with the side AB of the triangle ABC , it will be easily seen that the circle DEF passes through two fixed points, viz., the vertex O and U . The system whose centre of similitude is the centre of the circum-circle ABC , does not seem to call for special notice. The foci of its double contact inscribed conic are the centre of the circumcircle and orthocentre of ABC .

On a Problem of Conformal Representation.

By Prof. W. BURNSIDE. Read April 13th, 1893.

1. *Introductory.*

The conformal representation of a plane finite polygon, no two of whose sides cross each other, on the half of an infinite plane or on a circle, is due originally to Schwartz. If $a_1\pi, a_2\pi, \dots a_n\pi$ are the internal angles of the polygon in order, and if w and z are complex variables moving respectively in the plane of the polygon and in the infinite half-plane by the functional relation between which the conformal representation is given, it is shown first that

$$\frac{d}{dz} \left(\log \frac{dw}{dz} \right)$$

is everywhere finite, continuous, and uniform in the positive half of the z -plane, and that it takes real values along the axis of real quantities. Hence, by a general theorem, also due to Schwartz, it follows that this function can be continued across the real axis, so that it takes conjugate imaginary values at conjugate points; and therefore that its value at every point of the z -plane is determined by its discontinuities, which all lie on the real axis, and are the points of that axis corresponding to the angular points of the original polygon. If $a_1, a_2, \dots a_n$ are these points, the value of the function thus obtained is given by the equation

$$\frac{d}{dz} \left(\log \frac{dw}{dz} \right) = \sum_1^n \frac{a_r - 1}{z - a_r}.$$

That the quantities a_1, a_2 , &c. are actually determinable in terms of

the lengths of the sides of the polygon has been proved by Herr Schläfli (*Orelle's Journal*, Vol. LXXVIII.).

The formal integral of the above equation is

$$Aw + B = \int (z - a_1)^{a_1-1} \dots (z - a_n)^{a_n-1} dz.$$

The quantity on the right-hand side of this equation is an Abelian integral, so that, although when the region of variation of z is suitably restricted w and z are uniform functions of each other, there will generally be no integral functional relation between w and z for unrestricted variation of the latter.

It may, however, happen in particular cases that the integral is capable of being algebraically transformed into an elliptic integral, and when this is the case, the conformal representation in question will be given by an integral equation of the form

$$f[z, \wp(Aw + B)] = 0,$$

where f is a symbol for a rational function.

Thus, in the simplest and most familiar case of all,

$$z = \wp w$$

gives the conformal representation of a rectangle on a half-plane, the absolute invariant of the elliptic function being real and greater than unity.

The number of essentially different cases in which this may happen is clearly unlimited, but they are capable of being divided into a limited number of well-defined classes, and it is my object to deal with this question with such generality as the nature of the subject allows.

2. The General Problem.

The determination whether in any particular case the integral is capable of being transformed into an elliptic integral, depends, in fact, on geometrical considerations. Suppose that AB and CD are any two sides of the original polygon, and ab and cd the corresponding segments of the real axis in the z -plane.

So long as the variation of z is restricted to the positive half of the z -plane, w is a uniform function of z , and moves within the original polygon. If now z be allowed to pass into the negative half of its plane by crossing the segment ab of the real axis, the rest of the real axis being excluded from its range of variation, w will still be a uni-

form function of z , and it is immediately obvious that the values of w corresponding to the various positions of z in its negative half-plane will be represented by taking the reflection of the original polygon in AB . Similarly, if z be allowed to pass from the positive to the negative half-plane by crossing the segment cd of the axis, the values of w , corresponding to the various positions of z in the negative half-plane, will be represented by the reflection of the original polygon in CD . The process indicated here may be continued indefinitely; and it follows that the various branches of w (each of which is a uniform function so long as the variation of z is restricted to one half-plane) are represented by the successive reflections of the original polygon, and those obtained from it in their sides; those arising from an odd number of reflections corresponding to values of z in the negative half-plane, while an even number of reflections give the branches of w corresponding to values of z in the positive half-plane. Let P now be any point inside the original polygon; then, when the reflecting process is carried out completely, there may be either a finite or an infinite number of different polygons which contain the point P . If the latter is the case, there will correspond an infinite number of values of z to a given value of w , and there can therefore be no relation of the form

$$f(z, w) = 0$$

connecting z and w . But if it is found that there are only a finite number of different polygons containing P , the Abelian integral *must* be capable of transformation into an elliptic integral, as otherwise, when w is given, there would be an infinite number of different values of z .

The problem which must be solved is therefore the following:—To determine the forms of all rectilinear polygons such that any given point of the plane lies within a finite number only of polygons formed by repeated reflections from the original one. It is convenient slightly to alter the form of this question as follows:—A plane polygon and a point P within it are given, and from P is constructed an infinite series of points by successive reflections in any order in the sides of the polygon; it is required to determine whether or no points in the series can be found such that their distances from P , without being zero, can be made less than any assignable quantity; or, in other words, whether the group of transformations arising from reflections in the sides of the polygon is, in M. Poincaré's phrase, a continuous or a discontinuous group. The case of a triangle may

first be dealt with, both as being simplest and as leading to the solution of the more general case.

If the angles of the triangle are not commensurable with two right angles, it is at once clear that the group must be continuous, and that the required reduction of the integral to elliptic form cannot take place.

Suppose, then, that a triangle ABC is placed with the angle A at the origin, and the angle B at the point $(1, 0)$. The effect of a pair of reflections taken successively in AB and AC is given by

$$z' = e^{2iA} z,$$

and that of a pair in BO and AB by

$$z'' - 1 = e^{2iB} (z - 1),$$

where z is $x + iy$ for the original point, and z' , z'' the corresponding quantities for the two transformed points. The result of any even number of reflections will be represented by repetitions of these two substitutions and their inverses; and it is sufficient to consider the group of points arising from an even number of reflections, since, by combining these with a single reflection, say in AB , all the others result.

If $\pi - B$ be written for B in the second of the above equations, the only alteration caused is to replace the second substitution by its inverse, so that the group of points is unchanged, while the angles of the triangle are altered from A, B, C to $A, C + A, B - A$; and therefore, if one of these triangles leads to a discontinuous group of points, so also does the other.

Now A, B, C are by supposition commensurable, and, if θ is the greatest angle of which they are all integral multiples, it is easy to see that a continued repetition of the process of keeping the smallest angle unchanged and replacing the next least by its supplement, will lead eventually to an isosceles triangle whose equal angles are both θ . It follows that unless this triangle leads to a discontinuous group of points, the original triangle cannot do so; and therefore it is sufficient first to deal with an isosceles triangle. The advantage so gained is that for an isosceles triangle a simple and interesting general expression is readily obtainable for the group of points in question.

Thus, taking now

$$A = B = p\pi/q,$$

where p/q is a proper fraction in its lowest terms, then

$$e^{2iA} = e^{2iB} = \omega,$$

a primitive q^{th} root of unity, and the two substitutions are

$$z' = \omega z, \quad z'' = 1 + \omega(z-1).$$

The general expression for any substitutions compounded of these two is easily seen to be

$$\zeta = \sum_1^q m_s \omega^s + \omega^s z, \quad s = 1, 2, \dots, q,$$

where the only relation between the positive and negative integers $m_1, m_2, \dots$ is that their sum is zero.

When $q = 2$, this expression is

$$2m \pm z,$$

where m is any positive or negative integer, and the group is evidently discontinuous. The corresponding triangle has two right angles and infinite area.

When $q = 3$, the expression becomes

$$m + (3n - m)\alpha + \alpha^s z, \quad s = 1, 2, 3,$$

where m, n are any positive or negative integers, and α an imaginary cube root of unity. The group again is obviously discontinuous, and the triangle equilateral.

When $q = 4$, the expression becomes

$$m + (2n - m)i + i^s z, \quad s = 1, 2, 3, 4.$$

The group is again discontinuous and the triangle isosceles and right-angled.

Passing over the case $q = 5$, the expression

$$\sum_1^6 m_s \omega^s + \omega^s z,$$

where ω is a primitive sixth root of unity, is easily thrown into the form

$$m + n\alpha \pm \alpha^s z, \quad s = 1, 2, 3,$$

where m, n are any positive or negative integers, and α an imaginary cube root of unity.

The group is once again discontinuous, while the triangle has two angles each equal to $\pi/6$.

For any value of q other than 2, 3, 4 or 6, the quantity

$$\sum_1^q m_s \omega^s$$

is expressible as the sum of three or more constant quantities, whose ratios are not real, each multiplied by an arbitrary integer, and, by a

known theorem, can be made less than any assignable quantity without being zero. Hence, for all such values of μ , the series of points will contain an infinite number lying indefinitely near P . It follows, therefore, that the only triangles satisfying the required condition are those whose angles are multiples either of

$$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4} \text{ or } \frac{\pi}{6}.$$

The only triangles satisfying these conditions are those whose angles are

$$(i.) \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}; \quad (ii.) \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3};$$

$$(iii.) \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}; \quad (iv.) \frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3};$$

and these are therefore the only triangles which can be conformally represented on a half-plane or circle by means of an integral equation between two variables.

It is convenient to add here one four-sided figure, namely a rectangle, for which, from elementary geometrical considerations, the same property evidently holds.

It follows immediately from the present geometrical point of view, that any simply-connected plane polygon which can be formed by the juxtaposition of any finite number of one of these five figures, all of equal size and with homologous sides contiguous, possesses the same property. For, by repeated reflection in its sides, no new sides and corners can be formed other than those which would arise from the same process applied to its component parts, and hence only a finite number of different polygons can be formed which will contain a given point P . I now propose to go on to show that no plane polygons other than these possess the property in question.

Let AB, BC, CD, DE be four consecutive sides of such a polygon. Then, considering first repeated reflections in AB, BC , and CD , it follows, from what has just been proved, that the angles ABC and BOD must both be multiples either of $\pi/6$ or of $\pi/4$; and the same holds for the angles BCD and ODE , and for each pair of consecutive angles of the polygon. Hence the polygon must be such that its angles are all multiples of $\pi/6$ or all multiples of $\pi/4$. For the sake of clearness of statement I now take a particular case, though it will be at once seen that the nature of the argument is such that it will apply to any case whatever. Suppose that A, B, C, D, E are consecutive angles of a polygon which leads to a discontinuous group

of points, and suppose that the angles ABC , BCD , DCE are respectively $3\pi/4$, $3\pi/4$, and $\pi/2$. Through O draw OF parallel to DE . Then reflections in BC , CD , and BC are equivalent to a single reflection in OF , so that among the operations which are equivalent to an even number of reflections in the sides of the polygon are pairs of reflections in AB , OF , and in AB , DE , respectively.

Now, unless the perpendicular distances between AB and OF and between AB and DE are commensurable with each other, the group of points arising from this repetition and combination of these two operations is continuous. Hence these two distances, or, in other words, $BC\sqrt{2}$ and OD , must be commensurable.

Continuing this process, it may be seen that OD and DE must be commensurable, and generally that the ratio of any two sides of the polygon will be commensurable either with unity or with $\sqrt{2}$; the former being the case when both sides have at both their extremities angles which are odd multiples of $\pi/4$, or when both sides have, at one of their extremities at least, angles which are even multiples of $\pi/4$, the latter occurring when one side has at both extremities odd multiples of $\pi/4$, and the other an even multiple at one extremity at least. But these are obviously the conditions that the polygon should be capable of being divided into a finite number of equal isosceles right-angled triangles. And in a precisely similar way it may be shown that the only other cases that can occur are those in which the polygon can be divided into a finite number of equal rectangles or a finite number of equal triangles with angles $\pi/6$, $\pi/3$, and $\pi/2$, the latter case evidently including those of figures formed from triangles (ii.) and (iv.) of p. 191.

The result thus obtained is a known one as regards the four triangles, but I am not aware that a direct proof has ever been given from the purely geometrical point of view taken in this paper; while, as far as I know, the theorem that every rectilinear polygon, that can be represented conformally on a circle or half-plane by means of an integral relation between two variables, is capable of being formed by the juxtaposition either of equal rectangles or equal triangles of one of the four forms, has not been formally proved before this.

In giving some illustrative examples of the actual equations which lead to representations of the sort here considered, it is convenient to deal separately, on account of the greater symmetry of the formulæ, with those cases which arise from the juxtaposition of equilateral triangles, though strictly they do not form a separate class

from those given by the triangles $\frac{\pi}{6}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$. No corresponding separation need be made in the other case, as two isosceles right-angled triangles, when placed together, form either the same figure again, or a square, which leads merely to a particular case of the class of figures arising from a rectangle.

3. *Method of Solution.*

The actual equation which gives the conformal representation of any such area as is being dealt with, on a half-plane, may be directly obtained by algebraical processes, when once the representation of the simplest constituent area, triangle or rectangle, has been carried out. It is necessary to deal separately with the two cases in which the simple area is either a triangle or a rectangle.

Suppose, then, that a triangle ABC has been represented conformally on a half-plane by an equation

$$f(\rho w, z) = 0,$$

where w is the variable in the plane of the triangle, and z that in the half-plane, and suppose that to the angular points A, B, C of the triangle correspond the points $0, 1, \infty$ of the z -plane—this involving no limitation, since, by a linear transformation performed on z , any three points on the real axis may be brought to coincide with $0, 1, \infty$.

If $A'BC$ be the reflection of ABC in BC , the complete figure $ABA'CA$ is represented by means of the equation

$$f(\rho w, z) = 0,$$

on a complete z -plane, which is bounded by a slit running along the real axis from 1 through 0 to ∞ ; or, say upon a positive half-plane and a negative half-plane, which are continuous only along that part of the real axis for which x is positive and greater than unity. If now $A'BC''$ be the reflection of $A'BC$ in $A'B$, the complete figure $ABC''A'CA$ will be represented by means of the same equation on two positive half-planes and one negative half-plane, the two former not being directly connected with each other, while they are continuous with the negative half-plane along the segments 1∞ and 01 of the real axis respectively. The area thus formed is a simply-connected plane area bounded by straight lines (namely, segments of the real axis), and has its angles multiples of two right angles—the angles actually being 2π at 0 , 3π at 1 , and 2π at ∞ . It may therefore be represented on a positive half ζ -plane by means of an equation between z and ζ which is algebraical in both variables, and

of the first degree in z . The elimination of z between this equation and $f(\wp, z) = 0$ will give the required representation of $ABC'A'CA$ on a half-plane. The process thus indicated for the figure formed by the juxtaposition of three triangles may clearly be carried out in any case, so long as the figure is simply-connected (not excluding cases in which parts of the w -plane are covered more than once). When, however, the number of constituent triangles is sufficiently great, it may happen that some of their angular points do not lie on the boundary of the resultant figure. For instance, in the regular hexagon formed by six equilateral triangles, of the eighteen angular points of the triangles, twelve only lie on the boundary of the hexagon, while the other six coincide at its centre. These six all give the same value of z , and belong to triangles which alternately are represented on positive and negative half-planes; and therefore, in the z -plane figure, the centre of the hexagon corresponds to a branch-point where three complete sheets of the z -plane are cyclically connected.

It may now be shown that the form of the required equation between z and ζ may be written down directly, from inspection of the figure to be represented, when divided into its constituent triangles. Thus, if the figure is formed of N triangles, z is a rational function of ζ of degree N ; and since, when ζ is real, z is also real, the coefficients must be real. When ζ is regarded as a function of z , its only branch-points are $0, 1, \infty$, since the coefficients of the equation are real, and the half of the corresponding Riemann's surface which is given by the system of half z -planes is only branched at these points; and hence, if the equation be written

$$z : z-1 : 1 :: A \prod_1^r (\zeta - \alpha_1)^{m_1} : B \prod_1^s (\zeta - \beta_1)^{n_1} : C \prod_1^t (\zeta - \gamma_1)^{p_1},$$

it follows that

$$\sum_1^r (m_1 - 1) + \sum_1^s (n_1 - 1) + \sum_1^t (p_1 - 1) = 2N - 2,$$

the branch-points being all put in evidence by the equation.

Also, since the three products are all of degree N , it follows that

$$\sum_1^r m_1 = \sum_1^s n_1 = \sum_1^t p_1 = N;$$

and therefore

$$r + s + t = N + 2.$$

If any one of the constants α, β, γ is infinity (more than one cannot be so) the corresponding factor must be replaced by a constant.

The values of the indices are given at once by an inspection of the triangle figure. If, at a point on the boundary of the resultant figure,

μ angular points of constituent triangles for which $z = 0$ coincide among the indices m , one will be μ ; and if, at any point within the figure, $2\mu'$ such angular points coincide, then μ' will occur twice among the m indices.

Thus, for the figure

the form of equation is

$$z : z-1 : 1 :: A (\zeta - \alpha_1)^2 : B (\zeta - \beta_1)^2 : C (\zeta - \gamma_1)(\zeta - \gamma_2);$$

while for the figure

the equation is

$$\begin{aligned} z : z-1 : 1 :: & A (\zeta - \alpha_1)^2 (\zeta - \alpha_2)^2 (\zeta - \alpha_3)^4 \\ & : B (\zeta - \beta_1)(\zeta - \beta_2)^2 (\zeta - \beta_3)^2 (\zeta - \beta_4)^2 \\ & : C (\zeta - \gamma_1)^2 (\zeta - \gamma_2)^2 (\zeta - \gamma_3)^2. \end{aligned}$$

The number of constants α, β, γ is $r+s+t$, or $N+2$, and the number of relations between them and A, B, C , arising from the identity

$$A \prod_1^r (\zeta - \alpha_i)^{m_i} - B \prod_1^s (\zeta - \beta_i)^{n_i} = C \prod_1^t (\zeta - \gamma_i)^{p_i},$$

is $N+1$. The ratios only of $A : B : C$ are involved, and these are given in terms of the α, β, γ by two of the equations, so that there are $N-1$ equations connecting the $N+2$ quantities α, β, γ .

When three are given the rest are therefore determinate. From the nature of the problem, it follows that those of these constants which correspond to points on the boundary of the figure are necessarily real, while those which correspond to angular points of constituent triangles that lie within the area occur as pairs of conjugate imaginaries. The various determinations of the constants when

three, for points on the boundary, are given, correspond to the various ways in which these points can be chosen.

Thus, for the second of the two figures given, if $\alpha_1, \beta_1, \gamma_1$ have the values $0, 1, \infty$ assigned to them, then, while C must correspond to the point ∞ on the real axis in the ζ -plane, either A or G may correspond to the point 0 , or either I' or II to the point 1 .

When the constituent area is a rectangle, the real axis in the z -plane will be divided into four segments, say by the points $0, 1, \kappa, \infty$ across each of which the successive positive and negative half-planes may be continuous. If the resultant figure is formed from N elementary rectangles, z will still be a rational function of ζ of degree N with real coefficients; but now ζ , regarded as a function of z , will have the four branch-points $0, 1, \kappa, \infty$, and the equation between z and ζ will be of the form

$$z : z-1 : z-\kappa : 1$$

$$:: A \prod_1^r (\zeta - \alpha_1)^{m_1} : B \prod_1^s (\zeta - \beta_1)^{n_1} : C \prod_1^t (\zeta - \gamma_1)^{p_1} : D \prod_1^u (\zeta - \delta_1)^{q_1},$$

again putting in evidence all the branch-points.

Exactly as before, the indices m, n, p, q are directly determined by an inspection of the figure. The relation between their numbers is now

$$r+s+t+u = 2N+2,$$

and when three of the constants $\alpha, \beta, \gamma, \delta$ are chosen arbitrarily, the rest are determinate.

In the particular case in which all the reflections of the original rectangle are taken in one pair of opposite sides, the problem becomes the same as that of the transformation of the corresponding elliptic functions.

It may be pointed out here that when the complete area has a line of symmetry, the equations for the determination of the constants may be made more simple by taking account of it. Thus,

if the figure $ABIKLMN$ has been represented on the half ζ -plane, the complete figure formed by this and its reflection in BI will be represented on a half ζ' -plane at once by taking ζ a suitable quadratic function of ζ' .

It is, however, obvious, from the known theory of the modular equations, that the equations giving the constants will not generally be algebraically soluble.

4. Examples.

For the sake of completeness, I give first the formulæ for the representation of the four triangles. These have already been given in a variety of forms, so that little more than the results, which will be useful for further applications, need here be written down.

Dealing first with elliptic functions for which the absolute invariant J is zero, and taking

$$g_2 = 0, \quad g_3 = -4,$$

the equation

$$4z = \wp'^2 w \dots\dots\dots(i.)$$

leads to

$$z-1 = \wp^3 w, \quad dz = 3\wp^2 w \wp' w dw,$$

and

$$\frac{dz}{z^{\frac{1}{2}}(z-1)^{\frac{1}{2}}} = 6 dw;$$

so that equation (i.) gives the representation of a triangle

$$\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6} \right)$$

on the half z -plane, the angles corresponding in the order written to 0, 1, ∞ .

By placing together two such triangles, so as to form an equilateral triangle, equation (i.) represents the equilateral triangle conformally on positive and negative half z -planes which are continuous along the negative part of the real axis; and the equation

$$z : z-1 : 1 :: (2\zeta-1)^3 : 4\zeta(\zeta-1) : 1$$

represents these two half z -planes on a half ζ -plane, so that

$$2(2\zeta-1) = \wp' w \dots\dots\dots(ii.)$$

represents the equilateral triangle on the half-plane, the angular points of the triangle again corresponding to 0, 1, ∞ .

Thirdly, by placing two of the original triangles together so as to form a triangle $\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3} \right)$, equation (i.) will again represent this triangle on two half z -planes continuous between 0 and 1; while the equation

$$z : z-1 : 1 :: (\zeta+1)^3 : (\zeta-1)^3 : 4\zeta$$

gives the necessary means of passage to a single half-plane. Hence

$$(\zeta + 1)^2 = \zeta \wp'^2 w \dots\dots\dots(\text{iii.})$$

represents the triangle $\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3}\right)$ on the half ζ -plane, the angles corresponding in order to 0, ∞ , 1.

Taking next elliptic functions for which J is unity, and making

$$g_2 = -4, \quad g_3 = 0,$$

the equation

$$1 - z = \left(\frac{\wp w - 1}{\wp w + 1}\right)^4 \dots\dots\dots(\text{iv.})$$

leads to
$$z = \frac{2\wp'^2 w}{(\wp w + 1)^4}, \quad dz = -8 \frac{(\wp w - 1)^4 \wp' w dw}{(\wp w + 1)^6},$$

and

$$\frac{dz}{z^4(1-z)^4} = -4\sqrt{2} dw;$$

so that equation (iv.) represents the triangle $\left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right)$ on a half-plane, the angles corresponding in order to 1, ∞ , 0.

It will not now be perhaps without interest to illustrate the method of the preceding section by dealing with a certain number of special cases of figures which can be constructed from these triangles. The numbers 0, 1, ∞ , placed inside the figures, indicate the points of the original z -plane to which the angles of the constituent triangle correspond, while the numbers placed outside the figures give the points of the final ζ -plane corresponding to angles of the whole figure.

(a) *Rhombus.*

The equation is

$$z : z - 1 : 1 :: \zeta(\zeta - \alpha) : (\zeta - 1)^2 : O;$$

therefore

$$\alpha = 2, \quad O = -1,$$

and the required equation [combining with (ii.)] is

$$4\zeta(2 - \zeta) - 2 = \wp' w.$$

(β) *Regular hexagon.*—Here considerations of symmetry will shorten the determination of the constants.

The figure may clearly be taken thus :—

and the equation will be

$$z : z-1 : 1 :: \zeta^2 (\zeta^2 - a^2)^2 : O (\zeta^2 - 1)^2 : (\zeta^2 + \gamma^2)^2;$$

therefore $a^2 = 10 + 3\sqrt{5}$, $\delta^2 = 3 + 2\sqrt{5}$,

$$O = -(29 + 12\sqrt{5});$$

and these, with

$$2(2z-1) = \rho'w,$$

complete the representation. In this case it is clear that the form of the resulting equation would be simpler were the hexagon represented on a circle instead of a half-plane. There is little difficulty in showing that the required equation then is

$$\zeta^2 + \frac{1}{\zeta^2} = \rho'w.*$$

(γ) *A two-sheeted equilateral triangle with the winding point at its centre.*—This figure may be formed by the juxtaposition of six triangles $\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3}\right)$, the obtuse angles all coinciding and continuity being assumed between the first side of the first triangle and the last side of the sixth. If, in this case, the figure in the ζ -plane be taken a unit circle, centre the origin, instead of a half-plane, the necessary equation may, from considerations of symmetry, at once be written in the form

$$z : z-1 : 1 :: (\zeta^3 + 1)^2 : 4\zeta^3 : (\zeta^3 - 1)^2,$$

which, with $z + \frac{1}{z} + 2 = \rho'^2 w$ [from (ii.)],

completes the representation.

* This case is given by Schartz (*Gesammelte Abhandlungen*, II., p. 252); he obtains it by the transformation of the corresponding integral.

As a verification, these equations give

$$4^{\frac{1}{2}}.3dw = \frac{dz}{z^{\frac{1}{2}}(z-1)^{\frac{1}{2}}}, \quad dz = -12 \frac{\zeta^3+1}{(\zeta^3-1)^3} \zeta^2 d\zeta;$$

therefore
$$-4^{-\frac{1}{2}}dw = \frac{\zeta d\zeta}{(\zeta^3-1)^{\frac{1}{2}}},$$

so that the figure in the w -plane has six angles, each of $\frac{\pi}{3}$, and a simple winding point within it, as it should have.

(δ) A six-rayed star.

If the figure be represented on a unit circle in the ζ -plane, the symmetry round the centre shows that the correspondence of angles and points indicated in the figure is a possible one. The equation will be

$$z : z-1 : 1 :: (\zeta^3-1)^4 : (\zeta^3-\beta^3)^4 : C\zeta^3 (\zeta^3-\gamma^3);$$

therefore $\beta^3 = -1, \quad \gamma^3 = -1, \quad C = -8,$

and

$$\frac{(\zeta^3-1)^4}{2\zeta^3(\zeta^3+1)} + 2 + \wp'w = 0$$

gives the representation.

(ϵ) The simplest case arising from a rectangle, other than those leading to the formulæ for the transformation of elliptic functions, is that indicated in the following figure:—

The corresponding equation must be of the form

$$z - e_1 : z - e_3 : e_3 - e_1 :: \zeta (\zeta - \alpha)^2 : (\zeta - 1)(\zeta - \beta)^2 : C,$$

$z = e_2$ being no branch-point of ζ .

The constants are at once found to be

$$\alpha = \frac{3}{4}, \quad \beta = \frac{1}{4}, \quad C = \frac{1}{16},$$

and hence the equation effecting the representation is

$$\frac{zw - e_1}{e_3 - e_1} = \zeta (4\zeta - 3)^2.$$

(ζ) As a last example it may be shown, in a manner similar to that used in the third, that a two-sheeted rectangle with a branch-point at its centre is represented conformally on the surface of a unit circle by the equation

$$zw - e_1 : zw - e_3 : e_3 - e_1 :: (\zeta^4 + 1)^2 : (\zeta^4 - 1)^2 : 4\zeta^4.$$

5. Conformal Representation of Multiply-connected Areas.

If the area formed from triangles or rectangles be not simply-connected, it is impossible to represent it conformally on a half-plane, but still a process closely parallel to that which has been explained will give the representation on a standard area of the same multiplicity of connection as the given area. Suppose, for instance, that the area formed from N constituent triangles is doubly-connected. The N half z -planes will then form one (symmetrical) half of a doubly-connected Riemann's surface with definite branch-points at $0, 1, \infty$, and at these points only. On such a surface there is no function of position which takes every value once only, but there are necessarily functions taking every value only twice.

If ζ is such a function it must be connected with z by an equation of the form

$$z^2 f_3(\zeta) + z f_2(\zeta) + f_1(\zeta) = 0 \quad \dots\dots\dots (i.),$$

where one at least of the functions f_1, f_2, f_3 must be of degree N . An inspection of the arrangement of the N triangles in the given figure will determine, as in the case of a simply-connected area, the nature

of the three branch-points, so that the forms of f_1, f_2, f_3 will be given by

$$f_1(\zeta) = A \prod_1^r (\zeta - a_1)^{m_1},$$

$$f_2(\zeta) = O \prod_1^s (\zeta - \gamma_1)^{p_1},$$

$$f_1(\zeta) + f_2(\zeta) + f_3(\zeta) = B \prod_1^t (\zeta - \beta_1)^{n_1},$$

where the indices m, n, p are determined by inspection, and where, of the constants a, β, γ , two, but not more, may be infinity, in which case the corresponding factors are replaced by unity.

Assuming for simplicity that ζ does not become infinite at any of the branch-points, it follows that

$$\sum_1^r m_1 = \sum_1^s n_1 = \sum_1^t p_1 = N,$$

$$\text{and} \quad \sum_1^r (m_1 - 1) + \sum_1^s (n_1 - 1) + \sum_1^t (p_1 - 1) = 2N,$$

so that

$$r + s + t = N.$$

Lastly, since the only irrationality that z contains is the square root of a quartic function, the expression

$$[f_2(\zeta)]^2 - 4f_1(\zeta)f_3(\zeta),$$

must have a square factor of degree $N - 2$.

This is equivalent to $N - 1$ equations of condition. Now, since the ratios only of A, B, O occur, the functions f_1, f_2, f_3 contain $N + 2$ arbitrary constants, and hence, when three of these (say the two points where ζ becomes infinite and one of the points where it becomes zero) are assigned, the determination of the constants becomes definite. When the area to be represented is given, equation (i.) can therefore be obtained by algebraical processes, and, in the symmetrical half of the corresponding surface on which the area was originally represented, ζ will take every value once, and once only. Hence, when (i.) is combined with the equation between z and $\wp w$, the resulting equation between ζ and $\wp w$ represents the area conformally on an infinite plane, rendered doubly-connected by two slits regarded as boundaries.

When the area has a line of symmetry, the process referred to at the end of § 3 may be taken advantage of to simplify its representation on a slit infinite plane. Thus, if the figure there given is represented on a half-plane by an equation

$$f[\zeta, \wp(w)] = 0,$$

so that the points M, N, D, E of the boundary correspond to points m, n, d, e of the real axis in the ζ -plane, the same equation will represent the doubly-connected figure formed of twelve rectangles, obtained by reflecting the given figure in the line of the sides MN and DE , on an infinite plane the positive and negative halves of which are continuous only along the segments mn and de of the real axis. When this method can be applied, the previous general one clearly cannot, since z and ζ are not independent functions of position on the corresponding Riemann's surface.

As an illustration, the actual equations may be given for the representation of the area between two similar, similarly situated, and concentric squares, the larger of which is four times the smaller on a slit plane.

Using the figure at the end of § 3 in conjunction with example (ϵ), e_1, e_2, e_3 are $-1, 0, 1$ when the rectangles are squares, and the equation connected with the figure $ABIJKLMN$ is

$$z+1 : z-1 : 2 :: \zeta(4\zeta-3)^2 : (\zeta-1)(4\zeta-1)^2 : 1.$$

The values of ζ at B, N, A are $0, 1, \infty$, and its values at I, K, M must be in ascending order of magnitude, while they correspond to $z = 0$, or to

$$1 = 2\zeta(4\zeta-3)^2.$$

They are therefore

$$\frac{2-\sqrt{3}}{4}, \quad \frac{1}{2}, \quad \frac{2+\sqrt{3}}{4}.$$

The equation connecting this figure and the complete figure of § 3 is therefore

$$\zeta^2 = O \frac{\zeta}{\zeta - \frac{2-\sqrt{3}}{4}},$$

where O may be chosen arbitrarily.

If O be taken unity the values of ζ are N, D , and at M, E , where ζ is 1 and $\frac{2+\sqrt{3}}{4}$ respectively, are $\pm \frac{2\sqrt{2}}{\sqrt{3}+1}$ and $\pm \frac{\sqrt{3}+1}{2 \cdot 3^{\frac{1}{2}}}$.

Hence, finally,

$$pw+1 = \frac{[(\sqrt{3}-1)\zeta\{(\sqrt{3}+1)\zeta^2-3\}]^2}{(\zeta^2-1)^2}$$

represents the area on an infinite plane bounded by straight slits

from $\frac{-2\sqrt{2}}{\sqrt{3}+1}$ to $\frac{2\sqrt{2}}{\sqrt{3}+1}$, and from $-\frac{\sqrt{3}+1}{2.3^{\frac{1}{2}}}$ through infinity to $\frac{\sqrt{3}+1}{2.3^{\frac{1}{2}}}$.

6. Conclusion.

Returning to the case in which the given area is simply-connected, it is interesting to consider the equation

$$f(z, \wp w) = 0,$$

which effects the representation from the point of view of Riemann's theory. The equation is of finite degree N in z , and transcendental in w . If, then, the surface be regarded as extended over the w -plane, it will be N -sheeted, while z is a function which takes every value an infinite number of times on the surface. If N is greater than unity, the surface is of an infinitely high degree of connectivity, but the branch-points and the distribution of the sheets in connection with each will be perfectly definite. To determine the nature of each branch-point and the possible arrangements of branch-lines, it is sufficient to deal with the value of z in the neighbourhood of each angular point of the original area. Thus, in the equation for the representation of a regular hexagon,

$$z^3 + \frac{1}{z^3} = \wp' w \quad (g_2 = 0, g_3 = -4),$$

the values of $\wp' w$ at three angular points of the hexagon are 2, and at the other three -2 . Near one of the first three

$$(z^3 - 1)^2 \propto w - w_0,$$

so that the six sheets are connected in pairs, and the same clearly holds for the other three angles.

From each branch-point three simple branch-lines must diverge, each connecting a different pair of sheets. A complete system of branch-lines, ensuring the necessary connection between the sheets, may therefore be represented by the figure

where the numbers indicate the two sheets that are connected by each branch-line.

In the case of figure (ε) it is clear that the only branch-points are at the re-entrant angles of the area, and that each of these is a double branch-point connecting three sheets cyclically. A complete system of branch lines will be given by the figure

continued, of course, as in the former case, indefinitely in both directions.

A consideration of the nature of the surface corresponding to the equation

$$f(z, \wp w) = 0$$

becomes of importance, from quite another point of view, when the z is regarded as a particular case of an automorphic function.

The double area obtained by taking together a given area and its reflection in any one of its sides, is a fundamental region for the group of substitutions arising from pairs of reflections in the sides of the given area. But this group is, in general, identical with that arising from pairs of reflections in the side of the constituent triangle or rectangle from which the area has been built up; and the simplest fundamental region is then formed by one pair of the triangles or rectangles. This fundamental region can be within certain limits altered in form (for the simple class of groups here under consideration it cannot be altered in area), but what is here made clear, by a simple instance, is that it depends entirely on the nature of the continuum in which it is to be taken, and that generally when a region is given whose sides are connected in pairs by substitutions belonging to the group (from which the complete group can be constructed) a continuum can be found on which it is a fundamental region.

Thursday, May 11th, 1893.

A. B. KEMPLE, Esq., F.R.S., President, in the Chair.

Señor Tamborrel, of Mexico, was elected a member, and Mr. A. L. Dixon was admitted into the Society.

The following communications were made:—

On the Collapse of Boiler Flues: Mr. A. E. H. Love.

On some Formulæ of Codazzi and Weingarten in relation to the application of Surfaces to each other: Prof. Cayley.

On the Expansion of certain Infinite Products: Prof. L. J. Rogers.

A Theorem for Bicircular Quartic Curves and for Cycloides, analogous to Ivory's Theorem for Conics and Conicoids: Mr. A. L. Dixon.

On the Linear Transformations between two Quadrics: Mr. H. Tuber.

On Maps and the Problem of the Four Colours: Prince C. de Polignac.

On Fermat's Proof that Primes of the Form $4n+1$ can be broken up into the Sum of Two Squares: Mr. S. Roberts.

Supplementary Note on Complex Primes formed with the Fifth Roots of Unity: Prof. Lloyd Tanner.

The following presents were received:—

A Cabinet Likeness of Mr. Love, presented by Mr. Love.

"Voordrachten over den Grondslag van een Bibliographisch Repertorium der Wiskundige Wetenschappen," E and M²3.

"Beiblätter zu den Annalen der Physik und Chemie," Band xvii., Stück 4.

"Proceedings of the Royal Society," Vol. LII., No. 320.

Lemoine, M. E.—"Application d'une Méthode d'Évaluation de la Simplicité des Constructions à la Comparaison de quelques Solutions du Problème d'Apollonius." (Extrait des *Nouvelles Annales de Mathématiques*.)

Lemoine, M. E.—"Application de la Géométrie graphique à l'examen de diverses Solutions d'un même Problème." (Extrait du *Bulletin de la Société Mathématique de France*.)

Lemoine, M. E.—"Résultats et Théorèmes divers concernant la Géométrie du Triangle, &c." (Congrès de Pau, 1892.)

"Nyt Tidsskrift for Mathematik," A. Fjerde Aargang, Nos. 1, 2; Copenhagen.

"Nyt Tidsskrift for Mathematik," B. Fjerde Aargang, No. 1; Copenhagen.

"Transactions of the Connecticut Academy of Arts and Sciences," Vol. viii., Part 2, 1893, and Vol. ix., Part 1; Newhaven, 1892.

- "Bulletin of the New York Mathematical Society," Vol. II., No. 7; April, 1893.
 "Index du Répertoire Bibliographique des Sciences Mathématiques," Paris, 1893.
 "Bulletin des Sciences Mathématiques," 2^{ème} Serie, Tome XVII.; Mars, 1893.
 "Bulletin de la Société Mathématique de France," Tome XXI., No. 3; Paris.
 "Rendiconti del Circolo Matematico di Palermo," Fasc. 1 and 2, Tome VII.; 1893.
 "Annales de la Faculté des Sciences de Toulouse," Fasc. 1, Tome VII., 1893; Paris.
 "Atti della Reale Accademia dei Lincei, 5^o Serie — Rendiconti," Vol. II., Fasc. 6, Sum. 1; Roma, 1893.
 "Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," XLI.-LV., Title page, Contents, and Index, for 1892.
 "Annali di Matematica," Serie II., Tome XXI., Fasc. 1, 1893; Milano.
 "Journal für die reine und angewandte Mathematik," Band CXI., Heft 3, 1893; Berlin.
 "Educational Times," May, 1893.
 "Indian Engineering," Vol. XIII., Nos. 12-16.

On the Collapse of Boiler Flues. By A. E. H. LOVE.

Read May 11th, 1893.

Abstract.

The problem consists in discovering the conditions of a collapse of a thin cylindrical shell under external pressure, when the ends are constrained to occupy fixed positions. Since all problems of collapse depend on the geometrical possibility of finite displacements being accompanied by only infinitesimal strains, it appears at the outset that, unless the shell can receive a displacement of pure bending without stretching of the middle surface, collapse is impossible. The assumed condition of no terminal displacement is equivalent to closing the ends of the shell, and, since a closed surface cannot be bent without stretching, this condition apparently precludes the possibility of collapse. On the other hand, it is well known that, if the external pressure exceed a certain value, an infinitely long cylindrical shell of given small thickness and given diameter will collapse under the pressure. The critical pressure has been determined by Bryan, and Basset, who find the same result. It is therefore to be expected that, if the cylinder is of sufficient length, the extensional displace-