XXXIX.—*On Centres, Faisceaux, and Envelopes of Homology.* By Rev. HUGH MARTIN, M.A., Member of the Mathematical Society of London, and Examiner in Mathematics in the University of Edinburgh. Communicated by Professor KELLAND.

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One of the theorems of a paper which Professor KELLAND did me the honour to read to the Society, in March 1865, opens up a field of geometrical investigation so interesting and fertile, that I venture to ask attention to some of the results of a partial examination of it in the following series of propositions. I think it right to explain, that I do not venture to expect attention to them on account of any importance attaching to them individually, but on account of their number and somewhat elegant relations. Considered individually, they may be of little importance, having no claim to rank, so to speak, among propositions of a *planetary* magnitude. But a system of *moons*, however diminutive, may become interesting if they present elegant relations among their mean motions and longitudes; and an orbit that would be grudged to a pigmy planet may be willingly accorded to a host of *planetoids*. If this is still too exalted language in which to speak of the following results, I can at least confidently affirm that they indicate a direction in which an analyst of very moderate attainments may easily discover for himself a shower of *meteors*.

It is well known that when straight lines are drawn from the angles of a triangle through any point in its plane, they intersect the sides in three points, which form the angular points of a triangle so related to the first that the intersections of their corresponding sides range in a straight line. As the triangles are said to be in homology,* we may conveniently designate the point as the Centre of Homology, and the resulting straight line as the Line of Homology,—the line represented in the former paper by $\phi(P_1)$. If we have a second point, P_2 , inverse to the former, we have a second line of homology, which may be called the inverse line.

Now, if the centre of homology is subjected to motion according to a given law, or in a given curve, the line of homology displaces itself, so as, in its varying positions, to constitute a *faisceau*. The Envelope of this *faisceau* may then be inquired for. Simultaneously the inverse centre of homology will move in a

* I fear this terminology may give an aspect of pretentiousness to the paper which is far from being intended. But if I was to avoid the indefinite title, "*On a Certain Class*," &c., I confess I could find no other sufficiently descriptive.

curve that may be found; and the inverse line of homology, displacing itself, will generate another *faisceau*, whose envelope also may be sought. Farther, the lines of homology, the direct and inverse, will in general intersect, and the point of intersection, partaking of the motion of the system, will describe a locus, which may also be made the subject of inquiry. The points in which perpendiculars from the centres of homology meet their respective, direct, and inverse lines of homology will also present certain loci when the centre of homology describes any given curve, and the perpendiculars themselves will generate *faisceaux*, whose envelopes may be sought.

Farther; this system of problems may be generalised and greatly extended. The line, called the line of homology, has originated in a particular geometrical consideration; but it may be considered as unshackled from its geometrical genesis. The conception may be idealised, and conceived of as not restricted to the straight line, the curve of the first degree, but as a curve of any degree whatever. It may be a function of any degree in the variables, and of any degree in the co-ordinates of the centre as parameters; and we may thus have *faisceaux* of curves of homology of any order, and envelopes corresponding to them as before. Moreover, the general problem may be inverted, and instead of inquiring, "What envelope will the *faisceau* of homology generate when the centre of homology moves in a given curve?" we may inquire, "In what curve must the centre of homology move, in order, with a given curve of homology, to beget a given envelope?"* Or the inquiry may take yet another direction, "What must be the form of the curve of homology in order that its *faisceau*, generated by the centre of homology moving in a given curve, may produce a given envelope?" Some instances of the problem, in all these forms, will be found in the following pages. It is evident that the general inquiry may be prosecuted in such directions as would task the uttermost resources of modern discoveries in the theory of linear transformation and of canonical forms. With the exception of an instance of elimination by the aid of the Jacobian and its differential coefficients, we shall not pursue it to the necessity of laborious calculations; and shall, for the most part, restrict ourselves to the conic sections, and to such forms of these as are most manageable, and do not demand intricate elimination. These forms are,—*First*, the conic circumscribing the triangle of reference, and of the general form,

$$\frac{u}{\alpha} + \frac{v}{\beta} + \frac{w}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1);$$

Second, the conic touching the three sides of the triangle, viz.,

$$\pm (u\alpha)^{\frac{1}{2}} \pm (v\beta)^{\frac{1}{2}} \pm (w\gamma)^{\frac{1}{2}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2);$$

* The idea of the inverse of the problem of the Envelope seems first to have occurred to BOOLE. See his paper—characterised by his usual high generality and beautiful originality—in "Cambridge and Dublin Mathematical Journal," vol. vii. p. 156.

Third, the conic with respect to which the triangle is self-conjugate, viz.,

$$u^2\alpha^2 + v^2\beta^2 + w^2\gamma^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3);$$

And, *fourthly*, the conic touching two sides of the triangle of reference in the points where the third side meets them, viz.,

$$k^2\alpha^2 = \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

SECTION I.—*The Faisceau of Homology being Straight Lines.*

I. If the centre of homology move in a straight line, the *faisceau* of homology (being straight lines) will envelope a conic touching the three sides of the triangle of reference.

Let the centre of homology move in the straight line

$$l\alpha + m\beta + n\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Retaining the co-ordinates of the centre in the form in which they appeared in the former paper, f^{-1} , g^{-1} , h^{-1} (co-ordinates of P_1), and substituting these in (5), we have

$$lf^{-1} + mg^{-1} + nh^{-1} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

The equation of the straight line of homology is,

$$f\alpha + g\beta + h\gamma = 0 = u \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Now, by the theory of envelopes,

$$du = 0 = \alpha df + \beta dg + \gamma dh \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Differentiating (6), $0 = -lf^{-2}df - mg^{-2}dg - nh^{-2}dh \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$

Multiplying (8) by an indeterminate coefficient p , and adding (9), we have

$$(p\alpha - lf^{-2})df + (p\beta - mg^{-2})dg + (p\gamma - nh^{-2})dh = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Now, as p is indeterminate, and as it is the ratios merely of the three parameters f , g , h , we are concerned with, reducing them virtually to two—with which the fact that they are homogeneously involved accords—we are at liberty to make two suppositions. Let these be that the coefficients of df and dg in (10) shall vanish. This makes the coefficient of dh also vanish; and we have,

$$f = \pm \sqrt{\frac{l}{p\alpha}}; \quad g = \pm \sqrt{\frac{m}{p\beta}}; \quad h = \pm \sqrt{\frac{n}{p\gamma}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Substituting these values in (7) gives,

$$\pm (l\alpha)^{\frac{1}{2}} \pm (m\beta)^{\frac{1}{2}} \pm (n\gamma)^{\frac{1}{2}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12);$$

which is the equation of a conic touching the three sides of the original triangle.

II. If the centre of homology move in a conic touching the three sides of the original triangle, the linear *faisceau* of homology will envelope the curve,

$$\pm (l\alpha)^{\frac{1}{2}} \pm (m\beta)^{\frac{1}{2}} \pm (n\gamma)^{\frac{1}{2}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

The proof of this is, *mutatis mutandis*, the same as before. And generally,—

III. If the centre of homology move in the curve

$$\pm (l\alpha)^{\frac{1}{n}} \pm (m\beta)^{\frac{1}{n}} \pm (n\gamma)^{\frac{1}{n}} = 0 \quad (14),$$

the linear *faisceau* of homology will envelope the curve,

$$\pm (l\alpha)^{\frac{1}{n+1}} \pm (m\beta)^{\frac{1}{n+1}} \pm (n\gamma)^{\frac{1}{n+1}} = 0 \quad (15).$$

IV. If the centre of homology move in a conic with respect to which the original triangle is self-conjugate, namely,

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 = 0 \quad (16),$$

the linear *faisceau* of homology will envelope the curve,

$$(l\alpha)^{\frac{2}{3}} + (m\beta)^{\frac{2}{3}} + (n\gamma)^{\frac{2}{3}} = 0 \quad (17).$$

The proof of this is the same as before; and it follows also from III., n being taken equal to $\frac{1}{2}$.

Of course, in (16), that the conic may not be imaginary, one or other of the terms l, m, n , must be affected with the coefficient $\sqrt{-1}$; but the equation is maintained in this form for the sake of symmetry.

V. If the centre of homology move in a conic circumscribing the original triangle, the linear *faisceau* of homology will be a *faisceau pivotante*.

The equation of the circumscribing conic is

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0 \quad (18).$$

Replacing α, β, γ , by the co-ordinates of the centre, we have.

$$lf + mg + nh = 0 \quad (19).$$

And, as before,

$$f\alpha + g\beta + h\gamma = 0 = u \quad (20).$$

And the result is that

$$\alpha : \beta : \gamma :: l : m : n \quad (21).$$

That is, the line of homology revolves round the fixed point l, m, n . This is a case of what the French writers call *courbes pivotantes*,* the curve here being of the first degree, the straight line. The number of *pivots* about which a *faisceau* of curves of the degree s revolves, or through which every member of the *faisceau pivotant* passes, is of course s^2 . We shall meet with a case of this subsequently, in reference to curves of the second degree—conic sections.

VI. If the centre of homology move in the circle circumscribing the original

* The reader will find this subject elegantly treated in "*Études Analytiques sur la Théorie Générale des Courbes Planes*," par M. FELIX LUCAS (Paris, 1864); where, among other results, a very pretty proposition concerning a property of conics circumscribing the same quadrilateral (due to M. LAMÉ), is thus generalised:—*Les polaires d'ordre quelconque d'un point du plan, relativement aux diverses courbes d'un faisceau pivotant, forment elles-mêmes un faisceau pivotant.*

triangle, the linear *faisceau* of homology will revolve round a point whose co-ordinates are in the ratios of the sides of the triangle.

This is an immediate corollary from the former, the equation of the circumscribing circle being, as is well known,

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

VII. If the centre of homology describe a straight line, the inverse centre will describe a conic circumscribing the original triangle; the linear *faisceau* will envelope a conic touching its three sides, while the inverse linear *faisceau* will revolve round a fixed point.

This singular system of movements is merely a combination of propositions I. and V.,—it being remembered that the co-ordinates of the two centres are mutually inverse.

VIII. If the centre of homology describe a conic touching two sides of the triangle in the points where the third side meets them, the inverse centre will describe a second conic similarly situated; the linear *faisceau* will envelope a third conic similarly situated, and the inverse linear *faisceau* a fourth conic, also similarly situated.

The equation of the conic which, by hypothesis, the centre describes is,

$$k^2 \alpha^2 = \beta \gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (23);$$

or, substituting the co-ordinates of the centre,

$$k^2 gh - f^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

Also

$$f\alpha + g\beta + h\gamma = 0 = u \quad . \quad . \quad . \quad . \quad . \quad (25).$$

Differentiating as before, multiplying by an indeterminate coefficient p , &c., we have,

$$f = \frac{p\alpha}{2}; \quad g = -\frac{p\gamma}{k^2}; \quad h = -\frac{p\beta}{k^2} \quad (26).$$

And substituting in (25), we find, as the equation of the envelope of the direct linear *faisceau*,

[illegible]

Farther, while the centre moves in $k^2\alpha^2 = \beta\gamma$, it is evident, inverting the co-ordinates, that the inverse centre moves in

$$\frac{\alpha^2}{k^2} = \beta\gamma . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

and that, by the portion of the proposition already proved, the inverse *faisceau* must envelope

$$\frac{\alpha^2}{k^2} = 4\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (29);$$

which completely proves the proposition.

Substituting these in (37), we have, as the envelope required, the conic section,

$$3\left(\frac{\alpha^2}{l^2} + \frac{\beta^2}{m^2} + \frac{\gamma^2}{n^2}\right) - 2\left(\frac{\beta\gamma}{mn} + \frac{\gamma\alpha}{nl} + \frac{\alpha\beta}{lm}\right) = 0 \quad (40).$$

Let it now be supposed that the centre of homology moves in this last curve, and let it be required to find the envelope of the inverse *faisceau*.

We have now, $3\{(lf)^{-2} + (mg)^{-2} + (nh)^{-2}\} - 2\{(mng h)^{-1} + (nlh f)^{-1} + (lmf g)^{-1}\} = 0$,

and $f^{-1} \cdot \alpha + g^{-1} \cdot \beta + h^{-1} \gamma = 0 = u$,

which, with the exception of the numerical coefficients, are identical with (36) and (40), the variables to be eliminated being merely inverted, and l, m, n also inverted. Moreover, the process reverses the change of numerical coefficients, as exhibited between (35) and (40), and the result is

$$l^2 \alpha^2 + m^2 \beta^2 + n^2 \gamma^2 + mn\beta\gamma + nl\gamma\alpha + lm\alpha\beta = 0 \quad (41);$$

a conic, and somewhat singularly related to the original curve, namely (35).

X. If the centre of homology move in a conic circumscribing the original triangle, required the locus of the intersection of the direct and inverse lines of homology.

As the point whose locus is required is on both the lines, we have simultaneously,

$$\alpha f + \beta g + \gamma h = 0 \quad (42),$$

$$\frac{\alpha}{f} + \frac{\beta}{g} + \frac{\gamma}{h} = 0 \quad (43).$$

Also the equation of the circumscribing conic is,

$$lf + mg + nh = 0 \quad (44).$$

From (42) and (44), we have,

$$\frac{f}{\begin{vmatrix} m, \beta \\ n, \gamma \end{vmatrix}} = \frac{g}{\begin{vmatrix} n, \gamma \\ l, \alpha \end{vmatrix}} = \frac{h}{\begin{vmatrix} l, \alpha \\ m, \beta \end{vmatrix}} \quad (45),$$

and, by (43),

$$\frac{\alpha}{\begin{vmatrix} m, \beta \\ n, \gamma \end{vmatrix}} + \frac{\beta}{\begin{vmatrix} n, \gamma \\ l, \alpha \end{vmatrix}} + \frac{\gamma}{\begin{vmatrix} l, \alpha \\ m, \beta \end{vmatrix}} = 0 \quad (46),$$

$$\text{or, } \frac{mn}{\beta\gamma}(\alpha^2 - \beta^2 - \gamma^2) + \frac{nl}{\gamma\alpha}(\beta^2 - \gamma^2 - \alpha^2) + \frac{lm}{\alpha\beta}(\gamma^2 - \alpha^2 - \beta^2) + l^2 + m^2 + n^2 = 0 \quad (47);$$

which is the locus required—a curve of the third order. If $\alpha = 0$, we have

$$(n\beta - m\gamma) \cdot (\beta^2 - \gamma^2) = 0 \quad (48);$$

that is to say, the curve cuts the side of the triangle in points where $\beta = \pm \gamma$, and $\beta : \gamma :: m : n$. And similarly for the intersections with the other sides. The geometrical interpretation is, that the curve cuts the sides of the triangle in the points where the bisectors of the interior and exterior opposite angles cut them; as also, in the points determined by drawing straight lines from the angles to the opposite sides respectively, through the points l, m, n ; that is, the pole of the line $\{l\alpha + m\beta + n\gamma = 0\}$ in which the inverse centre of homology moves, while the direct centre describes the circumscribing conic. Considering the triangle as a curve of the third order, the $3^2 = 9$ points, in which it intersects the third-order locus thus found, are in this manner somewhat elegantly determined.

XI. If the centre of homology move in a conic touching two sides of the triangle where the third side meets them, required the locus of the intersection of the direct and inverse lines of homology.

Here we have to eliminate from the three equations,

$$\left. \begin{aligned} f^2 - k^2gh &= 0, \\ \alpha f + \beta g + \gamma h &= 0, \\ \alpha gh + \beta hf + \gamma fg &= 0. \end{aligned} \right\} \quad (49);$$

and the result is,

$$\frac{(\beta^2 - \gamma^2)^2}{\alpha^2\beta\gamma} - \frac{\beta^2 + \gamma^2}{\beta\gamma} + k^2 + \frac{1}{k^2} = 0 \quad (49);$$

a curve of the fourth order, which is not altered by interchanging β and γ ,—as it evidently ought not to be; nor by inverting k ,—which evidently also it ought not to be, since while the centre of homology moves in

$$k^2\alpha^2 = \beta\gamma,$$

the inverse centre moves in

$$\frac{\alpha^2}{k^2} = \beta\gamma,$$

and if these are interchanged, the lines of homology are simply interchanged, and the result of elimination ought to be unaltered, as we see is the case.

XII. Required the locus of the intersection of the direct and inverse lines of homology, when the centre of homology moves in a conic with respect to which the triangle is self-conjugate.

Taking the equation of the curve in which the inverse centre moves, the square of the equation of the direct line of homology, and the equation of the inverse line cleared of fractions, we have to eliminate f, g, h , from the three following equations:—

$$u = lf^2 + mg^2 + nh^2 = 0 \quad (50);$$

$$v = \alpha^2f^2 + \beta^2g^2 + \gamma^2h^2 + 2\beta\gamma gh + 2\gamma\alpha hf + 2\alpha\beta fg = 0 \quad (51);$$

$$w = \alpha gh + \beta hf + \gamma fg = 0 \quad (52).$$

Forming the Jacobian,

$$\begin{vmatrix} \frac{du}{df}, & \frac{dv}{df}, & \frac{dw}{df} \\ \frac{du}{dg}, & \frac{dv}{dg}, & \frac{dw}{dg} \\ \frac{du}{dh}, & \frac{dv}{dh}, & \frac{dw}{dh} \end{vmatrix} = J \quad . \quad . \quad . \quad . \quad . \quad . \quad (53),$$

we have,

$$(af + \beta g + \gamma h) \times \begin{vmatrix} lf, & \alpha, & \beta h + \gamma g \\ mg, & \beta, & \gamma f + \alpha h \\ nh, & \gamma, & \alpha g + \beta f \end{vmatrix} = J \quad . \quad . \quad . \quad . \quad . \quad . \quad (54),$$

That is,—

$$(af + \beta g + \gamma h) \cdot \left\{ \begin{array}{l} l\alpha\beta^2g + l\beta^2f^2 - l\gamma^2f^2 - l\gamma\alpha hf + m\beta\gamma gh + m\gamma^2g^2 - \\ m\alpha^2g^2 - m\alpha\beta fg + n\gamma\alpha hf + n\alpha^2h^2 - n\beta^2h^2 - n\beta\gamma gh \end{array} \right\} = J \quad (55).$$

Multiplying the part within brackets by the coefficient af , and differentiating with respect to f , we have

$$\left\{ \begin{array}{l} 2l\alpha^2\beta g + 3l\alpha\beta^2f^2 - 3l\alpha\gamma^2f^2 - 2l\gamma\alpha^2hf + m\alpha\beta\gamma gh + m\gamma^2\alpha g^2 - \\ m\alpha^3g^2 - 2m\alpha^2\beta fg + 2n\gamma\alpha^2hf + n\alpha^3h^2 - n\alpha\beta^3h^2 - n\alpha\beta\gamma gh \end{array} \right\} \quad (56).$$

Differentiate, with respect to f , the part within the brackets of (55) in which f appears, and multiply by $(\beta g + \gamma h)$, and we have

$$\left\{ \begin{array}{l} l\alpha\beta^2g^2 + 2l\beta^3fg - 2l\beta\gamma^2fg - l\alpha\beta\gamma gh - m\alpha\beta^2g^2 + n\alpha\beta\gamma gh + \\ l\alpha\beta\gamma gh + 2l\beta^2\gamma hf - 2l\gamma^3hf - l\gamma^2\alpha h^2 - m\alpha\beta\gamma gh + n\gamma^2\alpha h^2 \end{array} \right\} \quad (57).$$

Hence

$$(56) + (57) = \frac{dJ}{df} \quad . \quad . \quad . \quad . \quad . \quad . \quad (58).$$

Now, in this sum, we find in $\frac{dJ}{df}$ —

$$\left. \begin{array}{l} \text{Coefficient of } f^2 = 3l\alpha(\beta^2 - \gamma^2). \\ \text{,, } g^2 = \alpha\{(l - m)\beta^2 + m(\gamma^2 - \alpha^2)\} \\ \text{,, } h^2 = \alpha\{(n - l)\gamma^2 + n(\alpha^2 - \beta^2)\} \\ \text{,, } gh = 0 \\ \text{,, } hf = 2\gamma\{(n - l)\alpha^2 + l(\beta^2 - \gamma^2)\} \\ \text{,, } fg = 2\beta\{(l - m)\alpha^2 + l(\beta^2 - \gamma^2)\} \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (59).$$

And symmetrically for $\frac{dJ}{dg}$ and $\frac{dJ}{dh}$.

Now, it is well known that

$$\frac{dJ}{df} = 0; \quad \frac{dJ}{dg} = 0; \quad \frac{dJ}{dh} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (60).$$

These, with $u = 0$; $v = 0$; $w = 0$, give 6 equations in $f^2, g^2, h^2, gh, hf, fg$: whence we have the following determinant:—

$$\left| \begin{array}{llll} l, \alpha^2, 0, & 3\alpha \cdot l(\beta^2 - \gamma^2), & \beta \cdot (\bar{l} - m)\alpha^2 + l(\beta^2 - \gamma^2), & \gamma \cdot (n - l)\alpha^2 + l(\beta^2 - \gamma^2) \\ m, \beta^2, 0, & \alpha \cdot (\bar{l} - m)\beta^2 + m(\gamma^2 - \alpha^2), & 3\beta \cdot m(\gamma^2 - \alpha^2), & \gamma \cdot (m - n)\beta^2 + m(\gamma^2 - \alpha^2) \\ n, \gamma^2, 0, & \alpha \cdot (n - l)\gamma^2 + n(\alpha^2 - \beta^2), & \beta \cdot (n - l)\gamma^2 + n(\alpha^2 - \beta^2), & 3\gamma \cdot n(\alpha^2 - \beta^2) \\ 0, 2\beta\gamma, \alpha, & 0, & 2\gamma \cdot (m - n)\beta^2 + m(\gamma^2 - \alpha^2), & 2\beta \cdot (n - l)\gamma^2 + n(\alpha^2 - \beta^2) \\ 0, 2\gamma\alpha, \beta, & 2\gamma \cdot (n - l)\alpha^2 + l(\beta^2 - \gamma^2), & 0, & 2\alpha \cdot (n - l)\gamma^2 + n(\alpha^2 - \beta^2) \\ 0, 2\alpha\beta, \gamma, & 2\beta \cdot (\bar{l} - m)\alpha^2 + l(\beta^2 - \gamma^2), & 2\alpha \cdot (\bar{l} - m)\beta^2 + m(\gamma^2 - \alpha^2), & 0 \end{array} \right| = 0 \quad (61).$$

which is the equation of the locus required, and which I have not yet calculated out. (See *Postscript*,—where l^2, m^2, n^2 , are written for l, m, n .)

Hitherto we have proceeded on the supposition of the *faisceaux* being curves of the first order,—that is, straight lines. We shall now suppose them to be conics, and we shall take the four special forms, which for brevity we shall call the circumscribing, the self-conjugate, the tri-tangent, and the bi-tangent conics.

SECTION II.—*The Faisceau of Homology being Circumscribing Conics.*

XIII. (1.) Required the envelope of a *faisceau* of circumscribing conics, while the centre of homology moves in a straight line.

Let the circumscribing conic of homology be

$$\frac{f^s}{\alpha} + \frac{g^s}{\beta} + \frac{h^s}{\gamma} = 0 = u \quad . \quad . \quad . \quad . \quad . \quad (62),$$

and the straight line in which the centre moves

$$l\alpha + m\beta + n\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

$$i.e. \quad \frac{l}{f} + \frac{m}{g} + \frac{n}{h} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (64).$$

Differentiating (62) and (64), introducing an indeterminate coefficient, adding, making the coefficient of the differentials to vanish as before, and substituting in (62) the values thus found for f, g, h , we have, as the envelope required,

$$\left(\frac{l^s}{\alpha}\right)^{\frac{1}{s+1}} + \left(\frac{m^s}{\beta}\right)^{\frac{1}{s+1}} + \left(\frac{n^s}{\gamma}\right)^{\frac{1}{s+1}} = 0 \quad (65).$$

Giving to s the appropriate values, $-2, -3, -\frac{3}{2}$; we find as follows:—

XIV. If the centre of homology move in a straight line, the *faisceau* of circumscribing conics

$$\frac{1}{f^2\alpha} + \frac{1}{g^2\beta} + \frac{1}{h^2\gamma} = 0$$

will envelope the straight line

$$l^2\alpha + m^2\beta + n^2\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad (66).$$

XV. If the centre of homology move in a straight line, the *faisceau* of circumscribing conics

$$\frac{1}{f^3\alpha} + \frac{1}{g^3\beta} + \frac{1}{h^3\gamma} = 0$$

will envelope the tri-tangent conic

$$\pm (l^3\alpha)^{\frac{1}{3}} \pm (m^3\beta)^{\frac{1}{3}} \pm (n^3\gamma)^{\frac{1}{3}} = 0 \quad . \quad . \quad . \quad (67).$$

XVI. If the centre of homology move in a straight line, the *faisceau* of circumscribing conics

$$\frac{1}{f^{\frac{3}{2}}\alpha} + \frac{1}{g^{\frac{3}{2}}\beta} + \frac{1}{h^{\frac{3}{2}}\gamma} = 0$$

will envelope the self-conjugate conic,

$$l^3\alpha^2 + m^3\beta^2 + n^3\gamma^2 = 0 \quad . \quad . \quad . \quad (68).$$

XVII. (2.) Required the envelope of a *faisceau* of circumscribing conics, when the centre of homology moves in a given circumscribing conic.

In this case, instead of (64), we have

$$lf + mg + nh = 0 \quad . \quad . \quad . \quad (69).$$

If, therefore, instead of f, g, h , we take as variables their inverses, and consider s as negative, our new variables will be involved precisely as in (62), and (64) and (65) will represent the envelope we now seek, provided for s we substitute $-s$, which gives

$$(l^s\alpha)^{\frac{1}{s-1}} + (m^s\beta)^{\frac{1}{s-1}} + (n^s\gamma)^{\frac{1}{s-1}} = 0 \quad . \quad . \quad . \quad (70).$$

Therefore, giving to s the appropriate values 2, 3, $\frac{3}{2}$; we find as follows:—

XVIII. If the centre of homology describe a circumscribing conic, the *faisceau* of circumscribing conics

$$\frac{f^2}{\alpha} + \frac{g^2}{\beta} + \frac{h^2}{\gamma} = 0$$

will envelope the straight line

$$l^2\alpha + m^2\beta + n^2\gamma = 0 \quad . \quad . \quad . \quad (71).$$

XIX. If the centre of homology describe a circumscribing conic, the *faisceau* of circumscribing conics

$$\frac{f^3}{\alpha} + \frac{g^3}{\beta} + \frac{h^3}{\gamma} = 0$$

will envelope the tri-tangent conic

$$\pm (l^3\alpha)^{\frac{1}{3}} \pm (m^3\beta)^{\frac{1}{3}} \pm (n^3\gamma)^{\frac{1}{3}} = 0 \quad . \quad . \quad . \quad (72).$$

XX. If the centre of homology describe a circumscribing conic, the *faisceau* of circumscribing conics

$$\frac{f^{\frac{3}{2}}}{\alpha} + \frac{g^{\frac{3}{2}}}{\beta} + \frac{h^{\frac{3}{2}}}{\gamma} = 0$$

will envelope the self-conjugate conic

$$l^3\alpha^2 + m^3\beta^2 + n^3\gamma^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (73).$$

XXI. *Cor.* From the preceding we see, that since the inverse centre of homology describes a circumscribing conic, when the direct centre describes a straight line, and *vice versa*, it follows that the direct and inverse *faisceaux* of circumscribing conics have the same envelope. Compare (66) and (71); (67) and (72); (68) and (73).

XXII. (3.) Required the envelope of a *faisceau* of circumscribing conics, when the centre of homology moves in a given self-conjugate conic.

In this case, instead of (64), we have

$$\frac{l}{f^2} + \frac{m}{g^2} + \frac{n}{h^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (74).$$

If, therefore, instead of f, g, h , we take as variables their squares, and consider s as having *half* its former value, our new variables will again be involved precisely as in (62) and (64), and (65) will represent the envelope we now seek, provided for s we read $\frac{1}{2}s$, which gives

$$\left(\frac{l}{\alpha^2}\right)^{\frac{1}{s+2}} + \left(\frac{m}{\beta^2}\right)^{\frac{1}{s+2}} + \left(\frac{n}{\gamma^2}\right)^{\frac{1}{s+2}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (75).$$

Hence giving s the appropriate values, -4 ; -3 ; -6 : we have as follows:—

XXIII. If the centre of homology move in the self-conjugate conic,

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

the *faisceau* of circumscribing conics

$$\frac{1}{f^4\alpha} + \frac{1}{g^4\beta} + \frac{1}{h^4\gamma} = 0$$

will envelope the straight line

$$l^2\alpha + m^2\beta + n^2\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad (76).$$

XXIV. If the centre of homology move in the same self-conjugate conic, the *faisceau* of circumscribing conics

$$\frac{1}{f^3\alpha} + \frac{1}{g^3\beta} + \frac{1}{h^3\gamma} = 0$$

will envelope another self-conjugate conic, namely,

$$l^3\alpha^2 + m^3\beta^2 + n^3\gamma^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (77).$$

XXV. If the centre of homology move in the same self-conjugate conic, the *faisceau* of circumscribing conics

$$\frac{1}{f^6\alpha} + \frac{1}{g^6\beta} + \frac{1}{h^6\gamma} = 0$$

will envelope the tri-tangent conic

$$\pm \left(\frac{\alpha}{l^s}\right)^{\frac{1}{2}} \pm \left(\frac{\beta}{m^s}\right)^{\frac{1}{2}} \pm \left(\frac{\gamma}{n^s}\right)^{\frac{1}{2}} = 0 \quad (78).$$

XXVI. (4.) Required the envelope of a *faisceau* of circumscribing conics, when the centre of homology moves in a given bi-tangent conic, $k^2\alpha^2 = \beta\gamma$.

Here we have

$$\left. \begin{aligned} \frac{f^s}{\alpha} + \frac{g^s}{\beta} + \frac{h^s}{\gamma} &= 0 = u \\ f^2 - k^2gh &= 0 \end{aligned} \right\} \quad (79).$$

Hence, differentiating, and merging the factor $(s-1)$ in the indeterminate coefficient, which we may suppress, as it divides out in the final substitution, we have

$$f^{s-2} = 2\alpha \quad (1')$$

$$g^{s-1} = -k^2h\beta \quad (2')$$

$$h^{s-1} = -k^2g\gamma \quad (3')$$

$$(2') \times (3') \quad (gh)^{s-2} = k^4\beta\gamma \quad (4')$$

$$(2') \div (3') \quad \left(\frac{g}{h}\right)^s = \frac{\beta}{\gamma} \quad (5')$$

$$(4') \text{ gives } (gh)^s = k^{\frac{4s}{s-2}} (\beta\gamma)^{\frac{s}{s-2}} \quad (6')$$

$$(5') \times (6') \quad g^{2s} = k^{\frac{4s}{s-2}} (\beta\gamma)^{\frac{s}{s-2}} \times \frac{\beta}{\gamma} \quad (7')$$

$$\begin{aligned} \frac{g^{2s}}{\beta^2} &= \frac{k^{\frac{4s}{s-2}} (\beta\gamma)^{\frac{s}{s-2}}}{\beta\gamma} \\ &= k^{\frac{4s}{s-2}} (\beta\gamma)^{\frac{2}{s-2}} \end{aligned} \quad (8')$$

$$\frac{g^s}{\beta} = \frac{h^s}{\gamma} = \pm k^{\frac{2s}{s-2}} (\beta\gamma)^{\frac{1}{s-2}} \quad (9')$$

$$\text{By (1')} \quad \frac{f^s}{\alpha} = 2^{\frac{s}{s-2}} \alpha^{\frac{2}{s-2}} \quad (10')$$

$$\text{By (79), (9'), (10')} \quad 2^{\frac{s}{s-2}} \alpha^{\frac{2}{s-2}} = \pm 2k^{\frac{2s}{s-2}} (\beta\gamma)^{\frac{1}{s-2}}$$

$$\alpha^2 = \left(\frac{k}{2}\right)^2 \beta\gamma \quad (80);$$

the required envelope, which is another bi-tangent conic, very neatly related to the former.

XXVII. Required the locus of the fourth point of intersection of the direct

and inverse circumscribing conics of homology, when the centre of homology describes a given circumscribing conic.

[It may be noted here, that the direct and inverse circumscribing conics of homology are two members of a *faisceau* of *courbes pivotantes*, passing through the $2^2 = 4$ points, namely, the three angles of the triangle and the point whose locus is now sought. If the direct circumscribing conic is

$$\frac{f}{\alpha} + \frac{g}{\beta} + \frac{h}{\gamma} = 0 \quad (81),$$

the inverse is

$$\frac{1}{f\alpha} + \frac{1}{g\beta} + \frac{1}{h\gamma} = 0 \quad (82);$$

and any member of the *faisceau pivotant* to which they belong will be represented by

$$\frac{f + \frac{\lambda}{f}}{\alpha} + \frac{g + \frac{\lambda}{g}}{\beta} + \frac{h + \frac{\lambda}{h}}{\gamma} = 0;$$

where λ may have any value from 0 to $\pm \infty$, the direct conic being represented when $\lambda = 0$, and the inverse conic when $\lambda = \pm \infty$.

To find the locus required, we have to eliminate f, g, h , between (81) and (82) and

$$lf + mg + nh = 0 \quad (83).$$

Now, these variables, the parameters, are involved in these three equations precisely as in (42), (43), (44); and the two sets of equations are identical, if for α, β, γ we read their inverses. Hence inverting α, β, γ in (47), we get the locus required, namely,

$$mn\left(\frac{\beta\gamma}{\alpha^2} - \frac{\beta^2 + \gamma^2}{\beta\gamma}\right) + nl\left(\frac{\gamma\alpha}{\beta^2} - \frac{\gamma^2 + \alpha^2}{\gamma\alpha}\right) + lm\left(\frac{\alpha\beta}{\gamma^2} - \frac{\alpha^2 + \beta^2}{\alpha\beta}\right) + l^2 + m^2 + n^2 = 0 \quad (84);$$

a curve of the 6th order, and which circumscribes the triangle.

XXVIII. Required the locus of the same point when the centre of homology moves in the bi-tangent conic $k^2\alpha^2 = \beta\gamma$. Here again the elimination is the same, as in (48') and α, β, γ have to be inverted in (49). This gives

$$\alpha^2\left(\frac{\beta^2 - \gamma^2}{\beta^3\gamma^3}\right)^2 - \frac{\beta^2 + \gamma^2}{\beta\gamma} + k^2 + \frac{1}{k^2} = 0 \quad (85),$$

and which is the same curve if the bi-tangent conic were not $k^2\alpha^2 = \beta\gamma$, but $\alpha^2 = k^2\beta\gamma$.

XXIX. Required the locus of the same point when the centre of homology moves in a self-conjugate conic.

The same consideration as that used in the last two propositions shows that this locus is to be got by inverting α, β, γ in the determinant (61).

SECTION III.—*The Faisceau of Homology being Self-Conjugate Conics.*

XXX. (1.) Required the envelope of a *faisceau* of self-conjugate conics, when the centre of homology moves in a straight line.

Here we have

$$f^s \alpha^2 + g^s \beta^2 + h^s \gamma^2 = 0 = u \quad . \quad . \quad . \quad . \quad . \quad (86),$$

and

$$\frac{l}{f} + \frac{m}{g} + \frac{n}{h} = 0 \quad . \quad . \quad . \quad . \quad . \quad (87).$$

Proceeding as before, the envelope is

$$(l^s \alpha^2)^{\frac{1}{s+1}} + (m^s \beta^2)^{\frac{1}{s+1}} + (n^s \gamma^2)^{\frac{1}{s+1}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (88).$$

Giving s appropriate values, namely, 1; -3 ; 3: we have as follows:—

XXXI. If the centre of homology move in a straight line, the *faisceau* of self-conjugate conics

$$f\alpha^2 + g\beta^2 + h\gamma^2 = 0$$

will envelope, or inscribe, the quadrilateral represented by

$$\pm (l)^{\frac{1}{2}} \alpha \pm (m)^{\frac{1}{2}} \beta \pm (n)^{\frac{1}{2}} \gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad (89).$$

XXXII. If the centre of homology move in a straight line, the *faisceau* of self-conjugate conics

$$\frac{\alpha^2}{f^3} + \frac{\beta^2}{g^3} + \frac{\gamma^2}{h^3} = 0$$

will envelope the circumscribing conic,

$$\frac{l^{\frac{2}{3}}}{\alpha} + \frac{m^{\frac{2}{3}}}{\beta} + \frac{n^{\frac{2}{3}}}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad (90).$$

XXXIII. If the centre of homology move in a straight line, the *faisceau* of self-conjugate conics

$$f^3 \alpha^2 + g^3 \beta^2 + h^3 \gamma^2 = 0$$

will envelope the tri-tangent conic

$$\pm (l^{\frac{3}{2}} \alpha)^{\frac{1}{2}} \pm (m^{\frac{3}{2}} \beta)^{\frac{1}{2}} \pm (n^{\frac{3}{2}} \gamma)^{\frac{1}{2}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (91).$$

XXXIV. (2.) Required the envelope of a *faisceau* of self-conjugate conics when the centre of homology moves in a circumscribing conic.

On the same consideration as before, s in (88) must be read $-s$, which gives the required envelope

$$\left(\frac{l^s}{\alpha^2}\right)^{\frac{1}{s-1}} + \left(\frac{m^s}{\beta^2}\right)^{\frac{1}{s-1}} + \left(\frac{n^s}{\gamma^2}\right)^{\frac{1}{s-1}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (92).$$

$$\pm \sqrt{l}\alpha \pm \sqrt{m}\beta \pm \sqrt{n}\gamma = 0 \quad (97).$$

XXI. If the centre of homology move in a self-conjugate conic, the *faisceau* of self-conjugate conics

$$\frac{\alpha^2}{f^6} + \frac{\beta^2}{g^6} + \frac{\gamma^2}{h^6} = 0$$

will envelope the circumscribing conic

$$\frac{l^{\frac{3}{2}}}{\alpha} + \frac{m^{\frac{3}{2}}}{\beta} + \frac{n^{\frac{3}{2}}}{\gamma} = 0 \quad (98).$$

XLII. If the centre of homology move in a self-conjugate conic, the *faisceau* of self-conjugate conics

$$f^6 \alpha^2 + g^6 \beta^2 + h^6 \gamma^2 = 0$$

will envelope the tri-tangent conic

$$\pm \left(l^{\frac{3}{2}} \alpha \right)^{\frac{1}{2}} \pm \left(m^{\frac{3}{2}} \beta \right)^{\frac{1}{2}} \pm \left(n^{\frac{3}{2}} \gamma \right)^{\frac{1}{2}} = 0 \quad (99).$$

XLIII. (4.) Required the envelope of a *faisceau* of self-conjugate conics when the centre moves in the bi-tangent conic

$$k^2 \alpha^2 = \beta \gamma.$$

Here we have

$$f^s \alpha^2 + g^s \beta^2 + h^s \gamma^2 = 0 = u.$$

and

$$f^2 - k^2 gh = 0.$$

Differentiating as before, suppressing the factor $(s-1)$ as merged in the indeterminate coefficient which is suppressed because it divides out in the final substitution, we have

$$f^{s-1} \alpha^2 = 2f \therefore f^s \alpha^2 = \frac{2^{s-2}}{\alpha^{\frac{4}{s-2}}} \quad [1]$$

$$g^{s-1} \beta^2 = -k^2 h$$

$$h^{s-1} \gamma^2 = -k^2 g$$

$$\therefore (gh)^{s-2} = \frac{k^4}{(\beta\gamma)^2} \therefore (gh)^{\frac{s}{2}} = \pm \left(\frac{k^2}{\beta\gamma} \right)^{\frac{s}{s-2}} \quad [2]$$

$$\text{and} \quad \left(\frac{g}{h} \right)^{\frac{s}{2}} = \pm \frac{\gamma}{\beta} \quad [3]$$

$$\text{and} \quad g^s \beta^2 = h^s \gamma^2 \quad [4]$$

$$[2] \times [3] \quad g^s = (k^2)^{\frac{s}{s-2}} \cdot (\beta\gamma)^{-\frac{s}{s-2}} \cdot \frac{\gamma}{\beta}$$

$$\therefore g^s \beta^2 = (k^2)^{\frac{s}{s-2}} \cdot (\beta\gamma)^{-\frac{2}{s-2}}$$

$$\text{Substituting in } u; \quad \frac{2^{s-2}}{\alpha^{\frac{4}{s-2}}} = -2 \cdot \left(\frac{k^s}{\beta\gamma} \right)^{\frac{2}{s-2}}$$

$$\therefore \frac{2^s}{\alpha^4} = \pm 2^{s-2} \cdot \left(\frac{k^s}{\beta\gamma} \right)^2 : \quad \pm \text{ as } s \text{ is even or odd.}$$

If s is even $k^s \alpha^2 = 2\beta\gamma : -$ the envelope required.

XLIV. It is evident that if the centre move in this last bi-tangent, the envelope will be

$$k^{\frac{s^2}{2}} a^2 = 2^{\frac{s+2}{2}} \beta \gamma.$$

If, again, in this, the envelope will be

$$k^{\frac{s^3}{4}} a^2 = 2^{\frac{s^2+2s+4}{4}} \beta \gamma.$$

And so on *en suite*; the law being obvious, namely, that the t^{th} envelope will be

$$k^{\frac{s^t}{2^{t-1}}} a^2 = \left(2 \cdot \frac{s^{t-1} + 2s^{t-2} + 4s^{t-3} + 8s^{t-4} + \&c.}{2^{t-1}} \right) \beta \gamma \quad (100).$$

or

$$\frac{k^{\frac{2 \cdot (\frac{s}{2})^t}{1 - (\frac{s}{2})^t}}}{2^{\frac{1 - \frac{s}{2}}{2}}} a^2 = \beta \gamma \quad (100').$$

—a somewhat curious result.

SECTION IV.—*The Faisceau of Homology being Tri-tangent Conics.*

XLV. (1.) Required the envelope of a *faisceau* of tri-tangent conics when the centre of homology moves in a straight line.

Here we have

$$\left. \begin{aligned} \pm f^s \cdot \alpha^{\frac{1}{2}} \pm g^s \beta^{\frac{1}{2}} \pm h^s \gamma^{\frac{1}{2}} &= 0 \\ \frac{l}{f} + \frac{m}{g} + \frac{n}{h} &= 0 \end{aligned} \right\} \quad (101).$$

and the envelope, found as before, is

$$\pm (l^{2s} \cdot \alpha)^{\frac{1}{2(s+1)}} \pm (m^{2s} \cdot \beta)^{\frac{1}{2(s+1)}} \pm (n^{2s} \gamma)^{\frac{1}{2(s+1)}} = 0 \quad (101').$$

Hence, giving to s the appropriate values, namely, $-\frac{1}{2}$; $-\frac{3}{2}$; $-\frac{5}{2}$: we find as follows:—

XLVI. If the centre of homology move in a straight line, the *faisceau* of tri-tangent conics

$$\pm \sqrt{\frac{\alpha}{f}} \pm \sqrt{\frac{\beta}{g}} \pm \sqrt{\frac{\gamma}{h}} = 0$$

will envelope the straight line

$$\pm \frac{\alpha}{l} \pm \frac{\beta}{m} \pm \frac{\gamma}{n} = 0 \quad (102).$$

XLVII. If the centre of homology move in a straight line, the *faisceau* of tri-tangent conics

$$\pm \sqrt{\frac{\alpha}{f^3}} \pm \sqrt{\frac{\beta}{g^3}} \pm \sqrt{\frac{\gamma}{h^3}} = 0$$

will envelope the circumscribing conic

$$\frac{l^3}{\alpha} + \frac{m^3}{\beta} + \frac{n^3}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad (103).$$

XLVIII. If the centre of homology move in a straight line, the *faisceau* of tri-tangent conics

$$\pm \frac{\alpha^{\frac{1}{2}}}{f^{\frac{1}{2}}} \pm \frac{\beta^{\frac{1}{2}}}{g^{\frac{1}{2}}} \pm \frac{\gamma^{\frac{1}{2}}}{h^{\frac{1}{2}}} = 0$$

will envelope the self-conjugate conic

$$\frac{\alpha^2}{l^3} + \frac{\beta^2}{m^3} + \frac{\gamma^2}{n^3} = 0 \quad . \quad . \quad . \quad . \quad . \quad (104).$$

XLIX. In general; in order that the envelope may be of the form

$$\lambda \alpha^t + \mu \beta^t + \nu \gamma^t = 0$$

we must evidently have

$$\frac{1}{2s+2} = t \text{ or } s = \frac{1-2t}{2t}$$

And the *faisceau*

$$\pm f^{\frac{1-2t}{2t}} \alpha^{\frac{1}{2}} \pm g^{\frac{1-2t}{2t}} \beta^{\frac{1}{2}} \pm h^{\frac{1-2t}{2t}} \gamma^{\frac{1}{2}} = 0$$

gives the envelope

$$\pm l^{1-2t} \alpha^t \pm m^{1-2t} \beta^t \pm n^{1-2t} \gamma^t = 0 \quad . \quad . \quad . \quad . \quad . \quad (105).$$

L. (2.) Required the envelope of a *faisceau* of tri-tangent conics when the centre of homology moves in a conic circumscribing the triangle.

Here, by the same consideration as we have already employed in like cases, the envelope required is got from (101') by reading $-s$ for s , namely,

$$\pm \left(\frac{l^{2s}}{\alpha}\right)^{\frac{1}{2(s-1)}} \pm \left(\frac{m^{2s}}{\beta}\right)^{\frac{1}{2(s-1)}} \pm \left(\frac{n^{2s}}{\gamma}\right)^{\frac{1}{2(s-1)}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (106).$$

Hence, giving s the appropriate values $\frac{1}{2}$; $\frac{3}{2}$; $\frac{5}{2}$: we find as follows:—

LI. If the centre of homology move in a circumscribing conic, the *faisceau* of tri-tangent conics

$$\pm \sqrt{f\alpha} \pm \sqrt{g\beta} \pm \sqrt{h\gamma} = 0$$

will envelope the straight line

$$\pm \frac{\alpha}{l} \pm \frac{\beta}{m} \pm \frac{\gamma}{n} = 0 \quad . \quad . \quad . \quad . \quad . \quad (107).$$

LII. If the centre of homology move in a circumscribing conic, the *faisceau* of tri-tangent conics

$$\pm \sqrt{f^3\alpha} \pm \sqrt{g^3\beta} \pm \sqrt{h^3\gamma} = 0$$

will envelope the circumscribing conic

$$\frac{l^3}{\alpha} + \frac{m^3}{\beta} + \frac{n^3}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad (108).$$

LIII. If the centre of homology move in a circumscribing conic, the *faisceau* of tri-tangent conics

$$\pm \sqrt{f^{\frac{2}{3}}\alpha} \pm \sqrt{g^{\frac{2}{3}}\beta} \pm \sqrt{h^{\frac{2}{3}}\gamma} = 0$$

will envelope the self-conjugate conic

$$\frac{\alpha^2}{l^3} + \frac{\beta^2}{m^3} + \frac{\gamma^2}{n^3} = 0 \quad . \quad . \quad . \quad . \quad . \quad (109).$$

LIV. (3.) Required the envelope of a *faisceau* of tri-tangent conics when the centre of homology moves also in a given tri-tangent conic, say,

$$\pm l\alpha^{\frac{1}{3}} \pm m\beta^{\frac{1}{3}} \pm n\gamma^{\frac{1}{3}} = 0.$$

Here we have

$$\left. \begin{aligned} &\pm f^{\frac{1}{3}}\alpha^{\frac{1}{3}} \pm g^{\frac{1}{3}}\beta^{\frac{1}{3}} \pm h^{\frac{1}{3}}\gamma^{\frac{1}{3}} = 0 \\ \text{and} \quad &\pm \frac{l}{f^{\frac{1}{3}}} \pm \frac{m}{g^{\frac{1}{3}}} \pm \frac{n}{h^{\frac{1}{3}}} = 0 \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad (110).$$

Now, $f^{\frac{1}{3}}, g^{\frac{1}{3}}, h^{\frac{1}{3}}$ are involved in equations (110) precisely as f, g, h are involved in equations (101), provided in equations 101 we substitute $2s$ for s . Making, therefore, this substitution in (101'), we have the envelope now required, namely,

$$\pm (l^{4s}\alpha)^{\frac{1}{2(2s+1)}} \pm (m^{4s}\beta)^{\frac{1}{2(2s+1)}} \pm (n^{4s}\gamma)^{\frac{1}{2(2s+1)}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (111).$$

Hence, giving s the appropriate values, $-\frac{1}{4}$; $-\frac{3}{4}$; $-\frac{5}{4}$: we find as follows:—

LV. If the centre of homology move in a tri-tangent conic, the *faisceau* of tri-tangent conics

$$\pm \sqrt{\frac{\alpha}{f^{\frac{1}{3}}}} \pm \sqrt{\frac{\beta}{g^{\frac{1}{3}}}} \pm \sqrt{\frac{\gamma}{h^{\frac{1}{3}}}} = 0$$

will envelope the straight line

$$\pm \frac{\alpha}{l} \pm \frac{\beta}{m} \pm \frac{\gamma}{n} = 0 \quad . \quad . \quad . \quad . \quad . \quad (112).$$

LVI. If the centre of homology move in a tri-tangent conic, the *faisceau* of tri-tangent conics

$$\pm \sqrt{\frac{\alpha}{f^{\frac{2}{3}}}} \pm \sqrt{\frac{\beta}{g^{\frac{2}{3}}}} \pm \sqrt{\frac{\gamma}{h^{\frac{2}{3}}}} = 0$$

will envelope the circumscribing conic

$$\pm \frac{l^3}{\alpha} \pm \frac{m^3}{\beta} \pm \frac{n^3}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad (113).$$

LVII. If the centre of homology move in a tri-tangent conic, the *faisceau* of tri-tangent conics

$$\pm \sqrt{\frac{\alpha}{f^{\frac{1}{3}}}} \pm \sqrt{\frac{\beta}{g^{\frac{1}{3}}}} \pm \sqrt{\frac{\gamma}{h^{\frac{1}{3}}}} = 0$$

will envelope the self-conjugate conic

$$\pm \frac{\alpha^2}{l^3} \pm \frac{\beta^2}{m^3} \pm \frac{\gamma^2}{n^3} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (114).$$

LVIII. (4.) Required the envelope of a *faisceau* of tri-tangent conics when the centre of homology moves in a bi-tangent conic, say

$$k^2 \alpha^2 = \beta \gamma.$$

Here we have

$$\pm f^s \alpha^{\frac{1}{s}} \pm g^s \beta^{\frac{1}{s}} \pm h^s \gamma^{\frac{1}{s}} = 0,$$

and

$$f^2 - k^2 g h = 0.$$

Proceeding as before, the envelope is found to be

$$\left(\frac{k^s}{2}\right)^4 \alpha^2 = \beta \gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (115).$$

another bi-tangent elegantly related to the former.

LIX. If the centre of homology move in this bi-tangent, the bi-tangent envelope will of course be

$$\left(\frac{k^{2s^2}}{2^{2s+1}}\right)^4 \alpha^2 = \beta \gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (116).$$

If in this, the envelope will be

$$\left(\frac{k^{4s^3}}{2^{4s^2+2s+1}}\right)^4 \alpha^2 = \beta \gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (117).$$

If in this, the envelope will be

$$\left(\frac{k^{8s^4}}{2^{8s^3+4s^2+2s+1}}\right)^4 \alpha^2 = \beta \gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (118).$$

And so on, *en suite*, the law being obvious,—the t^{th} envelope being

$$\left(\frac{k^{\frac{(2s)^t}{2}}}{2^{\frac{1-(2s)^t}{1-2s}}}\right)^4 \alpha^2 = \beta \gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (118').$$

LX. (5.) Required the envelope of a *faisceau* of tri-tangent conics when the centre of homology moves in a self-conjugate conic, say $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$.

By a similar consideration to that used in LIV., in (101') for s read $\frac{1}{2}s$, and we have the envelope required, namely

$$\pm (l^s \alpha)^{\frac{1}{s+2}} \pm (m^s \beta)^{\frac{1}{s+2}} \pm (n^s \gamma)^{\frac{1}{s+2}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (119).$$

Hence, giving to s appropriate values, namely, -1 ; $-\frac{3}{2}$; -3 : we find as follows:—

$$\alpha^2 = \left(\frac{l^2}{4mn} \right)^{\frac{s}{s+2}} \beta \gamma \quad (126).$$

LXVII. (4.) Required the envelope of a *faisceau* of bi-tangent conics when the centre of homology moves in a tri-tangent conic. The envelope is

$$\alpha^2 = \left(\frac{4mn}{l^2} \right)^{\frac{2s}{2s+1}} \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad (127).$$

LXVIII. (5.) Required the envelope of a *faisceau* of bi-tangent conics when the centre of homology moves in the bi-tangent conic

$$\alpha^2 = k^2 \beta\gamma.$$

The envelope is

$$\alpha^2 = k^{2s} \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (128).$$

If the centre of homology move in this conic, the envelope is

$$\alpha^2 = k^{2s^2} \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (129).$$

If in this, the envelope is

$$\alpha^2 = k^{2s^3} \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (130).$$

And the t^{th} envelope is

$$\alpha^2 = k^{2s^t} \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (131).$$

If we equate the indices (inverted when necessary) of (124) and (126); (125) and (126); (124) and (127); (125) and (127); (126) and (127); we get the following somewhat elegant propositions:—

LXIX. Whether the centre moves in a straight line or in a self-conjugate conic, the *faisceau* of bi-tangent conics

$$(gh)^3 \alpha^2 = f^6 \beta\gamma$$

envelopes the same bi-tangent conic

$$(4mn)^3 \alpha^2 = l^6 \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (132).$$

LXX. Whether the centre moves in a circumscribing or in a self-conjugate conic, the *faisceau* of bi-tangent conics

$$gh \alpha^2 = f^2 \beta\gamma$$

envelopes the same bi-tangent conic

$$l^2 \alpha^2 = 4mn \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (133).$$

LXXI. Whether the centre moves in a straight line or in a tri-tangent conic, the *faisceau* of bi-tangent conics

$$f \alpha^2 = \sqrt{gh} \cdot \beta\gamma$$

envelopes the same bi-tangent conic

$$l \alpha^2 = 2\sqrt{mn} \cdot \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (134).$$

LXXII. Whether the centre moves in a circumscribing conic or in a tri-tangent conic, the *faisceau* of bi-tangent conics

$$(gh)^{\frac{2}{3}} \alpha^2 = f^3 \beta \gamma$$

envelopes the same bi-tangent conic

$$l^3 \alpha^2 = (4mn)^{\frac{2}{3}} \beta \gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (135).$$

LXXIII. Whether the centre of homology moves in a self-conjugate or a tri-tangent conic, the *faisceau* of bi-tangent conics

$$(gh)^{\frac{2}{3}} \alpha^2 = f^{\frac{2}{3}} \beta \gamma$$

envelopes the same bi-tangent conic

$$(l^2)^{\frac{2}{3}} \alpha^2 = (4mn)^{\frac{2}{3}} \beta \gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (136).$$

I have only to add, that while the general problem of this paper becomes very difficult in the case of the general equation of the second degree, by reason of the complicated elimination that is necessary, I have no doubt that this difficulty might be evaded by reducing the general equation to one or other of the forms of the conic which have engaged our attention, by an appropriate transformation of the triangle of reference. If leisure permit, I should be glad to verify this supposition, as well as give certain extensions of the theory, in another paper.

Postscript.—While this is passing through the press, I find that SILVESTER'S "Dialytic method of Elimination as applied to a Ternary System of Equations" (see "Cambridge Mathematical Journal," vol. ii. p. 234), may be used more efficiently than the Jacobian in the last problem of Section I. Writing the equations (50), (52), (51), thus:—

$$0 = l^2 \cdot f^2 + m^2 \cdot g^2 + n^2 \cdot h^2 + \begin{matrix} * & * & * \end{matrix} \quad . \quad . \quad [1],$$

$$0 = \begin{matrix} * & * & * \end{matrix} + \alpha \cdot gh + \beta \cdot hf + \gamma \cdot fg \quad . \quad . \quad [2],$$

$$0 = \alpha \cdot f + \beta \cdot g + \gamma \cdot h \quad . \quad . \quad [3];$$

we have to find four equations in $f^2, g^2, h^2, gh, hf, fg$. Re-write the three just given (taking twice the second) thus:—

$$0 = (l^2 f) \cdot f + (m^2 g) \cdot g + (n^2 h) \cdot h \quad . \quad . \quad [4],$$

$$0 = (\beta h + \gamma g) \cdot f + (\gamma f + \alpha h) \cdot g + (\alpha g + \beta f) \cdot h \quad . \quad . \quad [5],$$

$$0 = (\alpha) \cdot f + (\beta) \cdot g + (\gamma) \cdot h \quad . \quad . \quad [6].$$

Eliminating f, g, h , we have, dropping the parentheses,

$$\begin{vmatrix} l^2 f & m^2 g & n^2 h \\ \beta h + \gamma g & \gamma f + \alpha h & \alpha g + \beta f \\ \alpha & \beta & \gamma \end{vmatrix} = 0 \quad . \quad . \quad . \quad [7];$$

that is:—

$$0 = l^2(\beta^2 - \gamma^2) \cdot f^2 + m^2(\gamma^2 - \alpha^2) \cdot g^2 + n^2(\alpha^2 - \beta^2) \cdot h^2 + (m^2 - n^2)\beta\gamma \cdot gh + (n^2 - l^2)\gamma\alpha \cdot hf + (l^2 - m^2)\alpha\beta \cdot fg \quad [8].$$

Multiplying [3] successively by f, g, h , we have

$$0 = \alpha \cdot f^2 \quad * \quad * \quad * \quad + \gamma \cdot hf \quad + \beta \cdot fg \quad . \quad . \quad [9];$$

$$0 = \quad * \quad \beta \cdot g^2 \quad + \quad \gamma \cdot gh \quad + \quad \alpha \cdot fg \quad . \quad . \quad [10];$$

$$0 = \quad * \quad * \quad \gamma \cdot h^2 + \beta \cdot gh + \alpha \cdot hf \quad * \quad . \quad . \quad [11].$$

Forming now the determinant of elimination from equations [1], [2], [8], [9], [10], [11], we have, cleared of extraneous factors, the equation of the locus sought, namely,

$$\begin{vmatrix} l^2 & 0 & l^2(\beta^2 - \gamma^2) & \alpha & 0 & 0 \\ m^2 & 0 & m^2(\gamma^2 - \alpha^2) & 0 & \beta & 0 \\ n^2 & 0 & n^2(\alpha^2 - \beta^2) & 0 & 0 & \gamma \\ 0 & \alpha & (m^2 - n^2)\beta\gamma & 0 & \gamma & \beta \\ 0 & \beta & (n^2 - l^2)\gamma\alpha & \gamma & 0 & \alpha \\ 0 & \gamma & (l^2 - m^2)\alpha\beta & \beta & \alpha & 0 \end{vmatrix} = 0,$$

which is, when expanded, the very symmetrical curve,

$$\left(\frac{\beta^2 + \gamma^2 - \alpha^2}{l\beta\gamma}\right)^2 + \left(\frac{\gamma^2 + \alpha^2 - \beta^2}{m\gamma\alpha}\right)^2 + \left(\frac{\alpha^2 + \beta^2 - \gamma^2}{n\alpha\beta}\right)^2 = 4\left(\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2}\right) - \left(\frac{l^2 + m^2 + n^2}{lmn}\right)^2.$$

It seems scarcely necessary to add that in equation [1] or (50), the curve in which the *inverse* centre moves, is *legitimately* taken, rather than that in which the centre itself moves, since in the latter case the other two equations change places, and the same elimination has to be effected, the variables being merely inverted.