266. The Self-Reciprocal 13-gon

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The figure can equally well be obtained starting from P, Q, R, S ; and it may be described as consisting of a quadrilateral p, q, r, s and a quadrangle P, Q, R, S having a common diagonal $\triangle ABC$.

The proof that the 6 lines, stated above to concur by threes at P, Q, R, S , really do so, can be obtained using trilinear coordinates, with $\triangle ABC$ as $\triangle$ of reference.

Let the line s be $la + m\beta + n\gamma = 0$.

Then the join of A to (rs) is $m\beta + n\gamma = 0$.

Now $(rs), (pq)$ are harmonic conjugates to B, C ;

$\therefore$ the join of A to (pq) is $m\beta - n\gamma = 0$(i)

Again the join of B to (qs) is $la + n\gamma = 0$, and $(qs), (pr)$ are harmonic conjugates to A, C ;

$\therefore$ the join of B to (pr) is $la - n\gamma = 0$(ii)

Comparing this with (i) we see that the equation of p is $la + m\beta - n\gamma = 0$.

Similarly the equation of q is $la - m\beta + n\gamma = 0$,

and the equation of r is $-la + m\beta + n\gamma = 0$;

$\therefore$ the join of C to (qr) is $la - m\beta = 0$(iii)

Now the three lines (i) (ii) (iii) intersect at the point $la = m\beta = n\gamma$, which is the point S .

Similarly the join of A to (rs) viz. $m\beta + n\gamma = 0$,

the join of B to (qs) viz. $la + n\gamma = 0$,

and the join of C to (qr) viz. $la - m\beta = 0$

intersect in the point $la = m\beta = -n\gamma$ which is the point P .

Thus the four lines of the quadrilateral are

$$la \pm m\beta \pm n\gamma = 0,$$

and the four corners of the quadrangle are

$$\frac{1}{l}, \pm \frac{1}{m}, \pm \frac{1}{n}.$$

To reciprocate the figure into itself we can proceed as follows :

The equation of a conic for which ABC is self-polar is of the form

$$ua^2 + v\beta^2 + w\gamma^2 = 0.$$

The polar of $\frac{1}{l}, \pm \frac{1}{m}, \pm \frac{1}{n}$ with respect to this is

$$\frac{ua}{l} \pm \frac{v\beta}{m} \pm \frac{w\gamma}{n} = 0.$$

If this is the same as $la \pm m\beta \pm n\gamma = 0$,

we must have

$$\frac{u}{l^2} = \frac{v}{m^2} = \frac{w}{n^2}.$$

Thus if we reciprocate with respect to the (imaginary) conic

$$l^2a^2 + m^2\beta^2 + n^2\gamma^2 = 0,$$

A reciprocates into BC etc., and S into s etc.

Conics which either pass through P, Q, R, S or touch p, q, r, s are necessarily of the form $ua^2 + v\beta^2 + w\gamma^2 = 0$.

For conics which both pass through P, Q, R, S and touch p, q, r, s the equations for u, v, w are

$$\frac{u}{l^2} + \frac{v}{m^2} + \frac{w}{n^2} = 0 \text{ and } \frac{l^2}{u} + \frac{m^2}{v} + \frac{n^2}{w} = 0,$$

thus the only such conics are of the form

$$l^2a^2 + m^2\omega\beta^2 + n^2\omega^2\gamma^2 = 0$$

and

$$l^2a^2 + m^2\omega^2\beta^2 + n^2\omega\gamma^2 = 0,$$

when ω and ω^2 are the imaginary cube-roots of unity.

The 13-gon is not a closed figure in the sense that every pair of points is joined: I believe the $\triangle$ is the only figure of this kind possible, though a figure such as a square with its diagonals drawn may appear to be so owing to some of the points of intersection of the sides having moved to infinity.

Each of the six lines such as PS intersects two of the original lines viz. p and s at fresh points: These points are 12 in number, and their coordinates will be found to be

$$\begin{aligned} \frac{2}{l}, \quad \pm \frac{1}{m}, \quad \pm \frac{1}{n}, \\ \pm \frac{1}{l}, \quad \frac{2}{m}, \quad \pm \frac{1}{n}, \\ \text{and} \quad \pm \frac{1}{l}, \quad \pm \frac{1}{m}, \quad \frac{2}{n}. \end{aligned}$$

If anyone has the energy, as one of my pupils Mr. D. F. Ferguson has had, to work out the equations of the 66 joins of these points he will be rewarded by some curious results.

18 of the 66 joins coincide with one or other of the lines already found: 12 more, of the forms $la \pm 2m\beta = 0$, $la \pm 2n\gamma = 0$ etc., pass through one of the points A, B, C : 12 more, of the forms $3la \pm m\beta \pm n\gamma = 0$, pass through one of the 6 corners of the original quadrilateral. The remaining 24 are of the form $\lambda la \pm \mu m\beta \pm n\gamma = 0$, when λ, μ, ν stand for 1, 3, 5 in any order.

These do not pass through any of the original 13 points, but a large number of concurrences on the original lines can be detected.

One particular case of the 13-gon is obtained by taking $l=m=n=1$.

In this case the corners of the quadrangle are the inscribed and escribed centres of $\triangle ABC$, and the six corners of the quadrilateral are the points where the internal and external bisectors of the angles meet the opposite sides.

Again if l, m, n are taken proportional to a, b, c , the sides of the quadrilateral become the line at infinity and the joins of the mid-points of the sides of $\triangle ABC$: while the sides of the quadrangle are the 3 medians, and the 3 parallels to the opposite sides through the corners: these 6 lines determine four points of intersection, one being the c.g. of the $\triangle$.

If any member of the Association finds his pupils inclined to get mixed between a quadrangle and a quadrilateral, may I recommend him to make them draw the 13-gon beginning first from the quadrilateral and then from the quadrangle.

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267. [V. a.] Supposing a^m to be defined by the property $a^m \times a^n = a^{m+n}$, to prove $(a^m)^n = a^{mn}$.

From the definition we have

$$\text{if } \phi(x) \times \phi(y) = \phi(x+y).$$

Then

$$\phi(x) = x^x, \quad \phi(y) = y^y.$$

Now let

$$a^{mx} = f(x).$$

Then

$$f(x) \cdot f(y) = a^{mx} \cdot a^{my}$$

$$= a^{mx+my}$$

$$= a^{m(x+y)} \quad \{\text{Law of multiplication}\}$$

$$= f(x+y);$$

$$\therefore f(x) = x^x;$$