Presents received during the Recess :—

“Proceedings of the Royal Society,” No. 272.

“The Educational Times” for January.

“Proceedings of the Cambridge Philosophical Society,” Vol. vi., Part iv.; Cambridge, 1888.

“Mathematics from the ‘Educational Times,’” Vol. XLIX.

“Royal Irish Academy—Proceedings,” Vol. i., No. 1 (Third Series), “Transactions,” Vol. XXIX., Parts III. and IV.

“Bulletin de la Société Mathématique de France,” Tome XVI., No. 5; Paris, 1888.

“Leçons sur la Théorie Générale des Surfaces,” Deuxième Partie, par Gaston Darboux, 8vo; Paris, 1889.

“Beiblätter zu den Annalen der Physik und Chemie,” Band XII., Stück 11; Leipzig, 1888.

“Jornal de Sciencias Mathematicas e Astronomicas,” Vol. VIII., No. 6; Coimbra, 1887.

“Rendiconti del Circolo Matematico di Palermo,” Tomo II., Fasc. VI., Nov.-Dic., 1888.

“Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa,” Nos. 71 and 72; Firenze.

“Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti,” Tomo V., Ser. VI., Disp. 10; Tomo VI., Ser. VI., Disp. 1—9.

“Journal of the College of Science,” Imperial University, Japan, Vol. II., Part IV.; Tokio, Japan, 1888.

“American Journal of Mathematics,” Vol. XI., No. 2.

Results of Ternary Quadric Operators on Products of Forms of any Orders. By J. J. WALKER, F.R.S.

[Read Jan. 10th, 1889.]

I. *Abstract.*

Considering any ternary form expressed in the quadric shape

$$(abcfgh \mathfrak{X} xyz)^2 \equiv s,$$

and the explicit xyz replaced by symbols of differentiation

$$x \text{ by } \partial_x, \quad y \text{ by } \partial_y, \quad z \text{ by } \partial_z,$$

the operator so obtained may be termed a “pure” quadric operator; but, if the variables are replaced

$$x \text{ by } n\partial_y - m\partial_z, \quad y \text{ by } l\partial_x - n\partial_z, \quad z \text{ by } m\partial_x - l\partial_y,$$

l, m, n being the constant coefficients of a linear form, say,

$$L \equiv lx + my + nz,$$

the latter may be termed a "mixed" quadric operator.

The objects of the present Note are to show :—

(1) How the results of the "pure" operator on the product of any two forms—say, v, w —of orders q, r respectively, may be exhibited in three distinct shapes, viz.—

(a) The "crudo" shape ;

(β) That from which the first differential coefficients of v, w have been eliminated by certain substitutions giving rise to terms reinforcing those of the "crudo" shape, in which v, w are either multiplied by the result of operating on the other, the residue being expressed in the quadric shape

$$(bC_1 + cB_1 - 2fF_1, \dots gH_1 + hG_1 - aF_1 - fA_1, \dots \mathfrak{X}xyz)^2 \dots (i.),$$

wherein $A_1 \dots F_1 \dots$ are the coefficients of the contravariant quadric of v, w , themselves expressed in the quadric form ;

(γ) And a third shape from which either v or w —or both, if forms of the same order—have been eliminated :

(2) How the results (β), (γ) may be extended to cases where the operand is the product of more than two forms :

(3) How the corresponding results may be best expressed when the "mixed" operator is employed on the product of two forms, viz., the result (β) becomes the sum of five terms in three of which s, v, w are each multiplied by the quadric contravariant of the other two ; a fourth term in which L^2 is multiplied by the invariant of svw ,

$$aA_1 + \dots + 2fF_1 + \dots ;$$

and lastly, a term in which L is multiplied by the sum of the products of the first differential coefficients of s with respect to x, y, z by those of the quadric contravariant of v, w with respect to l, m, n , respectively :

(4) How the last result is extended to the product of more than two forms.

The only lengthy piece of work involved being the development of the expression (i.) above, when the product of two functions is substituted for either v or w in it, the investigation of this is made a preliminary Lemma, which has a sort of intrinsic interest of its own.

II. Preliminary.

Supposing, then, v, w to be ternary forms of orders p, q , and representing, for brevity,

$$\left. \begin{aligned} \partial v/p \partial x \text{ by } v_1, \quad \partial w/q \partial x \text{ by } w_1, \quad \partial v/p \partial y \text{ by } v_2 \dots \\ \partial^2 v/p \cdot p-1 \partial x^2 \text{ by } a', \dots \partial^2 v/p \cdot p-1 \partial y \partial z \text{ by } f' \dots \\ \partial^2 w/q \cdot q-1 \partial x^2 \text{ by } a'', \dots \partial^2 w/q \cdot q-1 \partial y \partial z \text{ by } f'' \dots \end{aligned} \right\} \dots\dots\dots(1),$$

$$\begin{aligned} 2v_1w_1 &= 2(a'x + h'y + g'z)(a''x + h''y + g''z) \\ &= a''v + a'w \\ &\quad - (c'a'' + c''a' - 2g'g'')z^2 - (a'b'' + a''b' - 2h'h'')yz \\ &\quad + 2(g'h'' + g''h' - a'f'' - a''f')yz \end{aligned}$$

$$\text{or, say,} \quad = a''v + a'w - B_1z^2 - G_1yz + 2F_1yz \dots\dots\dots(2),$$

and, similarly,

$$v_1w_3 + v_3w_1 = f''v + f'w + A_1yz + F_1x^2 - G_1xy - H_1zx \dots\dots\dots(3),$$

where $A_1 = b'c'' + b''c' - 2f'f'', \dots\dots H_1 = f'g'' + f''g' - c'h'' - c'h'$;

with analogous values for $2v_3w_3 \dots v_3w_1 + v_1w_3 \dots$.

Next, let u, v, w be three ternary forms (in xyz) of orders p, q, r , respectively, and

$$A_1 = \left(\frac{\partial^2 v}{\partial y^2} \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 v}{\partial z^2} \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 v}{\partial y \partial z} \frac{\partial^2 w}{\partial y \partial z} \right) / qr(q-1)(r-1),$$

$$B_1 = \dots, \quad G_1 = \dots,$$

$$F_1 = \left(\frac{\partial^2 v}{\partial z \partial x} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial y} \frac{\partial^2 w}{\partial z \partial x} - \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 w}{\partial y \partial z} - \frac{\partial^2 v}{\partial y \partial z} \frac{\partial^2 w}{\partial x^2} \right)$$

$$/ qr(q-1)(r-1),$$

$$G_1 = \dots, \quad H_1 = \dots,$$

while $A_3 \dots H_2$ stand for the analogous functions of w, u ; and $A_3 \dots H_3$ for those of u, v .

Further, let any other form s be written in the shape

$$s \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy;$$

then, if in the formula

$$[s, v, w] \equiv (bG_1 + cF_1 - 2fF_1)x^2 + \dots + 2(gH_1 + hG_1 - aF_1 - fA_1)yz + \dots$$

the product uv be substituted for v , it may be shown that

$$- [s, uv, w] = pqw [s, u, v] / (p+q) (p+q-1) \\ - pv [s, w, u] / (p+q) - qu [s, v, w] / (p+q) \dots \dots (4).$$

To prove this:

$$\frac{\partial^2 (uv)}{\partial x^2} = u \frac{\partial^2 v}{\partial x^2} + v \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}$$

$$(2), \dots \dots \dots = (p+q-1)(qua' + pva) - pq (B_3 z^2 + C_3 y^2 - 2F_3 yz);$$

and (3),

$$\frac{\partial^2 (uv)}{\partial y \partial z} = (p+q-1)(quf' + pvf) - pq (-A_3 yz - F_3 x^2 + G_3 xy + H_3 xy),$$

with analogous values for $\partial^2 (uv) / \partial y^2 \dots \partial^2 (uv) / \partial x \partial y$.

If now $A \dots I' \dots$ (unmarked) are what $A_1 \dots I'_1$ respectively become by the substitution of uv for v , and, consequently, of $p+q$ for q , writing, as before,

$w_1 \dots$ for $\partial w / r \partial x \dots$, $a'' \dots f'' \dots$ for $\partial^2 w / r.r-1 \partial x^2 \dots$, $\partial^2 w / r.r-1 \partial y \partial z \dots$,

$$(p+q)(p+q-1) A = (p+q-1)(qu A_1 + pv A_1) - pqw A_3$$

$$- pq (A_3 a'' + \dots + 2F_3 f'' + \dots) x^2$$

$$- 2pq (A_3 w_1 + H_3 w_2 + G_3 w_3) x,$$

$$(p+q)(p+q-1) B = (p+q-1)(qu B_1 + pv B_1) - pqw B_3$$

$$- pq (A_3 a'' + \dots + 2F_3 f'' + \dots) y^2$$

$$- 2pq (H_3 w_1 + B_3 w_2 + F_3 w_3) y,$$

$$(p+q)(p+q-1) C = (p+q-1)(qu C_1 + pv C_1) - pqw C_3$$

$$- pq (A_3 a'' + \dots + 2F_3 f'' + \dots) z^2$$

$$- 2pq (G_3 w_1 + F_3 w_2 + C_3 w_3) z,$$

$$(p+q)(p+q-1) I' = (p+q-1)(qu I'_1 + pv I'_1) - pqw I'_3$$

$$- pq (A_3 a'' + \dots + 2F_3 f'' + \dots) yz$$

$$- pq \{ (G_3 w_1 + F_3 w_2 + C_3 w_3) y + (H_3 w_1 + B_3 w_2 + I'_3 w_3) z \},$$

$$(p+q)(p+q-1) G = (p+q-1)(qu G_1 + pv G_1) - pqw G_3$$

$$- pq (A_3 a'' + \dots + 2F_3 f'' + \dots) zw$$

$$- pq \{ (A_3 w_1 + H_3 w_2 + G_3 w_3) z + (G_3 w_1 + F_3 w_2 + C_3 w_3) w \},$$

$$\begin{aligned}
 (p+q)(p+q-1)H &= (p+q-1)(quH_1 + pvH_2) - pqwH_3 \\
 &\quad - pq(A_3a'' + \dots + 2F_3f'' + \dots)xy \\
 &\quad - pq\{(H_3w_1 + B_3w_2 + F_3w_3)x + (A_3w_1 + H_3w_2 + G_3w_3)y\}.
 \end{aligned}$$

From the above equalities

$$\begin{aligned}
 &\quad -(p+q)(p+q-1)(Bc + Cb - 2fF) \\
 &= -(p+q-1)\{qu(bC_1 + cB_1 - 2fF_1) + pv(bC_2 + cB_2 - 2fF_2)\} \\
 &\quad + pqw(bC_3 + cB_3 - 2fF_3) \\
 &\quad + pq(A_3a'' + \dots + 2F_3f'' + \dots)(bz^2 + cy^2 - 2fyz) \\
 &\quad + 2pq(H_3w_1 + B_3w_2 + F_3w_3)(fx - cy) + 2pq(G_3w_1 + F_3w_2 + C_3w_3)(fy - bz), \\
 &\quad -(p+q)(p+q-1)(fG + gF - cH - hC) \\
 &= -(p+q-1)\{qu(fG_1 + gF_1 - cH_1 - hC_1) + pv(fG_2 + gF_2 - cH_2 - hC_2)\} \\
 &\quad + pqw(fG_3 + gF_3 - cH_3 - hC_3) \\
 &\quad + pq(A_3a'' + \dots + 2F_3f'' + \dots)(-cxy + fzx + gyz - hz^2) \\
 &\quad + pq(A_3w_1 + H_3w_2 + G_3w_3)(cy - fz) + pq(H_3w_1 + B_3w_2 + F_3w_3)(cx - gz) \\
 &\quad + pq(G_3w_1 + F_3w_2 + C_3w_3)(-fx - gy + 2hz), \\
 &\quad -(p+q)(p+q-1)(hF + fH - bG - gB) \\
 &= -(p+q-1)\{qu(hF_1 + fH_1 - bG_1 - gB_1) + pv(hF_2 + fH_2 - bG_2 - gB_2)\} \\
 &\quad + pqw(hF_3 + fH_3 - bG_3 - gB_3) \\
 &\quad + pq(A_3a'' + \dots + 2F_3f'' + \dots)(-bzx + fxy - gy^2 + hyz) \\
 &\quad + pq(A_3w_1 + H_3w_2 + G_3w_3)(bz - fy) + pq(H_3w_1 + B_3w_2 + F_3w_3) \\
 &\quad \times (-fx + 2gy - hz) + pq(G_3w_1 + F_3w_2 + C_3w_3)(bx - hy).
 \end{aligned}$$

The sum of the last three equalities, multiplied by x, y, z , respectively, is

$$\begin{aligned}
 &\quad -(p+q)(p+q-1)\{(bC + cB - 2fF)x + (fG + gF - cH - hC)y \\
 &\quad + (hF + fH - bG - gB)z\} \\
 &= -(p+q-1)qu\{(bC_1 + cB_1 - 2fF_1)x + (fG_1 + gF_1 - cH_1 - hC_1)y \\
 &\quad + (hF_1 + fH_1 - bG_1 - gB_1)z\} \\
 &\quad - (p+q-1)pv\{(bC_2 + \dots)x + (fG_2 + \dots)y + (hF_2 + \dots)z\} \\
 &\quad + pqw\{(bC_3 + \dots)x + (fG_3 + \dots)y + (hF_3 + \dots)z\} \\
 &\quad + pq\{(A_3w_1 + \dots)(bz^2 + cy^2 - 2fyz) + (H_3w_1 + \dots)(-cxy + fzx + gyz - hz^2) \\
 &\quad + (G_3w_1 + \dots)(-bzx + fxy - gy^2 + hyz)\}.
 \end{aligned}$$

Analogously,

$$\begin{aligned}
 & -(p+q)(p+q-1) \{ (fG+gF-cH-hC) x + (cA+aC-2gG) y \\
 & \qquad \qquad \qquad + (gH+hG-aF-fA) z \} \\
 & = -(p+q-1) qu \{ (fG_1+\dots) x + (cA_1+\dots) y + (gH_1+\dots) z \} \\
 & \quad - (p+q-1) pv \{ (fG_2+\dots) x + (cA_2+\dots) y + (gH_2+\dots) z \} \\
 & \quad + \quad pqw \{ (fG_3+\dots) x + (cA_3+\dots) y + (gH_3+\dots) z \} \\
 & \quad + pq \{ (A_3w_1+\dots)(-cxy+fxz+gyz-hz^2) \\
 & \quad + (H_3w_1+\dots)(cx^2+az^2-2gzx) + (G_3w_1+\dots)(-ayz-fx^2+gxy+hzx) \}, \\
 & \text{and}
 \end{aligned}$$

$$\begin{aligned}
 & -(p+q)(p+q-1) \{ hF+fH-bG-gB) x + (gH+hG-aF-fA) y \\
 & \qquad \qquad \qquad + (aB+bA-2hH) z \} \\
 & = -(p+q-1) qu \{ (hF_1+\dots) x + (gH_1+\dots) y + (aB_1+\dots) z \} \\
 & \quad - (p+q-1) pv \{ (hF_2+\dots) x + (gH_2+\dots) y + (aB_2+\dots) z \} \\
 & \quad + \quad pqw \{ (hF_3+\dots) x + (gH_3+\dots) y + (aB_3+\dots) z \} \\
 & \quad + pq \{ (A_3w_1+\dots)(-bxz+fax-gy^2+hyz) \\
 & \quad + (H_3w_1+\dots)(-ayz-fx^2+gxy+hzx) + (G_3w_1+\dots)(ay^2+bx^2-2hxy) \}.
 \end{aligned}$$

Finally, the sum of the last three equalities, multiplied by x, y, z , respectively, is

$$\begin{aligned}
 & -(p+q)(p+q-1) \{ (bC+cB-2fF) x^2 + \dots \\
 & \qquad \qquad \qquad \dots + 2 (gH+hG-aF-fA) yz + \dots \} \\
 & = -(p+q-1) qu \{ (bC_1+\dots) x^3 + \dots + 2 (gH_1+\dots) yz + \dots \} \\
 & \quad - (p+q-1) pv \{ (bC_2+\dots) x^3 + \dots + 2 (gH_2+\dots) yz + \dots \} \\
 & \quad + \quad pqw \{ (bC_3+\dots) x^3 + \dots + 2 (gH_3+\dots) yz + \dots \} \dots (4 \text{ bis}),
 \end{aligned}$$

which, divided by $(p+q)(p+q-1)$, is the result (4), p. 113.

III. Results of operating on $vw, uvw \dots$ with

$$(abcfgh \, \mathfrak{X} \, \partial_x, \partial_y, \partial_z)^2.$$

Still employing the notation (1) p. 112, the "crude" shape of

$$\begin{aligned}
 & (a \dots f \dots \mathfrak{X} \partial_x, \partial_y, \partial_z)^2 (vw) \\
 & = r(r-1) v (aa'' + \dots + 2ff'' + \dots) + q(q-1) w (aa' + \dots) \\
 & \quad + 2qr \{ av_1w_1 + \dots + f(v_2w_3 + v_3w_2) + \dots \} \dots \dots \dots (5).
 \end{aligned}$$

Substituting in this for $2v_1w_1 \dots v_2w_3 + v_3w_2 \dots$ from (2), (3), p. 112,
1 2

$$\begin{aligned}
 (a \dots f \dots \mathfrak{X} \partial_x \partial_y \partial_z)^2 (vw) &= (q+r-1) \{rv (aa'' + \dots) + qw (aa' + \dots)\} \\
 &\quad -qr \{bC_1 + cB_1 - 2fF_1\} x^2 + \dots + 2 \{gH_1 + hG_1 - aF_1 - fA_1\} yz + \dots \} \\
 &\dots\dots\dots(6),
 \end{aligned}$$

which is the shape (β) of the Abstract.

Between (5), (6), either of the terms

$$v (aa'' + \dots) \quad \text{or} \quad w (aa' + \dots)$$

may be eliminated, or both if v, w are of the same order, i.e., if $q=r$; and so the shape (γ) of the Abstract be obtained.

Now, suppose the product uv to be substituted for v in (6), u being a form of order p ; then, q becoming $p+q$,

$$\begin{aligned}
 &(a \dots f \dots \mathfrak{X} \partial_x \partial_y \partial_z)^2 (uvw) \\
 &= (p+q+r-1) \{ruv (aa'' + \dots) + w (a \dots f \dots \mathfrak{X} \partial_x \partial_y \partial_z)^2 (uv) / (p+q-1)\} \\
 &\quad - (p+q) r \{bC + cB - 2fF\} x^2 + \dots,
 \end{aligned}$$

wherein $A \dots F' \dots$ are what $A_1 \dots F'_1 \dots$ of (6) become when v is replaced by uv .

Hence, by (5), (4), using the notation of p. 112,

$$\begin{aligned}
 &(a \dots f \dots \mathfrak{X} \partial_x \partial_y \partial_z)^2 (uvw) \\
 &= (p+q+r-1) \left[\{p v w (aa + \dots) + q w u (aa' + \dots) + r u v (aa'' + \dots)\} \right. \\
 &\quad \left. - p q [s, u, v] / (p+q-1) \right] \\
 &\quad + p q r w [s, u, v] / (p+q-1) - r q u [s, v, w] - r p v [s, w, u] \\
 &= (p+q+r-1) \{p v w (aa + \dots) + q w u (aa' + \dots) + r u v (aa'' + \dots)\} \\
 &\quad - q r u [s, v, w] - r p v [s, w, u] - p q w [s, u, v] \dots\dots\dots(7),
 \end{aligned}$$

viz., the coefficient of $[s, u, v]$

$$= \{-(p+q+r-1) p q + p q r\} / (p+q-1) = -p q.$$

By similar steps, the result of operating on the product of four functions may be derived from the case when the operand is the product of three.

Let t be a form of order k , and let

$$\partial^2 t / k . k-1 . \partial x^2 = a''' \dots \partial^2 t / k . k-1 . \partial y \partial z = f''' \dots,$$

then substituting tu for u , $k+p$ for p , and (4)

$$k p w [s, u, t] / (k+p)(k+p-1) - k u [s, w, t] / (k+p) - p t [s, w, u] / (k+p);$$

for $-(s, w, u)$, and for $-[s, u, v]$,

$$kpv [sut]/(k+p)(k+p-1) - ku [svt]/(k+p) - pt [snv]/(k+p),$$

in (7), there results

$$\begin{aligned} & (a \dots \mathfrak{X} \partial_x \partial_y \partial_z)^2 (tuvw) \\ &= (k+p+q+r-1) [vw (a \dots f \dots \mathfrak{X} \partial_x \partial_y \partial_z)^2 (tu)/(p+k-1) \\ & \quad + qvut (a\alpha' + \dots) + ruvt (a\alpha'' + \dots)] \\ & \quad - grut [svw] - rput [swu] - pqut [s, u, v] \\ & \quad + (krp + kpq) vw [sut]/(k+p-1) - kq [svt] - kr [swt]. \end{aligned}$$

$$\text{But (6) } (a \dots \mathfrak{X} \partial_x \dots)^2 (tu)/(k+p-1) = pt (au + \dots) + ku (a\alpha''' + \dots) \\ - kp [s, u, t]/(k+p-1),$$

substituting which in the preceding equation, the coefficient of $vw [s, u, t]$ becomes

$$\{kpr + kpq - (k+p+q+r-1) kp\}/(k+p-1) \quad \text{or} \quad kp,$$

and

$$\begin{aligned} & (a \dots f \dots \mathfrak{X} \partial_x \partial_y \partial_z)^2 (tuvw) \\ &= (k+p+q+r-1) \{kuvw (a\alpha''' + \dots + 2ff''' + \dots) \\ & \quad + p vut (a\alpha + \dots) + q wtu (a\alpha' + \dots) + ruv (a\alpha'' + \dots)\} \\ & \quad - grut [s, v, w] - rput [s, w, u] - pqut [s, u, v] \\ & \quad - kpvw [s, u, t] - kquw [s, v, t] - kruv [s, w, t] \dots \dots \dots (8). \end{aligned}$$

From (7), (8) the general law is evident, whatever be the number of forms in the product operated on; and similarly it may be shown that, if it holds for the product of $i-1$ functions, it will hold for that of i functions.

IV. Results of operating on $vw, uvw \dots$ with

$$(abcfgh \mathfrak{X} n \partial_y - m \partial_z, l \partial_z - n \partial_x, m \partial_x - l \partial_y)^2.$$

In the next place consider the operator formed by substituting in

$$s \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

$$n \frac{\partial}{\partial y} - m \frac{\partial}{\partial z} \text{ for } x, \quad l \frac{\partial}{\partial z} - n \frac{\partial}{\partial x} \text{ for } y, \quad m \frac{\partial}{\partial x} - l \frac{\partial}{\partial y} \text{ for } z;$$

it is evident that the result of operating with this will be deduced from the corresponding result of operating with the operator of the

first part of this paper by substituting in that result

$$\begin{aligned} &bn^2 + cm^2 - 2fmn \text{ for } a, \quad cl^2 + an^2 - 2gnl \text{ for } b, \quad am^2 + bl^2 - 2hlm \text{ for } c, \\ &-amn - fl^2 + glm + hnl \text{ for } f, \quad -bnl + flm - gm^2 + hmn \text{ for } g, \\ &-clm + fnl + gmn - hn^2 \text{ for } h. \end{aligned}$$

Thus (4)

$$\begin{aligned} &\left\{ n \left(n \frac{\partial}{\partial y} - m \frac{\partial}{\partial z} \right)^2 + \dots + 2f \left(l \frac{\partial}{\partial z} - n \frac{\partial}{\partial x} \right) \left(m \frac{\partial}{\partial x} - l \frac{\partial}{\partial y} \right) + \dots \right\} (vw) \\ &= (q+r-1) \left[rv \left\{ (bc'' + cb'' - 2ff'') l^2 + \dots \right. \right. \\ &\quad \left. \left. \dots + 2(-af'' - fa'' + glh'' + hlg'') mn + \dots \right\} \right. \\ &\quad \left. + qw \left\{ (bc' + cb' - 2ff') l^2 + \dots \right\} \right] \\ -qr\Sigma &\left| \begin{aligned} &x^2 \{ B_1 (bl^2 + am^2 - 2hlm) + C_1 (cl^2 + am^2 - 2gnl) \\ &\quad + 2F_1 (amn + fl^2 - glm - hnl) \} \\ &+ xy \{ F_1 (-bnl + flm - gm^2 + hmn) + G_1 (-amn - fl^2 + glm + hnl) \\ &\quad + C_1 (clm - fnl - gmn + hn^2) + H_1 (-am^2 - bl^2 + 2hlm) \} \\ &+ zx \{ H_1 (-amn - \dots + hnl) + F_1 (-clm + \dots - hn^2) \\ &\quad + B_1 (bnl + \dots + hmn) + G_1 (-cl^2 - am^2 + 2gnl) \} \end{aligned} \right| \end{aligned}$$

or, identically,

$$\begin{aligned} -qr\Sigma &\left| \begin{aligned} &ax^2 (A_1 l^2 + \dots + 2F_1 mn + \dots) + l^2 x^2 (A_1 a + \dots + 2F_1 f + \dots) \\ &\quad - 2axlx (A_1 l + H_1 m + G_1 n) - 2hxlx (H_1 l + B_1 m + F_1 n) \\ &\quad \quad - 2gxlx (G_1 l + F_1 m + C_1 n) \end{aligned} \right\} \\ &+ hxy \left\{ \begin{aligned} &(A_1 l^2 + \dots + 2F_1 mn + \dots) + lmax (A_1 a + \dots + 2F_1 f + \dots) \\ &\quad - (axmy + hylx) (A_1 l + H_1 m + G_1 n) \\ &\quad - (hxm y + bylx) (H_1 l + B_1 m + F_1 n) \\ &\quad - (gxmy + fylx) (G_1 l + F_1 m + C_1 n) \end{aligned} \right\} \\ &+ gzx \left\{ \begin{aligned} &(A_1 l^2 + \dots + 2F_1 mn + \dots) + nlzx (A_1 a + \dots + 2F_1 f + \dots) \\ &\quad - (axnz + gzx) (A_1 l + \dots) - (hxnz + fzx) (H_1 l + \dots) \\ &\quad \quad - (gxnz + czlx) (G_1 l + \dots) \end{aligned} \right\} \end{aligned}$$

or, if $s_1 = \partial s / 2 \partial x \dots$, $I_1 = lx + my + nz$,

$$\begin{aligned} -qr\Sigma &\left| \begin{aligned} &s_1 x (A_1 l^2 + \dots + 2F_1 mn + \dots) + Ll x (A_1 a + \dots + 2F_1 f + \dots) \\ &\quad - (axL + s_1 lx) (A_1 l + \dots) + (h xL + s_1 lx) (H_1 l + \dots) \\ &\quad \quad + (g xL + s_1 lx) (G_1 l + \dots) \end{aligned} \right| ; \end{aligned}$$

whence, finally,

$$\begin{aligned} & \left\{ a \left(n \frac{\partial}{\partial y} - m \frac{\partial}{\partial z} \right)^2 + \dots \right\} (vw) \\ = & (q+r-1) [rv \{bc'' + cb'' - 2ff''\} l^2 + \dots] + qw \{ (bc' + cb' - 2ff') l^2 + \dots \} \\ & -qr [(A_1 l^2 + \dots + 2F_1 mn + \dots) s + (A_1 a + \dots + 2F_1 f + \dots) L \\ & -2 \{ (A_1 l + H_1 m + G_1 n) s_1 + (H_1 l + B_1 m + F_1 n) s_2 \\ & + (G_1 l + F_1 m + C_1 n) s_3 \} L] \dots\dots\dots(9). \end{aligned}$$

From the foregoing investigation it is evident that (7)

$$\begin{aligned} & \left\{ a \left(n \frac{\partial}{\partial y} - m \frac{\partial}{\partial z} \right)^2 + \dots \right\} (uvw) \\ (p+q+r-1) [& p v w \{ (bc + cb - 2ff) l^2 + \dots \} + q w u \{ (bc' + \dots) l^2 + \dots \} \\ & + r u v \{ (bc'' + \dots) l^2 + \dots \}] \\ & - \{ q r u (A_1 l^2 + \dots) + r p v (A_1' l^2 + \dots) + p q w (A_1'' l^2 + \dots) \} s \\ & - \{ q r u (A_1 a + \dots) + r p v (A_1' a + \dots) + p q w (A_1'' l^2 + \dots) \} L^2 \\ + 2 [& \{ q r u (A_1 l + H_1 m + G_1 n) + r p v (A_1' l + \dots) + p q w (A_1'' l + \dots) \} s_1 \\ & + \{ q r u (H_1 l + B_1 m + F_1 n) + r p v (H_1' l + \dots) + p q w (H_1'' l + \dots) \} s_2 \\ & + \{ q r u (G_1 l + F_1 m + C_1 n) + r p v (G_1' l + \dots) + p q w (G_1'' l + \dots) \} s_3] L \\ & \dots\dots\dots(10); \end{aligned}$$

and that according to the same law the result of operating on the product of four, or more, functions may be at once written down.

A Theorem in Combinations. By R. W. D. CHRISTIE.

[Read Jan. 10th, 1889.]

Let $\Sigma ({}_n S_r)^m$ represent the sum of the m^{th} powers of the sums of the letters which form the several combinations of n letters taken r at a time; then

$$\Sigma ({}_n S_1)^m - \Sigma ({}_n S_2)^m + \Sigma ({}_n S_3)^m - \Sigma ({}_n S_4)^m \dots \pm \Sigma ({}_n S_n)^m \equiv 0,$$

where $m < n$, e.g., if $n = 5$, $m = 4$,

$$\begin{aligned} & a^4 + b^4 + c^4 + d^4 + e^4 - (a+b)^4 - (a+c)^4 - (a+d)^4 - (a+e)^4 - (b+c)^4 \\ & - (b+d)^4 - (b+e)^4 - (c+d)^4 - (c+e)^4 - (d+e)^4 + (a+b+c)^4 \end{aligned}$$

$$\begin{aligned}
& + (a+b+d)^4 + (a+b+e)^4 + (b+c+d)^4 + (b+c+e)^4 + (b+d+e)^4 \\
& + (c+d+e)^4 + (a+c+d)^4 + (a+c+e)^4 + (a+d+e)^4 - (a+b+c+d)^4 \\
& - (a+b+c+e)^4 - (a+b+d+e)^4 - (a+c+d+e)^4 - (b+c+d+e)^4 \\
& + (a+b+c+d+e)^4 \equiv 0.
\end{aligned}$$

$$\begin{aligned}
\text{Here } \Sigma({}_6S_6)^4 &= \Sigma a^4 + 4\Sigma a^3b + 6\Sigma a^2b^2 + 12\Sigma a^2bc + 24\Sigma abcd \\
\Sigma({}_6S_4)^4 &= 4\Sigma a^4 + 12\Sigma a^3b + 18\Sigma a^2b^2 + 24\Sigma a^2bc + 24\Sigma abcd \\
\Sigma({}_6S_3)^4 &= 6\Sigma a^4 + 12\Sigma a^3b + 18\Sigma a^2b^2 + 12\Sigma a^2bc \\
\Sigma({}_6S_2)^4 &= 4\Sigma a^4 + 4\Sigma a^3b + 6\Sigma a^2b^2 \\
\Sigma({}_6S_1)^4 &= \Sigma a^4.
\end{aligned}$$

Thus the number of times that

$$\begin{aligned}
\Sigma a^4 \text{ or } a^4, b^4, c^4 \dots \text{ will occur will be } {}_4C_0 - {}_4C_1 + {}_4C_2 - {}_4C_3 + {}_4C_4 &= (1-1)^4 = 0, \\
4\Sigma a^3b \text{ or } 4a^3b, 4a^2c \text{ or } 6\Sigma a^2b^2 \text{ or } 6a^2c^2 \dots -{}_3C_0 + {}_3C_1 - {}_3C_2 + {}_3C_3 &= -(1-1)^3 = 0, \\
12\Sigma a^2bc \dots \dots \dots {}_2C_0 - {}_2C_1 + {}_2C_2 &= (1-1)^2 = 0, \\
24\Sigma abcd \dots \dots \dots -{}_1C_0 + {}_1C_1 &= -(1-1)^1 = 0,
\end{aligned}$$

and there is no term containing the five letters.

In the general case, the number of times that $a^m, b^m, c^m \dots$ will occur, will be

$${}_{n-1}C_0 - {}_{n-1}C_1 + {}_{n-1}C_2 - \dots {}_{n-1}C_{n-1} = (1-1)^{n-1} = 0;$$

that $\frac{m!}{p! q! r!} a^p b^q c^r \dots$ will occur, will be

$$(-1)^{k-1} [{}_{n-k}C_0 - {}_{n-k}C_1 + \dots] = (-1)^{k-1} (1-1)^{n-k} = 0,$$

where $p+q+r \dots = m$, and the letters $a, b, c \dots$ are k in number.

The general explanation is, that the number of ways of selecting r things out of n so that each selection shall contain k given things is

$${}_{n-k}C_{r-k}.$$

The theorem will enable us to obtain an unlimited supply of equations such as the following:—

$$\text{Let } a = 1, b = 2, c = 3, d = 5, e = 7, f = 8.$$

Then, after cancelling terms like O^m with $-(a+b)^m$, &c., we get

$$0^m + 4^m + 9^m + 17^m + 22^m + 26^m \equiv 1^m + 2^m + 12^m + 14^m + 24^m + 25^m,$$

where $m = 1, 2, 3, 4$, or 5 ; and therefore for the same magnitudes

of m

$$n^m + (n+4)^m + (n+9)^m + (n+17)^m + (n+22)^m + (n+26)^m$$

$$\equiv (n+1)^m + (n+2)^m + (n+12)^m + (n+14)^m + (n+24)^m + (n+25)^m$$

for all magnitudes of n .

Thursday, February 14th, 1889.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

Mr. Baker was admitted into the Society.

The following communications were made:—

On the Diophantine Relation $y^2 + y'^2 = \text{square}$: Prof. Cayley, F.R.S.

Sur la Transformation des Équations Algébriques: Signor Brioschi.

On Projective Cyclic Concomitants or Surface Differential Invariants: E. B. Elliott, M.A.

On Secondary Invariants: Prof. L. J. Rogers, M.A.

Remarks upon Algebraical Symmetry, with particular reference to the Theory of Operations, and the Theory of Distributions: Major Macmahon, R.A.

The following presents were received:—

"Proceedings of the Royal Society," Vol. xlv., Nos. 273 and 274.

"Educational Times," for February.

"Greek Geometry, from Thales to Euclid," by Dr. G. J. Allman, 8vo; Dublin and London, 1889.

"Bulletin des Sciences Mathématiques," November and December, 1888, and January, 1889.

"Beiblätter zu den Annalen der Physik und Chemie," Band xii., Stück 12; Band xiii., Stück 1.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," No. 73; Index to Ditto, for 1888 (part only).

"Memorias de la Sociedad Científica—'Antonio Alzate,' " Tomo ii., No. 5; Mexico, 1888.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. iv., Fasc. 6, 7, 8, 9.

"Annales de l'Ecole Polytechnique de Delft," Tome iv., Livraison 3^{me}; Leide, 1888.

"Annali di Matematica," Tome xvi., Fasc. 3.

"Journal für die reine und angewandte Mathematik," Band 104, Heft. ii.

"Abhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften," Band 14, Nos. 10, 11, 12, 13; Leipzig, 1888.