2+n'^2)^3 + \dots\} \\ & + 2m'n'y\{\beta_1 + 2\beta_2(m'^2+n'^2) + 3\beta_3(m'^2+n'^2)^2 + \dots\} \\ & + \text{terms involving higher powers of } y. \end{aligned}$$

The coma terms in the expressions for y' and z' are those involving y to the first power as a factor when m' and n' assume the values they bear for a ray from the axial point of the object. Let the direction cosines for such a ray be denoted by capital letters L, M, N, L', M', N' . Although m' and n' may differ very little from M' and N' , it is not permissible to replace the small letters by capitals in the equations for y' and z' , for m' and n' are only actually equal to M' and N' when $y=0$. The exact expressions will be—

$$\begin{aligned} m' = & M' + yp + y^2p' + \dots \\ n' = & N' + yq + y^2q' + \dots \end{aligned} \quad (1)$$

where $p, q, p', q', \dots$ are functions of M' and N' . In the present case only p and q are involved in the determination of

coma. The substitution of $M'+yp$ and $N'+yq$ for m' and n' leads to

$$y' = gy + M' \{ \alpha_1 (M'^2 + N'^2) + \alpha_2 (M'^2 + N'^2)^2 + \alpha_3 (M'^2 + N'^2)^3 + \dots \} \\ + y [(\beta_1 + \alpha_1 p) (3M' + N'^2) + (\beta_2 + \alpha_2 p) (5M'^2 + N'^2) (M'^2 + N'^2) \\ + (\beta_3 + \alpha_3 p) (7M'^2 + N'^2) (M'^2 + N'^2)^2 + \dots] \\ + 2qM'N' \{ \alpha_1 + 2\alpha_2 (M'^2 + N'^2) + 3\alpha_3 (M'^2 + N'^2)^2 + \dots \} \\ + \text{terms involving } y^2, \&c.,$$

and

$$z' = N' \{ \alpha_1 (M'^2 + N'^2) + \alpha_2 (M'^2 + N'^2)^2 + \alpha_3 (M'^2 + N'^2)^3 + \dots \} \\ + y [2M'N' \{ \beta_1 + \alpha_1 p + 2(\beta_2 + \alpha_2 p) (M'^2 + N'^2) + 3(\beta_3 + \alpha_3 p) (M'^2 + N'^2)^2 + \dots \} \\ + q \{ \alpha_1 (M'^2 + N'^2) + \alpha_2 (M'^2 + 5N'^2) (M'^2 + N'^2) + \alpha_3 (M'^2 + 7N'^2) (M'^2 + N'^2)^2 + \dots \}] \\ + \text{terms involving } y^2, \&c.$$

Comparison with Chalmers' initial equations shows that he has assumed $p = \text{constant}$ and $q = 0$.

The direction cosines of the incident ray are

$$m = gm' + gy + m' \{ \beta_1 (m'^2 + n'^2) + \beta_2 (m'^2 + n'^2)^2 + \beta_3 (m'^2 + n'^2)^3 + \dots \} \\ + 2y \{ \gamma_1 (m'^2 + n'^2) + \gamma_2 (m'^2 + n'^2)^2 + \gamma_3 (m'^2 + n'^2)^3 + \dots \} \\ + 2m'^2 y \{ \delta_1 + \delta_2 (m'^2 + n'^2) + \delta_3 (m'^2 + n'^2)^2 + \dots \} + \dots$$

and

$$n = gn' + n' \{ \beta_1 (m'^2 + n'^2) + \beta_2 (m'^2 + n'^2)^2 + \beta_3 (m'^2 + n'^2)^3 + \dots \} \\ + 2m'n'y \{ \delta_1 + \delta_2 (m'^2 + n'^2) + \delta_3 (m'^2 + n'^2)^2 + \dots \} + \dots$$

or, substituting for m' and n' in terms of M' and N' —

$$m = gM' + gy(1+p) + M' \{ \beta_1 (M'^2 + N'^2) + \beta_2 (M'^2 + N'^2)^2 + \beta_3 (M'^2 + N'^2)^3 + \dots \} \\ + y \{ (2\gamma_1 + \beta_1 p) (M'^2 + N'^2) + (2\gamma_2 + \beta_2 p) (M'^2 + N'^2)^2 + \dots \} \\ + 2M'^2 y \{ \delta_1 + \delta_2 (M'^2 + N'^2) + \delta_3 (M'^2 + N'^2)^2 + \dots \} \\ + 2M'y(M'p + N'q) \{ \beta_1 + 2\beta_2 (M'^2 + N'^2) + 3\beta_3 (M'^2 + N'^2)^2 + \dots \} \\ + \dots$$

and

$$n = gN' + gqy + N' \{ \beta_1 (M'^2 + N'^2) + \beta_2 (M'^2 + N'^2)^2 + \beta_3 (M'^2 + N'^2)^3 + \dots \} \\ + 2M'N'y \{ \delta_1 + \delta_2 (M'^2 + N'^2) + \delta_3 (M'^2 + N'^2)^2 + \dots \} \\ + qy \{ \beta_1 (M'^2 + N'^2) + \beta_2 (M'^2 + N'^2)^2 + \dots \} \\ + 2N'(M'p + N'q)y \{ \beta_1 + 2\beta_2 (M'^2 + N'^2) + 3\beta_3 (M'^2 + N'^2)^2 + \dots \} \\ + \dots$$

It will be convenient to adopt a more concise notation.
Put

$$\begin{aligned} a &= \alpha_1(M'^2 + N'^2) + \alpha_2(M'^2 + N'^2)^2 + \alpha_3(M'^2 + N'^2)^3 + \dots \\ a &= 2 \{ \alpha_1 + 2\alpha_2(M'^2 + N'^2) + 3\alpha_3(M'^2 + N'^2)^2 + \dots \} \\ l &= g + \beta_1(M'^2 + N'^2) + \beta_2(M'^2 + N'^2)^2 + \beta_3(M'^2 + N'^2)^3 + \dots \\ b' &= 2 \{ \beta_1 + 2\beta_2(M'^2 + N'^2) + 3\beta_3(M'^2 + N'^2)^2 + \dots \} \\ c &= g + 2 \{ \gamma_1(M'^2 + N'^2) + \gamma_2(M'^2 + N'^2)^2 + \gamma_3(M'^2 + N'^2)^3 + \dots \} \\ d &= 2 \{ \delta_1 + \delta_2(M'^2 + N'^2) + \delta_3(M'^2 + N'^2)^2 + \dots \} \end{aligned}$$

so that

$$y' = aM' + y \{ ap + b + M'(M'p + N'q)a' + M'^2b' \} + \dots \quad (2)$$

$$z = aN' + y \{ aq + N'(M'p + N'q)a' + M'N'b' \} + \dots \quad (3)$$

The quantities corresponding to y' , z' , m , n for the central ray are denoted by Y' , Z' , M , N . They are obtained by putting $y=0$ into equations (2) and (3). The quantity a defines the central spherical aberration and b is the sine ratio, which is constant when the sine condition is satisfied.

Equations (1), (2) and (3) show that p and q serve to define with which ray from $(y, 0)$ the central ray is compared. Suppose the two rays intersect in the object space at a point distant ρ from the origin. The conditions

$$\frac{l}{L} = \frac{m}{M - y/\rho} = \frac{n}{N}$$

are then satisfied.

The relation $\frac{l}{L} = \frac{n}{N}$ give:

$$p(b^3 + b') + b^2c + d; M'N' + q(b - b^3M'^2 + b'N'^2) = 0, \quad (4)$$

and $\frac{m}{M - y/\rho} = \frac{n}{N}$ reduces to

$$\frac{1}{\rho} + c = b \left(\frac{M'}{N'q - p} \right). \quad (5)$$

Similarly, if the emergent rays meet in a point distant ρ' from the intersection of the central ray with the paraxial image plane,

$$\frac{\rho' L'}{l'} = \frac{Y' - y' + \rho' M'}{m} = \frac{Z' - z' + \rho' N'}{n'}.$$

The equality of the last two expressions yields

$$(\rho' + a) \left(\frac{M'}{N'q - p} \right) = b, \quad (6)$$

and the other condition leads to

$$\left\{ \frac{\rho'}{1-M'^2-N'^2} + a + (M'^2+N'^2)a' \right\} (M'p+N'q) + M'b + M'(M'^2+N'^2)b' = 0. \quad (7)$$

If both incident and emergent rays meet, equations (5) and (6) give

$$\left(\frac{1}{\rho} + c \right) (\rho' + a) = b^2, \quad (8)$$

as the equation for finding the point of intersection of the refracted rays when that for the incident rays is known. In order that the rays should actually intersect in this way p and q must receive values which enable both the conditions (4) and (7) to be satisfied. In general, whether the rays meet or not (8) is the relation determining the position of the radial focal line for a given object point.* This result does not hold for rays lying in a primary or axial plane, for which case q should be eliminated between (4) and (5) and N' put equal to zero in (7). The corresponding equations are, therefore,

$$\frac{1-b^2M'^2}{r} + p(b+M'^2b') + c + M'^2d = 0, \quad (9)$$

$$\frac{r'p}{1-M'^2} + p(a+M'^2a') + b + M'^2b' = 0, \quad (10)$$

or, eliminating p ,

$$\left(\frac{1-b^2M'^2}{r} + c + M'^2d \right) \left(\frac{r'}{1-M'^2} + a + M'^2a' \right) = (b+M'^2b')^2, \quad (11)$$

where r and r' have been written instead of ρ , ρ' to distinguish the different cases. Equation (11) fixes the position in which the central ray is met by the transverse focal line for a given object point.

Since from equation (4) q involves $M'N'$ as a factor, the value of q is seen from equations (2) to be quite immaterial as regards the refraction of rays associated with the ray $M'=0$. The points, therefore, of the comatic ring lying in the plane of the object point are determined by p alone. The value of q , however, does affect the shape of the ring. Equation (4) shows that the ring from a zone cannot be assumed to be circular unless the zone is so defined that $q=0$. Moreover, under these conditions p can only be a constant if certain relations hold among the coefficients of the system, and these coefficients

* This equation may be compared with equations (9) of p. 508, Proc. Phys. Soc., Vol. XXVII.

evidently involve the curvature and astigmatism in the system. It is, therefore, not permissible to assume that coma can be divided into terms of various orders which will automatically be resolved into components having displacements for rays in primary and secondary planes in the ratios 3:1, 5:1, 7:1, &c., by expanding in powers of the sine of the semi-angular aperture of a symmetrically placed circular zone. Chalmers' formula, though probably reliable in most cases, will consequently not be exact for large apertures. Assuming, however, a definition of a zone which leads to $q=0$, it is possible to find the two image points in the primary or axial plane, and so determine exactly the amount of the coma present.

Take first the ray in the secondary plane $M'=0$, for which

$$y'=y(ap+b), \quad z'-Z'=0.$$

From equation (6)

$$ap+b=-p\rho'. \quad (12)$$

The distances from the axis of the conjugate points on the ray are, for the object space,

$$\rho.V=\rho b.N',$$

and for the image space

$$(\rho'+a).N'.$$

Let the ratio of these heights be σ ; then

$$\rho'+a=\rho b\sigma,$$

and by comparison with (12)

$$p\rho\sigma=-1,$$

so that

$$\frac{y'}{y}=ap+b=\frac{\rho'}{\rho\sigma}. \quad (13)$$

Now, consider rays in the primary plane for which $N'=0$. Equations (2) give

$$z'=0,$$

and

$$\begin{aligned} y'-Y' &= y(p+b+M^2(a'p+b')) \\ &= -y_1 \frac{r'p}{1-M'^2}, \end{aligned}$$

by equation (10).

The p in this instance is as a rule not the p of the previous case. From equations (3) and (9)

$$\begin{aligned} m-M &= m-b.M' \\ &= y(b\rho+M'^2b'p+c+M'^2d) \\ &= -y \cdot \frac{1-M^2}{r}. \end{aligned}$$

Also from (1)

$$m' - M' = yp,$$

and, therefore,

$$\frac{m - M}{m - M'} = -\frac{1 - M^2}{p},$$

and

$$\begin{aligned} \frac{y' - Y'}{y} &= \frac{r'(1 - M^2)}{r(1 - M'^2)} \cdot \frac{m' - M'}{m - M}, \\ &= \frac{r'}{rs} \sqrt{\frac{1 - M^2}{1 - M'^2}}, \end{aligned} \quad (14)$$

where s is the reciprocal of the angular magnification at the conjugate points determined by r and r' . If $\sin \theta$ and $\sin \theta'$ are written for M and M' respectively the conditions for freedom from coma may evidently be written,

$$\frac{\rho'}{\sigma \rho} = \frac{r' \sec \theta'}{sr \sec \theta} = \text{constant for all rays.} \quad (15)$$

These conditions hold whatever the spherical aberration on the axis may be.

The quantities

$$\frac{r' \sec \theta'}{sr \sec \theta} \quad \text{and} \quad \frac{\rho'}{\sigma \rho}$$

may evidently be looked upon as the magnifications in the primary and secondary directions for small pencils, of which the ray (M', N') is a member. If these are greater than the magnification for paraxial rays the distance of the comatic ring from the axis will be greater than that of the image point for a ray through the centre of the objective. It is possible for one of these magnifications to be greater and the other less than the paraxial magnification, a case which shows the importance of finding both quantities if a correct estimate of the magnitude of the comatic patch is to be obtained.

If the external refractive indices differ it is necessary to regard g , s and σ as the reciprocals of angular magnifications. The quantities which have hitherto been called lengths will then be lengths multiplied by refractive indices. As, however, only the ratios of $\frac{y' - Y'}{y}$, $\frac{r'}{r}$, $\frac{\rho'}{\rho}$ to one another are involved in the final result, and these are all altered in the same proportion, it will be permissible to look upon the quantities that have been called lengths as absolute lengths, making g a linear magnification, while s and σ remain as before, reciprocals of angular magnifications, or linear magnifications multiplied by μ'/μ ,

where μ and μ' are the refractive indices of the object and image spaces respectively.

The final conditions in their general form might well have been suggested by the theory of oblique pencils of rays. Let Fig. 1 represent an axial plane of the system OO' is the axis, and OP , OQ incident rays arising from the axial point O , which are refracted as $P'O'$, $Q'O'$. Other rays, YP and YQ , start from Y , a point in the plane of the diagram on the perpendicular to the axis through O , and after refraction emerge as $P'Y'$ and $Q'Y'$. If the distance OY is small, and P and Q are equidistant from the axis, and in the same normal plane, P' and Q' will lie on another normal plane,

FIG. 1.

and be equidistant from the axis, and Y' will lie on the perpendicular to the axis through O' . P , P' and Q , Q' are pairs of conjugate points, and provided the angles are small the ratio of the angle between the two incident rays OP , YP to the angle between the corresponding emergent rays $P'O'$, $P'Y'$ is constant. Call this ratio s , so that $\widehat{OPY} = s \cdot \widehat{O'P'Y'}$. Let ON , $O'N'$ be perpendiculars on to the rays PY , $P'Y'$. Then $ON = OP \cdot \widehat{OPY}$ and $O'N' = P'O' \cdot \widehat{O'P'Y'}$, or

$$\frac{ON}{O'N'} = s \cdot \frac{OP}{P'O'}.$$

Now, if the angles between OP and $P'O'$ and the axis are θ , θ' the angles $\widehat{YON}$ and $\widehat{Y'O'N'}$ are also θ , θ' . Thus, $ON = OY \cos \theta$, $O'N' = O'Y' \cos \theta'$, or

$$\frac{O'Y'}{OY} = \frac{P'O' \sec \theta'}{s \cdot OP \sec \theta}.$$

If, instead of considering the length $O'Y'$ in the plane in which the central ray meets the axis another normal plane, such as that meeting the axis in I were taken, the result obtained would be—

$$\frac{JK}{OY} = \frac{P'J \sec \theta'}{s \cdot OP \sec \theta}.$$

Figs. 2 and 3 relate to rays in a secondary plane. In Fig. 2 the rays from O and Y are seen superposed. Fig. 3 is a view in which the planes containing both central and oblique secondary rays appear as straight lines. The secondary conjugate foci on the central rays are p , p' and q , q' . The point

FIG. 2.

FIG. 3.

of union of the two oblique rays is H on the line from O' through Y' (Fig. 1). The ratio of the angles $\widehat{OpY}$, $\widehat{O'p'H}$ is a constant σ so long as these angles are small. Thus,

$$\frac{HO'}{OY} = \frac{p'O' \cdot \widehat{O'p'H}}{Op \cdot \widehat{OpY}} = \frac{p'O'}{\sigma \cdot Op}$$

or, if measurements are to be made in the plane through I and the rays $p'O'$ and $p'H$ meet this plane in j and k respectively,

$$\frac{jk}{OY} = \frac{p'j}{\sigma \cdot Op}$$

Now, if h is a short length on a straight line through p normal to the plane Opn , and h' is its image, and therefore a length on a straight line through p' normal to the same plane,

$$\sigma = \frac{\mu' h'}{\mu h}.$$

But the lengths h' and h can be determined by rotating the primary plane containing p through a small angle about the axis, that is to say,

$$\sigma = \frac{\mu' h'}{\mu h} = \frac{\mu' \cdot p'n'}{\mu \cdot pn}.$$

The length $O'Y'$ gives the distance from the axis at which the rays YP , YQ in a primary plane unite, and $O'H$ gives the distance at which the rays Yp , Yq in a secondary plane unite. When coma is present but there is no spherical aberration, as in the plane through O' , the difference between $O'Y'$ and $O'H$ gives the diameter of the comatic circle. The conditions for freedom from coma are the equality of $O'Y'$ and $O'H$ to one another and to the distance from O' of the paraxial image of Y . When spherical aberration is present, as when the measurements are made in the plane through I , the light from Y through the zone $PpQq$ will meet the image plane in a curve exactly similar to that associated with the rays from O through this zone if JK and jk are equal. The condition of freedom from coma throughout the lens aperture is that for every zone JK and jk shall be equal to the displacement for paraxial rays. The analysis based on the potential function B shows that the conditions to which this geometrical treatment leads hold rigorously no matter how great the inclination of the incident beams of light may be.

In the application of the results to the determination of the amount of coma present in a system, the ray (M' , N') will usually pass near the margin of the aperture. The limiting stop may be a component lens or a hole in an opaque screen placed in the object space, the image space, or in an intermediate position between some of the component lenses. One of the intersection points will usually be selected in the plane of the stop or an image position corresponding approximately to that plane. It then only remains to find the two conjugate points and the corresponding primary and secondary magnifications to determine the coma. The determination of these conjugate points and magnifications is a type of calculation with which optical computers are quite familiar, and so offers no difficulty.

The effective stop of many important systems is in the neighbourhood of the principal points. The properties usually ascribed to the principal planes will not hold for any but paraxial rays, neither in general can the planes be replaced by a pair of surfaces which stand to one another in the relation of object and image, and have for a greater number of rays the properties possessed by the principal planes for paraxial rays. For there will usually be astigmatism in the image of a surface through a principal point; in fact, the only way in which astigmatism can be removed from the image it is desired to

correct is to transfer it to the stop. The place of each principal plane must, therefore, be taken by two surfaces, one of which will relate to rays in a primary, and the other to rays in a secondary plane. The surfaces in the general case will only relate strictly to rays coming from a restricted axial region, and will alter in shape with changes in the part of the axis from which the rays considered arise, just as the real image of an object in a lens not free from spherical aberration and coma will have varying focal surfaces as the stop is moved along the axis. The results that have been proved may then be stated in the form that the coma is determined by the shape of these principal surfaces, and coma is removed from a system by ensuring that these surfaces are suitably curved. This is clearly seen in the case of any thin system of lenses where the principal surfaces are free from spherical aberration, coma and astigmatism as well as from distortion; an increase in the curvature of each lens surface by the same amount R causes an increase in the curvature of each principal surface by $R(1+\varpi)$ in the same sense, where ϖ is the Petzval sum. This, combined with the fact that the difference in the curvatures of the principal surfaces is ϖ times the power of the lens, accounts for the prominent part taken by the Petzval sum in the theory of the aberrations of thin lenses.*

In one particular case, which is of frequent occurrence, there is no need to make any special calculation to determine the amount of coma. This is when the object is at infinity. The values of r and ρ are then equal for all rays and the secondary second principal surface is determined by the intersection of the paths of the incident and refracted rays. The constancy of the length intercepted on the refracted ray between the incident ray produced and the image plane is the condition for freedom from coma in the secondary plane. If this condition is fulfilled for all rays it is easy to see from the equation for y' that the corresponding condition for rays in an axial plane must necessarily be satisfied also. When the condition is not satisfied the displacement due to coma of rays in a secondary plane is

$$-(\rho'_j + 1)y'_j,$$

and due to rays in a primary plane is

$$-\left(1 + \tan \theta' \frac{d}{d\theta'}\right)(\rho'_j + 1)y'_j,$$

* See Proc. Phys. Soc., Vol. XXVII, p. 494.

where y_0' is the distance from the axis at which the image due to paraxial rays is formed, f is the focal length and ρ' is the length intercepted in the way just described on the refracted ray inclined at θ' with the axis. It is evident that when there is no spherical aberration and the condition $\rho' = -f$ is satisfied the second condition also holds. In this form the conditions are of value in the calculation of telescope objectives by trigonometrical methods. Interesting alternative forms for the total primary and secondary displacements are

$$-\frac{d(\rho' \sin \theta')}{d(\sin \theta')} \cdot \frac{y_0'}{f} \quad \text{and} \quad -\frac{(\rho' \sin \theta')}{\sin \theta'} \cdot \frac{y_0'}{f},$$

where it will be noted that $\rho' \sin \theta'$ is the length on the image plane intercepted between the incident and refracted rays.

The conditions under which Conrady's statement is correct may be readily found from the relations that have been proved in the course of this investigation. Equation (2) shows that any equation which involves the sine ratio, but not its differential coefficient can only refer to secondary rays, and also that the sine ratio will only measure the magnification for secondary rays if either $a=0$ or $p=0$. In the former case the zone considered is free from spherical aberration, and in the latter case the effective stop is in the object space in the surface containing the radial focal lines for an object at infinity in the image space. When the stop is so placed the primary and secondary magnifications are

$$\frac{\mu}{\mu'} \frac{d(\sin \theta)}{d(\sin \theta')} \quad \text{and} \quad \frac{\mu \sin \theta}{\mu' \sin \theta'}$$

respectively, but this expression for the primary magnification will only apply to the case when there is freedom from spherical aberration and the stop is not in the focal surface described, if the further condition is satisfied that the value of the spherical aberration is stationary on passing through this zone.

Exact formulæ of the class to which Conrady's and Chalmers' results belong may be found from the general equations already established. Such formulæ are not as general as those of equations (13) and (14), since the co-ordinates of rays in the object space do not appear in them, and the stop must consequently be assumed to lie in the image space. Let the stop lie in a plane at a distance ξ from the image plane, the equation of this

plane being therefore $x' = \xi$. Denote by g_1 and g_2 the primary and secondary magnifications of the image, so that—

$g_1 = g + \text{primary magnification due to coma,}$

and $g_2 = g + \text{secondary magnification due to coma.}$

Eliminate p between equations (12) and (13). Then—

$$\frac{y}{y'} \cdot b - 1 = \frac{a}{\rho'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

or, if the longitudinal spherical aberration is denoted by A —

$$\frac{1}{g_2} \cdot \frac{\mu \sin \theta}{\mu' \sin \theta'} - 1 = \frac{A}{\xi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

showing that the result $g_2 = \frac{\mu \sin \theta}{\mu' \sin \theta'}$ holds when either $A = 0$,

or $\xi = \infty$, as has already been pointed out. Equation (17) is the form Chalmers' condition takes when the assumption that his coefficients i_1 , i_2 , &c., are independent of the zone of the lens is eliminated.* The corresponding relation for the primary magnification is obtained by writing $a + M'^2 a'$, $b + M'^2 b'$ and $r'/(1 - M'^2)$ for a , b and ρ' respectively in equation (16). The substitution of $\sin \theta'$ for M' , $A \sec \theta'$ for a , $\sin \theta/\sin \theta'$ for b , and $\xi \sec \theta'$ for r' , leads at once to

$$\frac{1}{g_1} \cdot \frac{\mu}{\mu'} \cdot \frac{d(\sin \theta)}{d(\sin \theta')} - 1 = \frac{1}{\xi} \left(A + \sin \theta' \cos \theta' \frac{dA}{d\theta'} \right), \quad . \quad . \quad (18)$$

as the equation for the determination of g , the primary magnification. The conditions under which the primary magnification is represented by $\frac{\mu}{\mu'} \frac{d(\sin \theta)}{d(\sin \theta')}$ may be verified from this equation. The formulæ (17) and (18) will in a number of cases be more readily applied to the results obtained in tracing rays for the determination of spherical aberration than those given earlier in the Paper. They are not, however, immediately applicable to the case of an object at infinity, and are not so suggestive as are equations (13) and (14) of the means that

* In Chalmer's notation the equation would be written

$$\frac{OO_1}{JJ_1} \cdot \frac{\mu_0 \sin B}{\mu_2 \sin A} - 1 = \frac{KI}{N_1 K'}$$

which is simply a rearrangement of the equation

$$JJ_1 = OO_1 \frac{\mu_0 \sin B}{\mu_2 \sin A} \cdot \frac{N_1 K'}{N_1 I'}$$

he has reached on the middle of p. 6 before introducing approximations.

should be taken to remove coma in excess of the limit that can be tolerated. It is important in applying all the results to make sure that correct signs are given to all the quantities. In the case which most frequently arises—

$\theta, \rho, r, \sigma, s$ are positive, and

$\theta', \rho', r', g, g_1, g_2, \xi, x$ are negative.

The spherical aberration A has the sign usually adopted—viz., positive for under-correction, negative for over-correction.

It may be noted that when the stop is in the image space the comatic displacements in the image of rays through a symmetrically-placed circular zone of the stop will necessarily obey the circular law described on p. 293. For, if $P - N'^2Q$ and $M'N'Q$ are written for p and q respectively, equation (7) becomes

$$\left\{ \frac{\rho'}{1 - M'^2 - N'^2} + a + (M'^2 + N'^2)a' \right\} P + b + (M'^2 + N'^2)b' = 0,$$

showing that P , like all the other variables, is a function of $(M'^2 + N'^2)$, and thus constant for the zone. The new form of equation (6),

$$(\rho' + a)\{(M'^2 + N'^2)Q - P\} = b,$$

shows that Q also is constant for the zone. Equations (2) may now be written in the form—

$$y' - Y' = -y\rho' \left\{ \frac{(1 - N'^2)P}{1 - M'^2 - N'^2} - N'^2Q \right\}$$

$$z' - Z' = -y\rho' \left\{ \frac{P}{1 - M'^2 - N'^2} + Q \right\} M'N',$$

which indicate displacements of the type considered. It is, therefore, unnecessary in this case to assume $q=0$. The generality of the results obtained by assuming $q=0$ is not, of course, affected since the displacements are determined by p for both $M'=0$ and $N'=0$.

If this substitution for p and q is made in (4) the result is—

$$P(b^3 + b') + b^2c + d + Q\{b - (M'^2 + N'^2)b^3 - N'^2(b^2c + d)\} = 0,$$

so that the corresponding incident rays will only intersect on a circle, if either $Q=0$ or $b^2c + d=0$. The latter alternative is evidently the condition for freedom from astigmatism in the image of the stop for pencils of rays meeting the axis near the origins of co-ordinates. It is interesting to notice that the condition for freedom from astigmatism in a zone should

contain the square of the sine ratio with respect to the centre of the stop for rays through that zone.

Equations (13) and (14) show at once why in first order coma the displacement of the intersection point of rays in a primary plane is three times that of the point for rays in a secondary plane. For it may be assumed that to a first approximation the principal surfaces are planes normal to the axis. Let j be the common limiting value for ρ'/ρ and r'/r when the rays become paraxial. Then $\rho'/\rho = r'/r = j \sec \theta' \cos \theta$.

Also $\sigma = \mu'/\mu$ and $s = \frac{\mu'}{\mu} \sec \theta \cos \theta'$, so that for secondary planes,

$$\frac{y'}{y} = j \frac{\mu'}{\mu} \sec \theta' \cos \theta,$$

and for primary planes

$$\frac{y' - Y'}{y} = j \frac{\mu'}{\mu} \sec^3 \theta' \cos^3 \theta,$$

or, since θ and θ' are small,

$$\frac{y'}{y} - j \frac{\mu'}{\mu} = \frac{1}{2} j \frac{\mu'}{\mu} (\theta'^2 - \theta^2)$$

for secondary rays, and

$$\frac{y' - Y'}{y} - j \frac{\mu'}{\mu} = \frac{3}{2} j \frac{\mu'}{\mu} (\theta'^2 - \theta^2)$$

for primary rays.

The connection which has been shown to hold between the coma of a real image and the curvature of the surfaces of constant magnification close to the stop is only one example of numerous reciprocal relations which hold for all lens systems. Another one has been mentioned incidentally in the course of the proof, viz., the necessity of transferring to the stop the astigmatism it may be desired to remove from the image for which the system is to be corrected. Other examples which are more obvious are the relation between the coma in the stop and the distortion of the image, and the spherical aberration in the stop and the change of distortion as the magnification is varied. The particular instance which forms the subject of this note is dealt with at some length because a knowledge of the results here established is likely to be of use to optical computers at the present time.

ABSTRACT.

The absence of coma from an optical system free from spherical aberration can be established by means of the sine condition, from the refraction of rays which originate at the centre of the object. In actual systems spherical aberration is always present to some extent, and the conditions which have to be secured in practical cases do not involve mathematical freedom from spherical aberration and coma, but rather their confinement within predetermined limits. Previous investigations on this subject have been published by Conrady and by Chalmers; the present Paper shows that the conclusions of both require some qualification. Conrady's conclusion, that the sine ratio for any zone gives the exact magnification, holds only for rays in a secondary plane, and one of the further conditions (a) that the zone is free from spherical aberration, or (b) that the stop is at the principal focus for rays travelling in the negative direction, must also be satisfied. Under these conditions the magnification for rays in a

primary plane is $\left(1 + \tan \theta' \frac{d}{d\theta'}\right)$ times that for the secondary rays, where $2\theta'$ is the angle subtended by the zone at the centre of the image. The conditions assumed by Chalmers involve the satisfaction of a relation between various quantities, including the curvature and astigmatism of the system; his results may in consequence not apply strictly where very large apertures are involved. Analysis of the general problem shows that the primary and secondary comatic displacements can be accurately derived from the properties of thin oblique pencils of rays. If a central incident ray inclined at θ to the axis is met by an oblique ray in the same axial plane at a distance r from its point of origin, and is refracted at an angle θ' with the axis and meets the image plane at a distance r' from the image of its intersection with the oblique ray, the primary comatic displacement of the oblique ray plus the aberrationless first order displacement is proportional to $r' \sec \theta' / sr \sec \theta$, where $\mu s / \mu'$ is the magnification in a primary plane for a small normal object at the point of intersection of the two rays. The corresponding secondary displacement is $\rho' / \sigma \rho$, where the Greek letters have the same meaning for secondary rays as the corresponding Roman letters for primary rays. These two quantities determine the coma exactly whatever the spherical aberration may be. It follows that coma is dependent upon the principal surfaces of a lens. When the object is at infinity the coma is determined by the locus of the point of intersection of a ray, incident parallel to the axis, with its refracted portion.

The results established in the Paper form one of a series of reciprocal relations which exist between the aberrations of an object and those of the effective stop.

DISCUSSION.

MR. S. D. CHALMERS pointed out the exact relation which he had given in the Paper quoted by Mr. Smith, and said that an assumption was necessary to apply this to the evaluation of coma. The assumption was that the two rays from an axial point and a neighbouring point to the secondary point on the stop would intersect in the image space substantially on the image of the stop. He thought this assumption was justified, though in

exceptional cases it might require careful examination. Apart from such special cases he thought it was perfectly justifiable to use the expansion for the coma in powers of $\sin A$, and to deduce the values in the primary plane from these in the secondary plane. It would be necessary to determine the values for sufficient apertures in the secondary plane to find each of the terms in the coma expansion.

Prof. F. J. CHESHIRE said he was pleased to see that the subject of optical designing was occupying an important place in the work of the Physical Society. The maker who used to turn out the best microscope objectives in Germany would sit with his microscope beside him and a large number of component lenses from which he would try one combination after another until a satisfactory result was reached. In this way some excellent objectives and a large number of mediocrities were obtained. Abbe set out to kill this trial and error method, and introduced at the Zeiss works the system of determining mathematically the complete structural data beforehand. In this country a trial and error method was still largely employed. An approximate design was computed and a trial instrument was made. This turned out badly, and the designer had another try, and so on. First-class results were impossible on these lines. It was true that we had two or three first-class designers in the country whose designs could safely be made up without trial; and this must ultimately be aimed at by every optician if we are to take a prominent position in optical work.

Mr. BLAKESLEY said that an idea was involved in Mr. Cheshire's remarks which he had long held—viz., that a lens should only be used to form an image in the one particular position for which it is designed.

Mr. T. SMITH, in reply to Mr. Chalmers, said that the exact relation mentioned led at once to an expression equivalent to the alternative formula (17). If expressed in this form the equation was exact. The transformation given in Chalmers' Paper, which involved the neglect of a term, was apparently made to enable the primary coma to be found on the supposition that expansion in terms of $\sin A$ would separate out the coma of different ratios. Such a method could be applied provided the spherical aberration were also analysed into terms of different orders in $\sin A$, but not otherwise. The process would then be a comparatively laborious method of reaching the result given immediately by either of the formulæ for the primary comatic displacement contained in the present Paper. The formula given by Chalmers was correct in the absence of spherical aberration; if the spherical aberration of the first n orders were negligible, no serious errors in the estimation of the first $(n+1)$ orders of coma from his formula would arise, but appreciable spherical aberration of the n th order would throw out the estimate of coma of the $(n+1)$ th and subsequent orders.