same thing, writing $f(z) = x' + iy'$, the condition in order that $x' + iy'$ may be a function of $x + iy$ is

$$dx' + idy' = (F + iG)(dx + idy),$$

where F and G are functions of x and y . It is not part of the condition that $F + iG$ shall be a function of $x + iy$, and it is only a long way further on that the authors prove that this is the case (see the definition of a "*fonction holomorphe*," p. 14; and the proof, p. 137). The last-mentioned equation $dx' + idy' = (F + iG)(dx + idy)$, where F and G are only assumed to be functions of x and y , has, if we represent $x + iy$ by means of the point P , coordinates (x, y) , and in like manner $x' + iy'$ by means of the point P' , coordinates (x', y') , the geometrical interpretation that the figures of the points P and P' are conformable figures, that is, figures similar as regards their infinitesimal elements. The foregoing theorem in regard to the Auxesis and Streblosis is that we can, by means of F and G , construct a third conformable figure,—in fact, the Auxesis and Streblosis are $= \sqrt{F^2 + G^2}$ and $\tan^{-1} \frac{G}{F}$ respectively; and, using these as an Extension and Rotation, we have the third conformable figure $x'' + iy'' = (F + iG)(x + iy)$; that is $(F + iG)(x + iy)$, and therefore also $F + iG$, is a function of $x + iy$,—and we have thus the derivative of a function of $x + iy$ as itself a function of $x + iy$.

It is to be remarked that, although the theorem of the Auxesis and Streblosis, considered as a property of conformable figures, is not by any means geometrically self-evident, yet the foregoing analytical proof is only a proof conducted by means of real quantities, of what (admitting the theory of imaginary quantities) is in fact self-evident; viz., the analytical conclusion really is that, F, G denoting functions of x, y , then, if $dx' + idy' = (F + iG)(dx + idy)$, that is, if $(F + iG)(dx + idy)$ be a complete differential, then $F + iG$ is a function of $x + iy$.

A Method of Constructing by Pure Analysis, functions X, Y , &c., which possess the property that $\int XY d\sigma = 0$, and such that any given function can be expanded in the form $\alpha X + \beta Y + \gamma Z + \dots$ By E. J. ROUTH, F.R.S.

[Read May 8th, 1879.]

In many physical theories we have to expand some given function in a series of the kind indicated at the head of this paper. During the Christmas vacation, a method occurred to me of deducing by pure analysis, from one principle, all the expansions known to me. I was

unable to complete the method before the commencement of the following term, and was obliged to postpone the subject until the present time. I now venture to lay it before the Mathematical Society.

The following introduction contains a statement of certain propositions required in the theory. Two different applications of these immediately presented themselves according as (1) the expansion of the given function in terms of X , Y , &c., is to be true only for values of the independent variable restricted to lie between certain limits, or (2) to be always true. The functions X , Y , &c., are found to be any which satisfy a certain equation of an even order, and also certain other conditions which are not the same in the two applications. Two examples are given to illustrate these results. Finally, the theory is extended to the case of two independent variables, and illustrated by an example taken from Laplace's Functions.

I hope on another occasion to lay a further communication before the Society.

INTRODUCTION.

1. Let (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) be the coordinates of the extremities of three conjugate diameters of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Construct the determinant

$$\begin{vmatrix} \frac{x_1}{a} & \frac{y_1}{b} & \frac{z_1}{c} \\ \frac{x_2}{a} & \frac{y_2}{b} & \frac{z_2}{c} \\ \frac{x_3}{a} & \frac{y_3}{b} & \frac{z_3}{c} \end{vmatrix}.$$

Then (i.) the sum of the products two and two of the constituents in any two horizontal lines is zero, but if the horizontal lines be the same the sum is unity.

(ii.) The sum of the products two and two of the constituents in any two vertical lines is zero, except the two vertical lines are the same, and then the sum is unity.

(iii.) Each constituent is equal to its minor, if the proper extremities of the conjugate diameters are chosen; and the whole determinant is equal to unity.

All these results follow at once from the consideration that the constituents of the determinant are the direction-cosines of three straight lines at right angles.

2. The chief object of the following paper is to show, that the first of the properties just mentioned corresponds to that property of Laplace's Functions, and certain other functions, which is usually written

$$\int Y_m Y_n d\sigma = 0,$$

where m and n are not equal ; that the second property corresponds to the proposition that any function can be expanded in a series of the form

$$a_0 Y_0 + a_1 Y_1 + a_2 Y_2 + \dots$$

The third property may be conveniently used to show that the expansion can be made in one way only.

3. In order to establish this correspondence, it will be necessary to extend our propositions to quadrics of any number of variables. Let

$$\text{then} \quad A_1^2 X_1^2 + A_2^2 X_2^2 + A_3^2 X_3^2 + \dots + A_n^2 X_n^2 = H^2$$

be any quadric function of the n variables $X_1, X_2, \dots, X_n$; $A_1, A_2, \&c.$, and H being any constants.

Let us call $(X_1 \dots X_n)$, $(Y_1 \dots Y_n)$, $\&c.$ the n coordinates of the n extremities of any set of conjugate diameters. That is, let these points be such that the diametral plane of each contains all the others. Construct now the determinant

$$\begin{vmatrix} A_1 X_1 & A_2 X_2 & A_3 X_3 & \&c. \\ A_1 Y_1 & A_2 Y_2 & A_3 Y_3 & \&c. \\ A_1 Z_1 & A_2 Z_2 & A_3 Z_3 & \&c. \\ \&c. & \&c. & \&c. \end{vmatrix}$$

Then the first of our preliminary propositions may be written analytically in the forms

$$A_1^2 X_1 Y_1 + A_2^2 X_2 Y_2 + \dots = 0,$$

$$A_1^2 X_1 Z_1 + A_2^2 X_2 Z_2 + \dots = 0,$$

$$\&c. = 0,$$

except when $X = Y$ or $X = Z$, $\&c.$, and then the sum is equal to H^2 .

The second preliminary proposition may be written in the forms

$$A_1 A_2 (X_1 X_2 + Y_1 Y_2 + \dots) = 0,$$

$$A_1 A_3 (X_1 X_3 + Y_1 Y_3 + \dots) = 0,$$

$$\&c. = 0,$$

except when the suffixes are the same, and then the sum is equal to H^2 .

The third preliminary proposition may be written in the same form as before.

4. The second of these properties really follows from the first, so that, whenever the first is true for any determinant, the second will be true also. To prove this, let us put

$$\begin{aligned} A_1 X_1 \xi + A_1 Y_1 \eta + \dots &= xH, \\ A_2 X_2 \xi + A_2 Y_2 \eta + \dots &= yH, \\ &\&c. = \&c. \end{aligned}$$

Then the first property is equivalent to the statement

$$\xi^2 + \eta^2 + \dots = x^2 + y^2 + z^2 + \dots$$

for all values of ξ , η , &c. But if we multiply the equations by $A_1 X_1$, $A_2 X_2$, &c., and add the results, we get

$$H\xi = A_1 X_1 x + A_2 X_2 y + \dots;$$

similarly

$$H\eta = A_1 Y_1 x + A_2 Y_2 y + \dots$$

Adding the squares of these equations, and equating the coefficients of the independent variables x , y , &c., we get the second property.

This proposition will show how closely the two properties put into the heading of this paper are connected with each other.

5. There is one other proposition relating to conjugate diameters which will be wanted. When the given quadric is not referred to its principal coordinates, the second property given above is true, but with some modification. We then have

$$\begin{aligned} X_1 X_2 + Y_1 Y_2 + \dots &= \text{constant}, \\ X_1^2 + Y_1^2 + \dots &= \text{constant}, \\ &\&c. \quad \&c. \end{aligned}$$

for all sets of conjugate diameters of the same quadric.

6. We have now to select one complete set of conjugate diameters. To do this we assume any second quadric we please. Let this be

$$U = \frac{1}{2} a_{11} X_1^2 + a_{12} X_1 X_2 + \dots = 1,$$

so that the constants a_{11} , a_{12} , &c. may be chosen at our pleasure. We shall then find the set of conjugate diameters common to this second quadric and the first quadric.

Let X_1 , X_2 , ... X_n be the coordinates of the extremity of one such diameter in the first quadric. Then we have

$$\left. \begin{aligned} \frac{dU}{dX_1} &= pA_1^2 X_1 \\ \frac{dU}{dX_2} &= pA_2^2 X_2 \\ &\&c. = \&c. \end{aligned} \right\}$$

where p is to be found from the determinantal equation

$$\begin{vmatrix} a_{11}-pA_1^2 & a_{12} & \&c. \\ a_{12} & a_{22}-pA_2^2 & \&c. \\ & \&c. & \end{vmatrix} = 0.$$

We thus have n values of p . Each value of p determines the n coordinates of the extremity of one of the conjugate diameters.

If we divide the constituents of the first row and the first column by A_1 , those of the second row and column by A_2 , and so on, we reduce this determinant to the form given in Art. 44 of Dr. Salmon's "Higher Algebra." The roots of this determinantal equation are therefore all real. The ratios of $X_1, X_2, \&c.$ are therefore also all real.

7. The case in which some of the roots are equal will be found to be important in what follows. It may be shown* that such equalities imply a corresponding indeterminateness in the values of the ratios of $X_1, X_2 \dots X_n$. If one of these ratios is arbitrary, two of the roots will be found equal; if two ratios are arbitrary, three of the roots will be equal, and so on.

8. In beginning with these properties of conjugate diameters, we are not really resting the propositions which follow on a geometrical basis.

* Let the determinantal equation be written $f(p) = 0$. It is proved in Dr. Salmon's "Sixth Lesson on Higher Algebra," that if a new determinantal equation be found by equating to zero the minor of any constituent in the leading diagonal, the roots of *this* equation separate the roots of $f(p)$. If, then, two of the roots of $f(p)$ are equal, it follows that all the minors of every constituent in the leading diagonal are zero. We can now prove that the minor of every constituent is zero. For, consider the constituent a_{12} . Since the determinant vanishes, the square of its minor is (by Ex. 1, Art. 32 of Salmon's Algebra) = (minor of $a_{11}-pA_1^2$) (minor of $a_{22}-pA_2^2$), which is zero. But any other constituent can be brought into the place of a_{12} by changes which keep the determinant symmetrical. Hence the minor of every constituent is zero. But, these are just the conditions that all the equations in Art. 3 can be satisfied in an infinite number of ways by the ratios $X_1 : X_2 : \&c.$ If there be three equal roots, we can show that all the second minors are zero, and that two ratios are indeterminate; see also pages 6 and 7 of the Author's Treatise on the "Stability of Motion."

Conversely, let there be an indeterminateness in the equations, so that one ratio is arbitrary. The analytical condition is that all the first minors are zero. Let us call these minors $B_{11}, B_{12}, \&c.$ Then

$$\frac{df(p)}{dp} = -B_{11}A_1^2 - B_{22}A_2^2 - \dots$$

If, therefore, all the minors of the constituents in the leading diagonal be zero, this differential coefficient is zero, and the determinant has two equal roots. If there be an indeterminateness so that two ratios are arbitrary, all the second minors are

zero. But we have $\frac{d^2f(p)}{dp^2} = -A_1^2 \frac{dB_{11}}{dp} - A_2^2 \frac{dB_{22}}{dp} - \&c.$

Hence this differential coefficient vanishes, and the equation has two equal roots.

We merely use the known results of geometry to shorten the demonstrations. We might, if we pleased, give purely analytical proofs of these properties without a word of allusion to any geometrical construction.

FIRST APPLICATION.

Case in which limits are given.

9. The quadric U is perfectly arbitrary, let us take

$$\begin{aligned} 2U = & a_1 X_1^2 + b_1 (X_2 - X_1)^2 \\ & + a_2 X_2^2 + b_2 (X_3 - X_2)^2 \\ & + \&c. \\ & + a_{n-1} X_{n-1}^2 + b_{n-1} (X_n - X_{n-1})^2 \\ & + a_n X_n^2. \end{aligned}$$

Then, taking any equation, say the x^{th} ,

$$\frac{dU}{dX_x} = pA_x^2 X_x,$$

we have $a_x X_x + b_{x-1} (X_x - X_{x-1}) - b_x (X_{x+1} - X_x) = pA_x^2 X_x$.

Suppose we regard X_x as a function of x and p , so that, when $x = h$, $X = X_1$, when $x = 2h$, $X = X_2$, and so on. Then, this equation is an equation of Differences, which may be also written in the form

$$a_x X_x - \Delta (b_{x-1} \Delta X_{x-1}) = pA_x^2 X_x$$

If we regard h as indefinitely small, and write b' for $h^2 b$, the equation takes the form

$$a_x X - \frac{d}{dx} \left(b'_x \frac{dX}{dx} \right) = pA_x^2 X,$$

which is a Differential equation of the second order.

10. But neither of these two general forms includes the first or the last equation. In order that the equation corresponding to $x = h$ may be included, we must suppose it to be derived by the law of the general term from the quadric

$$U + \frac{1}{2} \lambda X_1^2,$$

where λ is some quantity which is independent of p . That is, we

must have $a_1 X_1 + b_0 (X_1 - X_0) - b_1 (X_2 - X_1) = pA_1^2 X_1$

the same as $a_1 X_1 - b_1 (X_2 - X_1) + \lambda X_1 = pA_1^2 X_1$.

This requires $b_0 (X_1 - X_0) = \lambda X_1$,

or in the limit when h is indefinitely small

$$b'_0 \frac{dX}{dx} = \lambda' X,$$

where $\lambda' = \lambda h$. This equation holds when $x = 0$.

In just the same way the last equation corresponding to $x = nh$ will be included in the general form if we have

$$-b_n (X_{n+1} - X_n) = \mu X_n.$$

In the limit when h is indefinitely small, this becomes

$$-b'_n \frac{dX}{dx} = \mu' X,$$

where μ' has been written for μh . This equation holds when $x = nh$, which in the limit when n is infinite we shall write $x = l$.

11. For the sake of brevity, we shall often speak only of the Differential equation, though the remarks apply also, with slight modifications, to the equation of Differences.

Let then a function X be given, between the limits $x=0$ and $x=l$, by the differential equation

$$a_x X - \frac{d}{dx} \left(b'_x \frac{dX}{dx} \right) = p A_x^2 X,$$

where a_x , b'_x , and A_x are functions of x , but are independent of p . Also, let it be given that, when $x = 0$,

$$b'_0 \frac{dX}{dx} = \lambda' X;$$

and when $x = l$,

$$-b'_l \frac{dX}{dx} = \mu' X,$$

where λ' and μ' are independent of p .

If we solve the differential equation, we have

$$X = L\psi_1(x, p) + M\psi_2(x, p),$$

where L and M are undetermined constants. One of the two conditions at the limits gives the ratio of L to M . We thus have

$$X = P\phi(x, p),$$

where P is a function of p only. The condition at the other limit will now give an equation to determine p . If p , q , &c. be the values of p thus found, we may write

$$Y = Q\phi(x, q),$$

$$Z = R\phi(x, r),$$

&c.

12. This equation, to find p , must be the same as the determinantal equation given in Art. 6, though in an expanded form. For, though no mention has been made of that equation, yet it was deduced from the n equations to find $X_1 \dots X_n$, and is therefore virtually included in

those equations. These are the very equations from which we now deduce an equation to find p .*

13. It appears, from Art. 6, that the values of p , q , &c., thus found, are all real. Since there is no indeterminateness in the value of X , it also appears, from Art. 7, that no two of the roots are equal. In the case of the equation of Differences, the number of roots is exactly n . In the Differential equation, the number is infinite.

14. The values of P , Q , &c. may be easily found by substituting in the equation

$$A_1^2 X_1^2 + A_2^2 X_2^2 + \dots = H^2.$$

Thus, writing $Hh = H'$, we have

$$P^2 \int_0^2 \{\phi(x, p)\}^2 dx = H'^2,$$

with similar expressions for Q , R , &c.

15. It also appears, from Art. 3, that in the case of the equation of Differences, we have

$$\Sigma XY A_x^2 = 0,$$

$$\Sigma XZ A_x^2 = 0,$$

$$\&c. = 0,$$

the summations being taken from $x=0$ to $x=n$.

In the case of the Differential equation, we have

$$\int_0^2 XY A_x^2 dx = 0,$$

$$\int_0^2 XZ A_x^2 dx = 0,$$

$$\&c. = 0.$$

16. The quantities λ , μ are constants at our disposal, so that, by giving them different values, we may sometimes obtain from the same differential equation more than one set of functions. Each of these sets will possess the two properties mentioned in the heading to this paper. These properties are also enunciated in Art. 15 and Art. 17.

* In the case of the equation of Differences we get, by continual substitution, beginning at the first equation,

$$X_n = X_0 \psi_1(x, p) + X_1 \psi_2(x, p),$$

where X_0 and X_1 are undetermined, and ψ_1 , ψ_2 are functions of p of the $n-1^{\text{th}}$ degree. Introducing the given relation between X_0 and X_1 , we have

$$X_n = P\phi(x, p),$$

where ϕ is a function of p of the $n-1^{\text{th}}$ degree. Substituting, now, in the given relation between X_n and X_{n-1} , the factor P divides out, and we have left an equation of the n^{th} degree to find p . Thus the two methods of obtaining this equation are substantially the same, and lead to an equation of the same degree.

17. Let us next inquire whether any function, say $f(x)$, can be expanded in a series of the form

$$f(x) = \alpha \phi(x, p) + \beta \phi(x, q) + \dots$$

Let the values of $f(x)$, when $x = h, 2h, 3h \dots nh$, be $B_1, B_2, B_3 \dots B_n$. These values we assume to be all finite. Consider now the series

$$\begin{aligned} S = & X_x (B_1 A_1^2 X_1 + B_2 A_2^2 X_2 + \dots) \\ & + Y_x (B_1 A_1^2 Y_1 + B_2 A_2^2 Y_2 + \dots) \\ & + \&c. \end{aligned}$$

When $x = h$, we have $X_x = X_1$, $Y_x = Y_1$, &c.; and therefore, by Art. 3, $S = B_1 H^2$. Again, when $x = 2h$, we have $X_x = X_2$, $Y_x = Y_2$, &c.; and therefore $S = B_2 H^2$, and so on for the other values of x .

Substituting for X_x , Y_x , &c. their values given by Art. 11, we see that the expansion

$$f(x) = \alpha \phi(x, p) + \beta \phi(x, q) + \dots$$

has n terms, and is true for the values $x = h, x = 2h$, &c. up to $x = nh$, provided

$$\alpha = \frac{\sum \phi(x, p) A_x^2 f(x)}{\sum [\phi(x, p) A_x]^2}, \quad \beta = \frac{\sum \phi(x, q) A_x^2 f(x)}{\sum [\phi(x, q) A_x]^2}, \quad \&c.$$

The summations in the values of α, β , &c. extend from $x = h$ to $x = nh$.

18. In the limit, when h is indefinitely diminished, the expansion

$$f(x) = \alpha \phi(x, p) + \beta \phi(x, q) + \dots$$

has an infinite number of terms; but the number is equal to the number of roots in the equation to determine p . In this case the equation is true for all values of x between $x = 0$ and $x = l$, and the values of α, β , &c. are

$$\alpha = \frac{\int_0^l \phi(x, p) A_x^2 f(x) dx}{\int_0^l [\phi(x, p) A_x]^2 dx}, \quad \beta = \frac{\int_0^l \phi(x, q) A_x^2 f(x) dx}{\int_0^l [\phi(x, q) A_x]^2 dx}, \quad \&c.$$

19. It may be noticed that no assumption has been made as to the continuity of the function $f(x)$. It has, however, been assumed to be finite between the limits.

20. There can be but one expansion of $f(x)$ in a series of functions of the form $\phi(x, p)$; for, if there were another, let it be

$$f(x) = \alpha' \phi(x, p) + \beta' \phi(x, q) + \dots$$

We have, by subtraction,

$$0 = a\phi(x, p) + b\phi(x, q) + \dots,$$

where $a = a - a'$, $b = b - b'$, &c. Writing for x its values $h, 2h \dots nh$, we have n equations to find the n constants a, b, c , &c. These are

$$0 = aX_1 + bY_1 + \dots,$$

$$0 = aX_2 + bY_2 + \dots,$$

$$0 = \&c.$$

Unless a, b , &c. are all zero, we have their ratios equal to the ratios of the minors of the constituents of any line, say the first line. But these minors, by the third preliminary proposition, are respectively equal to X_1, Y_1 , &c. Substituting in the first line, we have

$$0 = X_1^2 + Y_1^2 + \dots,$$

which contradicts the third preliminary proposition.

21. None of the values of $X_1 X_2 \dots X_n$ can be infinite, since they satisfy the quadric

$$A_1^2 X_1^2 + A_2^2 X_2^2 + \dots + A_n^2 X_n^2 = H^2,$$

provided $A_1^2 A_2^2 \dots A_n^2$ all keep one sign. It immediately follows that the function $\phi(x, p)$ cannot be infinite for any value of x between the given limits. For P the multiplier is independent of x , and therefore $P\phi(x, p) = X$ could not be finite for all values of x . This argument applies whether the terminal conditions are identically satisfied or not. So that, if either ψ_1 or ψ_2 in

$$X = L\psi_1(x, p) + M\psi_2(x, p)$$

becomes infinite between the limits, then the coefficients L or M , or both, must be zero.

22. We may also notice that, if X_1 is finite and b_1 not zero, the first equation in Art. 9 requires X_2 to be finite; and, granting this, we have from the general equation in the same article

$$b_x X_{x+1} = (b_{x-1} + b_x + a_x - pA_x^2) X_x - b_{x-1} X_{x-1}.$$

If X_x, X_{x-1} are finite, and b_x not zero between the limits, then X_{x+1} is finite. Thus, if the terminal conditions exist, no infinite values of X can enter. Thus both integrals $\psi_1(x, p)$ and $\psi_2(x, p)$ will be finite throughout if they are finite when $x = 0$ and b_x keeps one sign.

We also notice that, when $X_x = 0$, X_{x-1} and X_{x+1} have opposite signs; hence X_x changes sign whenever it vanishes. It easily follows also from the equation that $X_x = 0$ has no equal roots.

23. *The functions which possess the property*

$$\int_0^1 \phi(x, p) \phi(x, q) A_x dx = 0$$

are not restricted to be the solutions of a differential equation of the second order. We may take a more general value for U , say

$$\begin{aligned} 2U = & a_1 X_1^2 + b_1 (X_2 - X_1)^2 + c_1 (X_3 - 2X_2 + X_1)^2 \\ & + e_1 (X_4 - 3X_3 + 3X_2 - X_1)^2 \\ & + \text{four similar terms beginning with } X_2, \\ & + \&c., \text{ beginning with } X_3, X_4, \&c. \end{aligned}$$

Then, taking any equation, say the x^{th} , we have

$$\frac{dU}{dX_x} = pA_x X_x.$$

For the sake of brevity, we shall write down only the terms which contain c on the left-hand side. These are

$$\begin{aligned} & c_{x-2} (X_x - 2X_{x-1} + X_{x-2}) \\ & \quad - 2c_{x-1} (X_{x+1} - 2X_x + X_{x-1}) \\ & \quad + c_x (X_{x+2} - 2X_{x+1} + X_x). \end{aligned}$$

These may be written in the usual notation

$$\Delta^2 (c_{x-2} \Delta^2 X_{x-2}).$$

In the limit when h is indefinitely small, we have

$$h^4 \frac{d^2}{dx^2} \left(c \frac{d^2 X}{dx^2} \right).$$

The result is that, if we form the equation of Differences

$$a_x X_x - \Delta (b_{x-1} \Delta X_{x-1}) + \Delta^2 (c_{x-2} \Delta^2 X_{x-2}) - \&c. = pA_x X_x,$$

or the Differential equation

$$a_x X - \frac{d}{dx} \left(b_x \frac{dX}{dx} \right) + \frac{d^2}{dx^2} \left(c_x \frac{d^2 X}{dx^2} \right) - \&c. = pA_x X;$$

then, in the first case, we have

$$\Sigma_1^n XYA_x^2 = 0,$$

the summation extending from $x = h$ to $x = nh$; and, in the second case,

$$\int_0^1 XYA_x^2 dx = 0.$$

24. *We have still to consider the equations at each end.* We observe that the terms with zero and negative suffixes are missing. But, exactly as in Art. 10, we may add on to the expression for $2U$ the terms

$$\lambda X_1^2 + \mu (X_2 - X_1)^2 + \nu (X_3 - 2X_2 + X_1)^2.$$

To find the conditions at the limits, we equate the missing terms to

the differential coefficients of these added terms with regard to X_1 , X_2 , and X_3 .

It seems unnecessary to repeat the argument. After some reduction we obtain the conditions

$$\begin{aligned} b_0 \Delta X_0 - \Delta (c_{-1} \Delta^3 X_{-1}) + \&c. &= \lambda X_1, \\ c_0 \Delta^3 X_0 - \Delta (e_{-1} \Delta^5 X_{-1}) + \&c. &= \mu \Delta X_1, \\ &\&c. = \&c. \end{aligned}$$

for the equation of Differences; and

$$\begin{aligned} b_x \frac{dX}{dx} - \frac{d}{dx} \left(c_x \frac{d^3 X}{dx^3} \right) + \&c. &= \lambda' X, \\ c_x \frac{d^3 X}{dx^3} - \frac{d}{dx} \left(e_x \frac{d^5 X}{dx^5} \right) + \&c. &= \mu' \frac{dX}{dx}, \\ &\&c. = \&c. \end{aligned}$$

for the Differential equation.

We have similar conditions at the other end, the constants λ' , μ' , &c. not being necessarily the same at both extremities.

25. If the differential equation be of the $2n^{\text{th}}$ order, we have thus n conditions at each end. All the constants of integration except one will therefore be determined. There will also be an equation to determine p , the roots of which are all real and unequal.

26. Exactly as in Art. 17, it follows that any function $f(x)$ can be expanded in a series of functions of the form

$$f(x) = \alpha \phi(x, p) + \beta \phi(x, q) + \dots,$$

where $X = P\phi(x, p)$ is the solution of the differential equation which satisfies the conditions at the limits, and is finite throughout.

SECOND APPLICATION.

Case in which no limits are given.

27. We may take another form for the quadric U which leads to another result. Suppose we have

$$\begin{aligned} 2U &= a_1 X_1^2 + b_1 (X_2 - X_1)^2 \\ &+ \&c. \\ &+ a_{n-1} X_{n-1}^2 + b_{n-1} (X_n - X_{n-1})^2 \\ &+ a_n X_n^2 + b_1 (X_1 - X_n)^2, \end{aligned}$$

which differs from that taken in Art. 9 by the addition of the last term. Let us also suppose that a_x , b_x , Δ_x , and X are functions which repeat so that $a_{n+1} = a_1$, $a_{n+2} = a_2$, and so on.

It is now evident that all the previous reasoning applies as before except that we have no conditions at the limits.

28. Thus, supposing we have the differential equation

$$a_x X - \Delta (b_x \Delta X) = p A_x X;$$

when solved, we have

$$X = L\phi(x, p) + M\psi(x, p).$$

But if X is to repeat, we have

$$X_1 = X_{n+1}, \quad X_2 = X_{n+2}.$$

Since the equation of Differences gives X_2 in terms of X_1 and X_0 , these are the necessary and sufficient conditions of repetition. In the case of a Differential equation, we should have $X_0 = X_n$, $\frac{dX_0}{dx} = \frac{dX_n}{dx}$. These equations determine the ratio of L to M , and also give an equation to find the values of p .

29. In the case considered in Art. 9, the equations to find X_1 , X_2 , &c. are such that each contains a letter which is in none of the other equations. In fact, if b_x does not vanish between the limits, the equations are all independent, and X_1 , X_2 , &c. are all determinate. We therefore infer, by Art. 7, that the equation to find p has no equal roots. But this argument does not apply to the quadric now under consideration. It is therefore possible that the two equations given in the last article may really leave X indeterminate with two arbitrary constants.

30. In this case, the method described in Art. 6 fails to separate one particular set of conjugate diameters. But a simple consideration will enable us to complete the process. For example, let there be two roots equal to p . Then we have

$$\left. \begin{aligned} X &= L\phi(x, p) + M\psi(x, p) \\ Y &= L'\phi(x, p) + M'\psi(x, p) \end{aligned} \right\}.$$

The extremity of one diameter is arbitrary, so that the ratio L to M may be what we please. But, when this has been chosen, the ratio of L' to M' must be such that

$$\Sigma A_x^2 XY = 0.$$

This gives

$$LL' \Sigma (A^2 \phi^2) + (LM' + L'M) \Sigma (A^2 \phi \psi) + MM' \Sigma (A^2 \psi^2) = 0.$$

Also, since the extremities of the diameters must lie on the quadric, we have, as in Art. 14,

$$\begin{aligned} L^2 \Sigma (A^2 \phi^2) + 2LM \Sigma (A^2 \phi \psi) + M^2 \Sigma (A^2 \psi^2) &= H^2, \\ L'^2 \Sigma (A^2 \phi^2) + 2L'M' \Sigma (A^2 \phi \psi) + M'^2 \Sigma (A^2 \psi^2) &= H^2, \end{aligned}$$

where H is practically a constant at our disposal.

31. It might be supposed that, having an undetermined constant L at our disposal, we might, by following the rule in Art. 17, obtain more than one expansion of any function in a series of the form

$$f(x) = \alpha [L\phi x + M\psi x] + \beta [L'\phi x + M'\psi x] + \&c.$$

But this is not the case, so that in general it is more convenient to give L any value we please.

To prove this we follow the same rule as in Art. 17, and obtain

$$\alpha = \frac{L\Sigma\phi fA_x^2 + M\Sigma\psi fA_x^2}{H^2}, \quad \beta = \frac{L'\Sigma\phi fA_x^2 + M'\Sigma\psi fA_x^2}{H^2}.$$

Hence, the coefficient of $\phi(x)$ in the expansion of $f(x)$ is

$$\frac{L^2 + L'^2}{H} \Sigma\phi fA_x^2 + \frac{LM + L'M'}{H} \Sigma\psi fA_x^2.$$

Now, looking at the equations given in the last article, we see that (L, M) (L', M') may be regarded as the coordinates of two points lying on the quadric

$$L^2 \Sigma(A^2\phi^2) + 2LM \Sigma(A^2\phi\psi) + M^2 \Sigma(A^2\psi^2) = H,$$

while the condition

$$LL' \Sigma(A^2\phi^2) + (LM' + L'M) \Sigma(A^2\phi\psi) + MM' \Sigma(A^2\psi^2) = H$$

expresses the fact that these points are the extremities of conjugate diameters. Hence, by Art. 5, $L^2 + L'^2$, $LM + L'M'$, and $M^2 + M'^2$ are the same for all values of L .

The case of several equal roots may be treated in the same way.

EXAMPLE 1.

32. *Under what circumstances can we expand any finite function $f(x)$ of x in a series of functions which satisfy*

$$\frac{d^2 X}{dx^2} + pX = 0?$$

We have, in this case,

$$X = L \sin \sqrt{p}x + M \cos \sqrt{p}x.$$

If the expansion is to be true for all values of x , say between $x=0$ and $x=2\pi$, and not to be necessarily true for any values of x outside these limits, we must have a relation

$$\frac{dX}{dx} = \lambda X$$

true at each end. If we take $\lambda=0$, we have

$$L \cos \sqrt{p}x - M \sin \sqrt{p}x = 0,$$

both where $x=0$ and $x=2\pi$. This excludes negative values of p , and gives $L=0$, $\sqrt{p}=\frac{1}{2}i$, where i is any integer. Thus we have

$$f(x) = \Sigma M \cos \frac{ix}{2},$$

where the summation extends to all integer values of i , and M can be found in the usual manner. If we take λ infinite, we obtain an expansion in sines alone. Any other value of λ will give an expansion in both sines and cosines.

It is important to notice that a series in sines or cosines alone does not necessarily repeat when x is increased by 2π .

33. If the function $f(x)$, whether continuous or not, repeats when x is increased by 2π , and the expansion is to be true for *all* values of x , then we must use the method of Art. 27. In this case p must be a square integer, and there is nothing to determine the ratio L to M . This indeterminateness, by Art. 7, shows that the roots $p=1^2$, $p=2^2$, &c. occur twice each time, but the root $p=0$ only once. We thus have

$$X = L_0,$$

$$Y = L_1 \sin x + M_1 \cos x,$$

$$Z = L_1' \sin x + M_1' \cos x,$$

with similar terms for the other values of p . We have also, by Art. 30, the conditions

$$L_1 L_1' + M_1 M_1' = 0,$$

with similar equations corresponding to the other values of p . These are satisfied by $L_1' = 0$, $M_1' = 0$. Hence the expansion

$$f(x) = \alpha X + \beta Y + \gamma Z + \dots$$

of Art. 31 becomes

$$f(x) = \alpha + \beta \sin x + \gamma \cos x + \delta \sin 2x + \epsilon \cos 2x + \dots,$$

where the coefficients are given by Art. 17, or may be found in the usual way.

EXAMPLE 2.

34. *Under what circumstances can we expand any finite function $f(x)$ of x in a series of functions which satisfy*

$$\frac{d}{dx} \left\{ (1-x^2) \frac{dX}{dx} \right\} + pX = 0,$$

between the limits $x=-1$ to $x=1$.

First we notice that this equation comes under the general form given in Art. 9, where $b=1-x^2$. But this case is peculiar, because at the limit $x=-1$, $b=0$, so that the condition

$$b \frac{dX}{dx} = \lambda X,$$

when λ is chosen to be zero, appears to be nugatory. We may, however, conclude that, if X be any solution of the equation, and Y be the value of X when q is written for p , and if $\frac{dX}{dx}$ and $\frac{dY}{dx}$ are finite at both

limits, then
$$\int_{-1}^{+1} XY dx = 0,$$

if p and q be unequal.

35. In order, however, to expand $f(x)$ in a series of the functions X , Y , &c., it will be necessary to find *all* the values of p which satisfy the limiting conditions.

To examine the condition at the lower limit, let us put $x = -1 + t$, and expand X in powers of t . We have

$$\frac{d}{dt} \left\{ (2t - t^2) \frac{dX}{dt} \right\} + pX = 0.$$

To find one solution, we put

$$X = A_0 + A_1 t + A_2 t^2 + \dots$$

We have, by substitution,

$$A_{n+1} = \frac{n(n+1) - p}{2(n+1)^2} A_n.$$

By taking the test fraction, we find the series is convergent up to $t=2$.

Let the solution thus found be $X = f(t)$. Then the general solution is

$$X = Lf(t) \int \frac{dt}{b \{f(t)\}^2} + Mf(t),$$

where $b = 2t - t^2$. We easily find, when $t=0$, that

$$b \frac{dX}{dt} = \frac{L}{A_0}, \quad X = \frac{L}{2A_0} \log t + MA_0.$$

The constant λ is at our disposal, except that by Art. 10 it must be independent of p . If λ be finite, it does not appear to be possible to

satisfy the condition
$$b \frac{dX}{dt} = \lambda X,$$

when $t=0$ with the value of X just found unless $L=0$. If we choose $\lambda=0$, the condition at the lower limit becomes

$$b \frac{dX}{dt} = 0,$$

and this requires $L=0$. If we choose $\lambda = \infty$, the condition becomes $X=0$, which requires $L=0$ and $M=0$. The function X is then always zero.

To satisfy the condition at the lower limit, we must therefore take for X the series represented above by $f(t)$.

36. If l be less than unity, we have p determined by the condition

$$(1-x^2) \frac{dX}{dx} = \mu X,$$

which holds at the upper limit. But if $l=1$, we have $b=1-x^2=0$.

By the same reasoning as before, it does not appear to be possible to satisfy this condition unless

$$b \frac{dX}{dx} = 0$$

at the upper limit, and, as before, we get the same results as if we put $\mu=0$.

If we choose $\mu=0$, this condition is satisfied if p be any integer of the form $m(m+1)$. At the same time, the condition may be satisfied by other values of p , and we must therefore examine the value of the left-hand side of this equation when $t=2$. Now

$$\begin{aligned} (2-t) \frac{dX}{dt} &= (2-t) (A_1 + 2A_2t + \dots) \\ &= \text{zero} + (2-t) \{rA_r t^{r-1} + (r+1)A_{r+1}t^r + \dots\} \\ &= t^{r-1} [2rA_r + \{2(r+1)A_{r+1} - rA_r\}t + \dots] \\ &= t^{r-1} p \left\{ \frac{2r}{p} A_r - \frac{A_r t}{r+1} - \frac{A_{r+1}}{r+2} t^2 - \dots \right\}, \end{aligned}$$

which is obtained by substituting for A_{r+1} its value in terms of A_r , and so on. Now r is any integer we please; let it be greater than p . Then every term after the first in this series has the same sign; and since

$$\frac{A_{n+1}}{n+2} > \frac{A_n}{n+1} \frac{n^2}{(n+1)(n+2)} \cdot \frac{1}{2},$$

the sum, when $t=2$, of all the terms after the first is greater than

$$\frac{A_r}{r+1} \left\{ 1 + \frac{r^2}{(r+1)(r+2)} + \dots \right\},$$

which is a divergent series. We, therefore, conclude that we cannot have $(1-x^2) \frac{dX}{dt} = 0$ unless p is an integer of the form $m(m+1)$.

37. The result is, that if we wish to expand $f(x)$ in a series of functions which satisfy the given equation, and if the expansion is to hold for all values of x between ± 1 , then the several terms of the expansion are given by $p=0$, $p=1.2$, $p=2.3$, &c.; and the general term by $p=m(m+1)$.

THIRD APPLICATION.

Case of two Independent Variables.

38. Let us now apply the method to the case of two independent variables, x and y . Suppose we take the quadric

$$2U = a_{11}X_{11}^2 + b_{21}(X_{21} - X_{11})^2 + c_{12}(X_{12} - X_{11})^2 + e_{11}(X_{21} - X_{11})(X_{12} - X_{11}) \\ + \text{similar terms with continued additions of unity to each suffix.}$$

Here we take with X_{11}^2 only the products of X_{11} with X_{12} and X_{21} and the product $X_{12}X_{21}$.

Taking any one of the equations in Art. 9, say

$$\frac{dU}{dX_{xy}} = pA_{xy}^2 X_{xy},$$

we find, by the same process as before, the equation of Differences

$$a_{xy}X_{xy} - \Delta_x(b_{x-1,y}\Delta_y X_{x-1,y}) - \Delta_y(c_{x,y-1}\Delta_x X_{x,y-1}) \\ - \Delta_x(e_{x-1,y}\Delta_y X_{x-1,y}) - \Delta_y(e_{x,y-1}\Delta_x X_{x,y-1}) = pA_{xy}^2 X_{xy}.$$

In the limit this becomes the differential equation

$$aX - \frac{d}{dx}\left(b\frac{dX}{dx}\right) - \frac{d}{dy}\left(c\frac{dX}{dy}\right) \left\{ \begin{array}{l} \\ - \frac{d}{dx}\left(e\frac{dX}{dy}\right) - \frac{d}{dy}\left(e\frac{dX}{dx}\right) \end{array} \right\} = pA^2 X,$$

where a, b, c, e and A are any functions of x, y , but must be independent of p .

39. The condition at the limits is as follows:—If s be the arc of the boundary of the area over which we integrate, then at every point of the boundary

$$b\frac{dX}{dx}\frac{dy}{ds} - c\frac{dX}{dy}\frac{dx}{ds} + e\left(\frac{dX}{dy}\frac{dy}{ds} - \frac{dX}{dx}\frac{dx}{ds}\right) = \lambda X.$$

But if we integrate over the whole area of any closed surface, so that x and y are capable of unlimited increase or decrease, this condition is not necessary. We have merely the conditions that X must be one valued and finite over the whole surface.

40. In the same way, if we start with a more general form for the quadric U , we may arrive at equations of a higher order, which it is sufficient that X should satisfy. Thus we may take the additional terms

$$f(X_{22} - X_{12} - X_{21} + X_{11}) + g(X_{21} - X_{11})(X_{12} - 2X_{12} + X_{11}) \\ + h(X_{12} - X_{11})(X_{21} - 2X_{21} + X_{11}) + k(X_{12} - 2X_{12} + X_{11})^2 \\ + l(X_{21} - 2X_{21} + X_{11})^2.$$

These give terms in the differential equation of the fourth order.

Generally, the differential equation which X must satisfy is of the form

$$\Sigma \frac{d^n}{dx^\alpha dy^{\beta-\alpha}} \left(P \frac{d^n X}{dx^\alpha dy^{\beta-\alpha}} \right) \\ + \Sigma \left[\frac{d^\alpha}{dx^\alpha} \left(Q \frac{d^\gamma X}{dy^\gamma} \right) + (-1)^{\alpha+\gamma} \frac{d^\gamma}{dy^\gamma} \left(Q \frac{d^\alpha X}{dx^\alpha} \right) \right] = p A^\beta X,$$

where P , Q , and A are any functions of x , y , and the summations apply to any values of n , α , β , γ we please.

41. Let us now take the whole surface of any sphere as the area of integration. Let the coordinates of any point be given by the usual angular coordinates θ , ϕ ; then we may put $x = \cos \theta$, $y = \phi$. Let X be any one valued finite function of xy which satisfies the differential equation of Art. 38. Now x is restricted to lie between ± 1 , while $y = \phi$ is unrestricted. Describe an indefinitely small circle whose centre is the origin from which θ is measured. Then along the circumference of this circle, *i.e.*, when $x = -1$, we must have, by Art. 39,

$$b \frac{dX}{dx} + c \frac{dX}{dy} = \lambda X.$$

A similar condition holds at the other limit, *i.e.*, when $x = +1$. Here the constants λ and μ are at our disposal, and in some cases, by giving to them different values, we may obtain different sets of values of p , and therefore different sets of functions X . Let the values of p , for one set thus found, be p , q , ..., and let X , Y , &c. be the corresponding values of X . Then we have

$$\iint A^2 XY dx dy = 0;$$

and any one valued finite function of θ , ϕ can be expanded in a series of the form

$$f(\theta, \phi) = aX + \beta Y + \dots$$

42. The case of Laplace's functions is included in this proposition, if we can discover the proper values of p , q , &c. Taking, then, the equation

$$\frac{d}{dx} \left\{ (1-x^2) \frac{dX}{dx} \right\} + \frac{1}{1-x^2} \frac{d^2 X}{d\phi^2} + pX = 0,$$

we have to determine what values of p will satisfy the conditions at the limits.

First, we notice that the independent variable ϕ has no limits. This is the case considered in Art. 33. We can therefore expand X in a series of the form

$$X = \Sigma (u \cos k\phi + v \sin k\phi),$$

with both sines and cosines in the expansion. Here u and v are functions of θ . Also, there is but one such expansion for X .

43. Next, we notice that $x = \cos \theta$ is restricted to lie between -1 and $+1$. Exactly as in Art. 34, we must have a condition of the form

$$(1-x^2) \frac{dX}{dx} = \lambda X$$

true at each limit. Here λ is a constant at our disposal.

If we substitute the expansion for X in the differential equation, we have

$$(1-x^2) \frac{d^2u}{dx^2} - 2x \frac{du}{dx} + \left(p - \frac{k^2}{1-x^2}\right) u = 0.$$

We may equate the coefficients of $\cos k\phi$ to zero, because there is but one expansion for zero in a series of sines and cosines of multiples of ϕ which is to be true for *all* values of ϕ . If $P = P$ be a solution of

$$(1-x^2) \frac{d^2P}{dx^2} - 2x \frac{dP}{dx} + pP = 0,$$

we find, by Leibnitz's theorem, that one value of u is

$$u = (1-x^2)^{\frac{1}{2}} \frac{d^k P}{dx^k}.$$

If this solution be called $f(x)$, the general solution is

$$u = Lf(x) \int \frac{dx}{f(x)(1-x^2)} + Mf(x).$$

Now, if we choose $\lambda = 0$, this value of u must make

$$(1-x^2) \frac{dX}{dx} = 0,$$

when $x = -1$ for all values of ϕ . By the same reasoning as in Art. 35, this is impossible unless $L = 0$. It does not appear that any other choice for λ will permit the condition to be satisfied.

44. Again, when $k = 0$, the equation to find u reduces to the equation considered in Art. 35, and this does not admit of a solution which satisfies the condition at the upper limit except p is of the form $n(n+1)$. None of these values of p make $\frac{du}{dx}$ or $\frac{dv}{dx}$ infinite, and are not therefore further limited. We therefore conclude that

$$p = n(n+1).$$

45. The general value of X is, under these circumstances, given by the form $X = a_0 P_n + (a_1 \cos \phi + b_1 \sin \phi) (1-x^2)^{\frac{1}{2}} \frac{dP_n}{dx} + \dots$

This value of X has $2n+1$ undetermined constants. This indeterminateness, by Art. 7, shows that the equation to find p has $2n+1$ roots equal to $n(n+1)$.

The root $p=0$ occurs once, and makes

$$X = P_n.$$

The root $p=1$ occurs three times; we have therefore

$$Y = a_0 P_1 + (a_1 \cos \phi + b_1 \sin \phi)(1-x^2)^{\frac{1}{2}} \frac{dP_1}{dx},$$

$$Z = a'_0 P_1 + (a'_1 \cos \phi + b'_1 \sin \phi)(1-x^2)^{\frac{1}{2}} \frac{dP_1}{dx},$$

$$W = a''_0 P_1 + (a''_1 \cos \phi + b''_1 \sin \phi)(1-x^2)^{\frac{1}{2}} \frac{dP_1}{dx}.$$

By Art. 30, we have the relations

$$a_0 a'_0 + a_1 a'_1 + b_1 b'_1 = 0,$$

$$a_0 a''_0 + a_1 a''_1 + b_1 b''_1 = 0,$$

$$a'_0 a''_0 + a'_1 a''_1 + b'_1 b''_1 = 0;$$

all these are satisfied by taking

$$Y = P_1,$$

$$Z = \cos \phi (1-x^2)^{\frac{1}{2}} \frac{dP_1}{dx},$$

$$W = \sin \phi (1-x^2)^{\frac{1}{2}} \frac{dP_1}{dx}.$$

The root $p=2$ occurs five times, and may be treated in the same way.

46. We therefore infer that any function of θ, ϕ which is finite, and one valued, may be expanded in a series of the form

$$f(\theta, \phi) = \alpha X + \beta Y + \gamma Z + \dots,$$

where $\alpha, \beta, \gamma, \dots$ are constants whose values may be found in the usual way, and are given in Art. 17.

If we collect together all the terms which arise from any one value of p , say $p = n(n+1)$, we find

$$P_n \frac{\iint P_n f dx d\phi}{\iint P_n^2 dx d\phi} + \cos \phi (1-x^2)^{\frac{1}{2}} \frac{dP_n}{dx} \frac{\iint \cos \phi (1-x^2)^{\frac{1}{2}} \frac{dP_n}{dx} f dx d\phi}{\iint \left[\cos \phi (1-x^2)^{\frac{1}{2}} \frac{dP_n}{dx} \right]^2 dx d\phi} + \&c.$$

Suppose we write $x'\phi'$ for $x\phi$ inside the integrals, then this may be written

$$\iint Q_n f(x'\phi') dx' d\phi',$$

where

$$Q_n = \frac{P_n P'_n}{\iint P_n^2 dx d\phi} + \frac{\frac{dP_n}{dx} \frac{dP'_n}{dx} \cos \phi \cos \phi' (1-x^2)^{\frac{1}{2}} (1-x'^2)^{\frac{1}{2}}}{\iint \left[\cos \phi (1-x^2)^{\frac{1}{2}} \frac{dP_n}{dx} \right]^2 dx d\phi} + \&c.$$

It is not difficult to show that Q_n is the expansion of a Laplace's co-

efficient, which is symmetrical about the axis whose angular coordinates are θ, ϕ , in a series of the form

$$aP_n + (a_1 \cos \phi + b_1 \sin \phi) (1-x^2)^{\frac{1}{2}} \frac{dP_n}{dx} + \dots,$$

where P_n is a Laplace's coefficient about the axis from which θ is measured.

Thus, the expansion arrived at for the expansion of $f(\theta, \phi)$ in Laplace's functions agrees with that usually given.

June 12th, 1879.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Mr. R. C. Rowe, B.A., Fellow of Trinity College, Cambridge, was proposed for election.

The following communications were made:—

"Notes on the Reduction of a System of Forces; and on Plane Curves:" by Mr. J. J. Walker.

"Notes on Determinants of n Dimensions:" by Mr. H. W. Lloyd Tanner.

"Curves for the Inscription of a regular Nonagon and Undecagon in a Circle:" by Rev. T. J. Freeth, LL.D.

"On Clifford's Graphs, and on the Twenty-one Coordinates of a Conic in Space:" by Dr. Spottiswoode, P.R.S.

"Two Geometrical Notes:" by Prof. H. J. S. Smith, F.R.S.

The following presents were received:—

"Educational Times," June, 1879.

"Journal of Institute of Actuaries," Vol. xxi., Pt. v., No. cxv., April, 1879.

"List of Fellows of the Meteorological Society," May 1, 1879.

Notes on Determinants of n Dimensions.

By H. W. LLOYD TANNER, M.A.

[Read June 12th, 1879.]

1. *Definition.*—An n -dimensional determinant of the r^{th} order is a function of r^n elements, such as

$$a_{ij\dots k} \text{ or } (ij\dots k), \text{ where } i, j\dots k = 1, 2\dots r;$$

each element being designated by n symbols $i, j\dots k$.* It is the sum of

* We shall have to refer so frequently to these symbols, that it is desirable to give them a special name. In the paper they are called "sub-elements"; a name which may be taken to allude to their position in the signature of the element $a_{ij\dots k}$, or to indicate that each element is, in a certain sense, made up of its sub-elements.