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Notes on Concomitants of Binary Quantics. By E. B. ELLIOTT.

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1. Too little attention has, perhaps, been devoted by expositors of the English method to any concomitants of binary quantics except covariants and invariants; the fact that the connexion between cogredience and contragredience in the case of pairs of variables is of a very simple nature, and that, for instance, any homogeneous contravariant is made a covariant by a mere substitution for the variables, having been regarded as making the consideration of other concomitants than covariants a matter of secondary interest and importance. Thus, while in respect to a *ternary* quantic it has been made a fundamental

consideration that the system of leading coefficients in concomitants pure and mixed is the system of invariants of an auxiliary binary system, and that the system of leading coefficients in covariants in particular is to be extracted from that system of invariants by means of an additional fact of annihilation by a differential operator, it has not, as it seems to me, been adequately enforced that the answer to the inquiry: What gradients in the coefficients lead concomitants pure and mixed of a *binary* quantic? is: *All gradients*. The elucidation and examination of this fact occupy Section I. of my notes.

Again, it has, perhaps, when noticed, been generally regarded as a mere coincidence that the whole number of seminvariants (including invariants) of any degree i of a binary p -ic is also the number of terms in the most general gradient of degree i and of weight equal to whichever is integral of $\frac{1}{2}ip$ and $\frac{1}{2}(ip-1)$; the fact being that the most general gradient in question, with arbitrary coefficients, may be separated into parts which are coefficients in the covariants led by all those seminvariants, and so have a one-to-one correspondence with them. Facts like this occupy my Sections II. and III. The separation into critical parts of general gradients is effected in Section II. by means of repeated differential operations; and, from any gradient with its full number of arbitraries, an analytical formula is exhibited for that same full number of covariants in which parts of the gradient are coefficients. In Section III. an almost purely algebraical method of effecting the separation is exhibited. The investigations may be regarded as complementary to a theorem of separability given and applied by Hilbert (*Mathematische Annalen*, Vol. xxx., pp. 15, &c.).

A good deal of preliminary exposition of known theory has been unavoidable. The so-called "English," or non-symbolic, method has been followed throughout; the aim of the paper distinctly being to call attention to some neglected lessons of that method and to pursue its development.

I.

2.* Let ξ, η be contragredient with x, y . Take K a rational integral function of $x, y; \xi, \eta; a_0, a_1, a_2, \dots, a_p$, which is a concomitant of

$$u \equiv (a_0, a_1, a_2, \dots, a_p)(x, y)^p.$$

* At the suggestion of a referee much of this article has been rewritten, so as to base what follows entirely on the infinitesimal transformation. At his request I have also slightly altered the prefatory article.

The essential properties of K can be written down by expressing its invariancy, but for a factor, when we apply any infinitesimal—and therefore when we apply any—linear transformation. If $1+\lambda$, μ and λ' , $1+\mu'$, where λ , μ , λ' , μ' are infinitesimal, be the coefficients of the new variables X , Y in the equations of transformation, we express this invariancy by saying that, when we substitute in K

$$(1-\lambda)x - \mu y, \quad -\lambda'x + (1-\mu')y \quad \text{for } x, y,$$

$$(1+\lambda)\xi + \lambda'\eta, \quad \mu\xi + (1+\mu')\eta \quad \text{for } \xi, \eta,$$

$$\text{and} \quad a_r + \mu r a_{r-1} + \lambda' (p-r) a_{r+1} + \{\lambda (p-r) + \mu' r\} a_r \quad \text{for } a_r \\ (r = 0, 1, 2, \dots, p),$$

we reproduce K multiplied by a factor which is, to the first order, of the form $1 + b\lambda + b'\lambda' + c\mu + c'\mu'$, where b , b' , c , c' are numerical. The necessary and sufficient conditions which follow, the infinitesimals being independent, are, from the coefficients of μ , λ' , λ , μ' respectively, that

$$\left(\Omega + \xi \frac{\partial}{\partial \eta} - y \frac{\partial}{\partial x} \right) K = cK,$$

$$\left(O + \eta \frac{\partial}{\partial \xi} - x \frac{\partial}{\partial y} \right) K = b'K,$$

$$\left\{ p a_0 \frac{\partial}{\partial a_0} + (p-1) a_1 \frac{\partial}{\partial a_1} + \dots + a_{p-1} \frac{\partial}{\partial a_{p-1}} - x \frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi} \right\} K = bK,$$

$$\left\{ a_1 \frac{\partial}{\partial a_1} + 2a_2 \frac{\partial}{\partial a_2} + \dots + p a_p \frac{\partial}{\partial a_p} - y \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial \eta} \right\} K = c'K,$$

where, in Sylvester's notation which I have used elsewhere,

$$\Omega \equiv a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + \dots + p a_{p-1} \frac{\partial}{\partial a_p},$$

$$O \equiv p a_1 \frac{\partial}{\partial a_0} + (p-1) a_2 \frac{\partial}{\partial a_1} + \dots + a_p \frac{\partial}{\partial a_{p-1}}.$$

The first two conditions can only be satisfied by taking $c = 0$, $b' = 0$, as the terms of highest weight in K (regarding $x : y$ and $\eta : \xi$ as of weight 1) cannot occur in the left-hand side of the first, or the terms of lowest weight in K in the left-hand side of the second. Thus

$$\Omega + \xi \frac{\partial}{\partial \eta} - y \frac{\partial}{\partial x} \tag{1}$$

and
$$O + \eta \frac{\partial}{\partial \xi} - x \frac{\partial}{\partial y} \quad (2)$$

annihilate K . Hence also their alternant

$$\begin{aligned} & \left(\Omega + \xi \frac{\partial}{\partial \eta} - y \frac{\partial}{\partial x} \right) \left(O + \eta \frac{\partial}{\partial \xi} - x \frac{\partial}{\partial y} \right) \\ & - \left(O + \eta \frac{\partial}{\partial \xi} - x \frac{\partial}{\partial y} \right) \left(\Omega + \xi \frac{\partial}{\partial \eta} - y \frac{\partial}{\partial x} \right), \end{aligned}$$

i.e., the operator

$$\begin{aligned} & \left\{ p a_0 \frac{\partial}{\partial a_0} + (p-2) a_1 \frac{\partial}{\partial a_1} + \dots - p a_p \frac{\partial}{\partial a_p} \right\} \\ & + \left(\xi \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta} \right) - \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right), \quad (3) \end{aligned}$$

annihilates K . But this is the difference of the two operators whose effects on K as above are to multiply it by b and c' respectively. Hence $c' = b$; and the factor in the expression of invariancy for the infinitesimal transformation is $1 + b(\lambda + \mu')$, i.e., is a power of the modulus $1 + \lambda + \mu'$. As the product of the determinants of a succession of linear substitutions is the determinant of the resultant substitution, the factor is still the b th power of the modulus in the expression of invariancy for any finite linear transformation.

Thus the necessary and sufficient conditions that K be a concomitant are that it be annihilated by (1) and (2), and therefore by (3)—a group of three operators of infinitesimal transformation—and that

$$a_1 \frac{\partial}{\partial a_1} + 2a_2 \frac{\partial}{\partial a_2} + \dots + p a_p \frac{\partial}{\partial a_p} - y \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial \eta} \quad (4)$$

have the effect on it of a numerical multiplier b .

Let $C_{r,\rho,\sigma}$ be the coefficient of $x^r y^\rho \xi^r \eta^\sigma$ in K . The annihilation by (3) tells us that the degree i and weight w of every term in $C_{r,\rho,\sigma}$ are connected with the indices by the relation

$$ip - 2w + \rho - \sigma - r + s = 0, \quad (5)$$

and the fact as to (4) tells us that

$$w + \sigma - s = b = \text{const.} \quad (6)$$

In particular if, as we shall see must be the case, there is a part of K free from y and η , then, for every term $C_{i\omega} x^w \xi^{\omega'}$ of such part,

$$\begin{aligned} w &= b \\ &= ip - w + \omega' - \omega \\ &= \frac{1}{2}(ip + \omega' - \omega). \end{aligned} \quad (7)$$

This constant is, we have seen, the index of the power of the modulus which occurs in the general expression of invariancy of K .

We now introduce appropriate conventions in supplement to the above necessities. Denote by degree, order, and class respectively dimensions in $a_0, a_1, a_2, \dots a_p$, in x, y , and in ξ, η . Linear transformation leaves degree, order, and class unaltered. If K contains parts of different degrees, orders, and classes, the transformed parts have in each case their old degree, order, and class. The property of invariancy possessed by K must be possessed by all those parts separately. Though, then, the only limitation on freedom of degree, order, and class which must perforce hold throughout K is

$$p \text{ times degree} + \text{class} - \text{order} = \text{const.}, \quad (8)$$

which is obtained from (5) and (6) by elimination of w , we introduce no real incompleteness of systems of concomitants by conventionally preferring to speak only of concomitants which are of one degree, one order, and one class throughout as simple concomitants, regarding more complex concomitants with the same constant in (8) as linear functions of simple concomitants with the same power of the modulus required in their expressions of invariancy. Henceforward by concomitant we mean, as a rule, "simple" concomitant.

3. Take then a concomitant

$$K \equiv C x^w \xi^{\omega'} + \dots \equiv (O, \dots)(x, y)^w (\xi, \eta)^{\omega'}, \quad (9)$$

which is of one degree i , order ω , and class ω' throughout.

The leading coefficient O cannot be wanting. For the annihilation of K by (1) tells us among other things that the terms of highest degree in ξ are free from η , and the annihilation by (2) tells us that these terms are annihilated by $O - x \frac{\partial}{\partial y}$, so that the part of them which is of highest dimensions in x is free from y . Thus $O \neq 0$.

Moreover, the K led by $Cx^{\varpi}\xi^{\varpi'}$ is unique. Otherwise there would be a concomitant $K_1 - K_2$ with no $x^{\varpi}\xi^{\varpi'}$ term.

C is of one weight throughout by (7), and this weight is given in terms of the other characteristics of K by

$$w = \frac{1}{2} (ip + \varpi' - \varpi). \quad (10)$$

We shall presently see that there is no other restriction on the gradient C .

Given C and the order and class, all the other coefficients in K can be derived. In fact, the whole concomitant may be written in either of the forms

$$K \equiv x^{\varpi} e^{(y/x)(0+\eta\partial/\partial\xi)} \xi^{\varpi'} e^{-\eta/\xi} C, \quad (11)$$

$$\equiv \xi^{\varpi'} e^{-(\eta/\xi)(0-\eta\partial/\partial x)} x^{\varpi} e^{y/x} C. \quad (12)$$

These expressions are afforded by an analysis of the annihilation of K by (1) and (2). The former expresses K as the covariant of u and $\xi x + \eta y$ which has for leading coefficient a seminvariant of u and $\xi x + \eta y$ derived from C , and the latter has the correlative construction. Other forms to which (11) and (12) respectively reduce by Taylor's theorem are

$$K \equiv x^{\varpi-\varpi'} (\xi x + \eta y)^{\varpi'} e^{y/x} e^{-\eta x/(\xi x + \eta y)} C, \quad (13)$$

$$\equiv \xi^{\varpi'-\varpi} (\xi x + \eta y)^{\varpi} e^{-\eta/\xi} e^{\xi y/(\xi x + \eta y)} C; \quad (14)$$

two expressions which are known (see my *Algebra of Quantics*, § 99) to be equal when C is any gradient for which (10) holds.

Here, by (10), $\varpi - \varpi' = ip - 2w$. Let ϖ, ϖ' be $ip - w + k, w + k$ respectively, where k is zero or a positive or negative integer. Then I have elsewhere shown (*Algebra of Quantics*, *loc. cit.*) that the equivalent forms (13), (14) have for another equivalent

$$K \equiv (\xi x + \eta y)^k C(A_0, A_1, A_2, \dots, A_p), \quad (15)$$

where $(A_0, A_1, \dots, A_p)(x', y')^p \equiv (a_0, a_1, \dots, a_p)(xx' - \eta y', yx' + \xi y')^p$, (16)

and $C(A_0, A_1, \dots, A_p)$ is the same function of A_0 , &c., as C is of a_0 , &c. [Here k can be negative only when $C(A_0, A_1, \dots, A_p)$ is divisible by a power of $\xi x + \eta y$.] Thus K can be at once written from $Cx^{\varpi}\xi^{\varpi'}$ either by differential operation or by substitution.

So much when C, ϖ, ϖ' are given.

When an appropriate C alone is given—we defer till the next article a proof that all gradients C of one degree and one weight

throughout are appropriate— ϖ and ϖ' are, of course, not arbitrary positive integers. Neither are they determinate, as (10) only gives their difference $\varpi - \varpi' = ip - 2w$. There are smallest admissible positive or zero values of ϖ, ϖ' ; but these smallest values may be increased by any, the same, number, and a concomitant K obtained as above.

By (11) or (13) ϖ' must be at least equal to the index of the highest power of Ω which does not annihilate C . For, if $\Omega^{\varpi'+k}C \neq 0$ but $\Omega^{\varpi'+k+1}C = 0$, where $k > 0$, (13) would contain a part which is a numerical multiple of

$$x^{\varpi-\varpi'}(\xi x + \eta y)^{-k}(\eta x)^{\varpi'+k}e^{y/x}O^{\varpi'+k}C,$$

i.e., a part of negative degree in $\xi x + \eta y$, against which no other part could cancel so as to leave the whole integral. In like manner ϖ must be at least equal to the index of the highest power of O which does not annihilate C . These lowest admissible values of ϖ', ϖ are certainly not greater than w and $ip - w$ respectively; but for particular C 's they are less. If, for a given appropriate C , $w - k'$ is the smallest value of ϖ' which, with the corresponding

$$\varpi = ip - 2w + \varpi' = ip - w - k',$$

suffices to make K integral, then $ip - w - k'$ is the smallest value of ϖ which, with the corresponding $\varpi' = \varpi - (ip - 2w) = w - k'$, suffices. Moreover, in this case neither $\Omega^{w-k'}$ nor $O^{ip-w-k'}$ can annihilate C . Otherwise, to take the first supposition, (13) would tell us that K has the factor $\xi x + \eta y$, which could be removed, and leave a concomitant $K \div (\xi x + \eta y)$ led by C and of lower class than the lowest possible; and, in like manner, as to the second supposition. We have then

$$\begin{aligned}\Omega^{w-k'}C &\neq 0, & \Omega^{w-k'+1}C &= 0, \\ O^{ip-w-k'}C &\neq 0, & O^{ip-w-k'+1}C &= 0.\end{aligned}$$

4. We have now to see that every gradient C is appropriate for some value of $\varpi - \varpi'$ or other; in other words, that every rational integral function of the coefficients $a_0, a_1, a_2, \dots, a_p$ which is of one degree i and one weight w throughout leads a concomitant. This will be proved by showing that the $A_0, A_1, A_2, \dots, A_p$ of (16) are concomitants, and have for their leading coefficients $a_0, a_1, a_2, \dots, a_p$ respectively. This will establish that every homogeneous isobaric function of $a_0, a_1, a_2, \dots, a_p$ leads a concomitant which is that same function of $A_0, A_1, A_2, \dots, A_p$.

From (16), we have

$$A_0 \equiv (a_0, a_1, \dots, a_p)(x, y)^p \equiv u, \quad (17)$$

and, for $r = 1, 2, \dots, p$,

$$A_r \equiv \frac{1}{p(p-1)\dots(p-r+1)} \left(\xi \frac{\partial}{\partial y} - \eta \frac{\partial}{\partial x} \right)^r u, \quad (18)$$

the leading term in which is $a_r x^{p-r} \xi^r$.

Now these are all concomitants of u , the power of the modulus M in the expression of invariancy of A_r being M^r . For the polarizing operator $\xi \frac{\partial}{\partial y} - \eta \frac{\partial}{\partial x}$ is an invariant operator of modulus M . In fact, from

$$\begin{aligned} x &= lX + mY, & y &= l'X + m'Y, \\ \Xi &= l\xi + l'\eta, & H &= m\xi + m'\eta, \end{aligned}$$

it follows at once that

$$\Xi \frac{\partial}{\partial Y} - H \frac{\partial}{\partial X} \equiv M \left(\xi \frac{\partial}{\partial y} - \eta \frac{\partial}{\partial x} \right).$$

We may also form the same series of concomitants backwards, starting from the last, the contravariant

$$A_p \equiv (a_p, a_{p-1}, \dots, a_0)(\xi, -\eta)^p, \quad (19)$$

and producing, for $r = 1, 2, \dots, p$,

$$A_{p-r} \equiv \frac{1}{p(p-1)\dots(p-r+1)} \left(y \frac{\partial}{\partial \xi} - x \frac{\partial}{\partial \eta} \right)^r A_p. \quad (20)$$

Hence, if we take any homogeneous rational integral function, of sum of suffixes w and of degree i , of $A_0, A_1, \dots, A_p$, we obtain a concomitant whose leading term is the same function, of weight w , of $a_0, a_1, \dots, a_p$ multiplied by $x^{i-p-w} \xi^w$, and for which the power of the modulus in the expression of invariancy is M^w .

In particular, by way of example, the cubic

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

has, besides itself, the concomitants

$$\begin{aligned} (bx^2 + 2cxy + dy^2) \xi - (ax^2 + 2bxy + cy^2) \eta, \\ (cx + dy) \xi^2 - 2(bx + cy) \xi \eta + (ax + by) \eta^2, \\ d\xi^3 - 3c\xi^2\eta + 3b\xi\eta^2 - a\eta^3, \end{aligned}$$

i.e., it has, including itself, concomitants with the leading terms

$$ax^3, \quad bx^2\xi, \quad cx\xi^2, \quad d\xi^3,$$

which may be rationally and integrally compounded so as to produce a concomitant with any homogeneous isobaric function of a, b, c, d for its leading coefficient.

There are then as many linearly independent concomitants of degree i , weight of leading coefficient w , order $ip-w$, and class w , as there are products of weight w of i of the letters $a_0, a_1, a_2, \dots, a_p$. This number is, as usual, written $(w; i, p)$.

The number of degree i , weight of leader w , order and class ϖ, ϖ' , where $\varpi = ip - w + k$, $\varpi' = w + k$, for a positive k , is the same. We have simply to multiply each by $(\xi x + \eta y)^k$. The set of $(w; i, p)$ thus obtained is certainly complete; for the leading coefficients are all linearly independent and as numerous as possible, and no two concomitants have the same leader $Cx^\varpi \xi^{\varpi'}$.

Linear functions O of the products of weight w can, as we well know, be so formed that powers of the universal concomitant $\xi x + \eta y$ occur as factors in the derived concomitants of order and class $ip-w, w$. These factors may be removed, and concomitants of order and class

$$\varpi = ip - w - k', \quad \varpi' = w - k'$$

result. In particular, if $ip-w > w$ the covariants of order $ip-2w$, if $ip-w < w$ the contravariants of class $2w-ip$, and if $ip-w = w$ the invariants, are thus obtained.

All concomitants of course, except invariants, become either covariants or zero when in them we replace ξ, η by $y, -x$. The fundamental p -ic concomitants $A_0, A_1, A_2, \dots, A_p$ all of them thus become u itself, but for numerical factors. Those concomitants which become zero on the substitution have $\xi x + \eta y$ for a factor.

5. We have still to consider what are the concomitants of lowest orders and classes which have leading coefficients of weight and degree w, i ; in other words, to consider what are the concomitants, with such leaders, that cannot be rationally and integrally expressed in terms of other concomitants and the universal concomitant $\xi x + \eta y$, or in terms of which and $\xi x + \eta y$ all concomitants, with such leaders, can be rationally and integrally expressed.

Take any covariant

$$(C_w, C_{w+1}, \dots, C_{ip-w})(x, y)^{ip-2w}, \quad (21)$$

in which suffixes indicate weight, and in which the degree is i . We will assure ourselves that the coefficients of weight w in such covariants are the leading coefficients of the concomitants in question.

The coefficients in the covariant before us lead concomitants for all of which $ip-2w'$ is the sum $\varpi+\varpi'$ of order and class. In fact, the invariant operator

$$\frac{1}{(ip-2w')(ip-2w'-1)\dots(ip-2w'-r+1)} \left(\xi \frac{\partial}{\partial y} - \eta \frac{\partial}{\partial x} \right)^r$$

$$(r = 1, 2, \dots, ip-2w') \quad (22)$$

derives from (21) a series of concomitants, with this sum of order and class, whose leading terms are

$$C_{w'+1} x^{ip-2w'-1} \xi,$$

$$C_{w'+2} x^{ip-2w'-2} \xi^2,$$

$$\dots \dots \dots$$

$$C_{ip-w'} \xi^{ip-2w'},$$

of which the last is the contravariant derived from the covariant (21) itself by putting $\xi, -\eta$ for y, x , since the concomitant led by $C_{ip-w'} \xi^{ip-2w'}$ must be unique. The general leading term may be written

$$C_{w'+r} x^{ip-2(w'+r)+r} \xi^r \quad (r = 0, 1, 2, \dots, ip-2w'). \quad (23)$$

If C_w is the most general seminvariant of its type (w', i), these concomitants all involve the same number of arbitraries as does C_w . For $C_{ip-w'}$ does, as C_w can be derived from it by repeated operations with Ω , and so do all the intermediaries in consequence or in like manner.

Now alter the notation by putting w for $w'+r$. We see that, if C_w is a coefficient in a covariant whose leading coefficient is C_{w-r} , then

$$C_w x^{ip-2w+r} \xi^r \quad (24)$$

leads a concomitant. This cannot be divisible by $\xi x + \eta y$, for the substitution in it of y for ξ and $-x$ for η produces not zero, but the covariant led by C_{w-r} . Moreover, no linear function of coefficients C_w in covariants led by different seminvariants C_{w-r} , for the same r , can be equal to a linear function of coefficients C_w in covariants led by seminvariants with smaller r 's: for operation with Ω^r would annihilate the right-hand side of any such supposed quantity, but not the left. Thus there are at least as many concomitants (24), with

linearly independent leading coefficients of type (w, i) , and such that no linear function of those of them of the same order and class, or, therefore, of all of them, is divisible by $\xi x + \eta y$, as there are linearly independent seminvariants of degree i and of all weights $w-r$ between 0 and w or 0 and $ip-w$ inclusive, according as $2w$ is not or is greater than ip . For every one $\varpi - \varpi' = ip - 2w$, and, for those with any admissible r , $\varpi + \varpi' = ip - 2(w-r)$. Multiplied each by the corresponding zero or positive power $(\xi x + \eta y)^{w-r}$ of the universal concomitant, they produce at least as many linearly independent concomitants for which $\varpi - \varpi' = ip - 2w$, $\varpi + \varpi' = ip$ as there are linearly independent seminvariants such as described.

First, suppose that $w \leq \frac{1}{2}ip$, so that $\varpi - \varpi'$ is not negative. We take it as established, by the usual consideration of the annihilation by Ω , that the number of seminvariants of degree i and weight $w-r$ ($r = 0, 1, 2, \dots, w$) is not less than

$$(w-r; i, p) - (w-r-1; i, p).$$

In this case, then, the number of the linearly independent concomitants is not less than a number which is itself not less than

$$\sum_{r=0}^{r=w} \{ (w-r; i, p) - (w-r-1; i, p) \},$$

where $(0; i, p) = 1$, $(-1; i, p) = 0$, i.e., than a number not less than

$$(w; i, p).$$

But it certainly cannot exceed this number, which is the full number of arbitraries in the most general gradient of the type (w, i) of the gradients O_w . Hence we draw all the following conclusions—for our present case of $ip \geq 2w$.

(a) Every concomitant whose leading coefficient is of weight w and degree i has for that leading coefficient either a coefficient C_w in a covariant whose leader is a seminvariant C_{w-r} of weight between 0 and w inclusive, or a sum of such a coefficient C_w and coefficients of weight w in other covariants led by seminvariants of weight greater than $w-r$ but not greater than w . The coefficients C_w themselves lead a complete system of concomitants of degree and weight of leader i, w , and the order and class of the concomitant led by a C_w derived from a seminvariant C_{w-r} , when that concomitant has no factor $\xi x + \eta y$, are $ip - 2w + r$ and r respectively. The concomitant whose leader is the most general gradient of type (w, i) consists of

the sum of arbitrary multiples of these concomitants given by coefficients C_w , each multiplied by $(\xi x + \eta y)^{w-r}$ for the corresponding value of r assigned by C_{w-r} , and the whole multiplied by an arbitrary power of $\xi x + \eta y$.

(β) The number of gradients C_w which are leading coefficients in concomitants for which $\varpi = ip - 2w + r$, and consequently $\varpi' = r$, and no linear function of which has $\xi x + \eta y$ for a factor—or, as is the same thing, the number of aszygetic coefficients of $x^{ip-2w+r}y^r$ in covariants of order $ip - 2(w - r)$ —is exactly

$$(w - r; i, p) - (w - r - 1; i, p).$$

Here r lies between 0 and w inclusive.

(γ) The number of seminvariants of type (w', i) [$w' \leq w$ and so $\leq \frac{1}{2}ip$] is exactly, and not merely not less than,

$$(w'; i, p) - (w' - 1; i, p).$$

We have further to consider cases when $w > \frac{1}{2}ip$, so that $\varpi - \varpi'$ is negative, and there are contravariants but not covariants led by particular gradients of the type of C_w . In such cases C_{w-r} , the seminvariant leader of a covariant in which a C_w is a coefficient, must be of weight between 0 and $ip - w$ inclusive. Thus r lies between $2w - ip$ and w inclusive, and the number of linearly independent concomitants with leaders C_w is not less than a number of seminvariants not less than

$$\sum_{r=2w-ip}^{r=w} \{ (w - r; i, p) - (w - r - 1; i, p) \},$$

i.e., than $(ip - w; i, p)$,

i.e., than $(w; i, p)$,

as in the previously considered cases. This number it also cannot exceed, for reasons as before. The number is then exact; and we draw the conclusions:

(δ) The statement (α) holds equally when $w > \frac{1}{2}ip$, except that the seminvariant leaders C_{w-r} , mentioned in the opening sentence and later, are of weights between 0 and $ip - w$ inclusive, instead of between 0 and w .

(ϵ) When $w > \frac{1}{2}ip$ the statement (β) is replaced by:—The number of gradients C_w which are leading coefficients in concomitants

for which $\varpi' = r$, and consequently $\varpi = r - (2w - ip)$, and of which no linear function has $\xi x + \eta y$ for a factor, or, which is the same thing, the number of asyzygetic coefficients of $x^{r-(2w-ip)} y^r$ in covariants of order $ip - 2(w - r)$, is exactly

$$(w - r; i, p) - (w - r - 1; i, p).$$

Here r lies between $2w - ip$ and w inclusive.

6. The above has all been stated in terms applicable to concomitants of a single binary quantic u only. But the results for a system of binary quantics

$$u_1 \equiv (a_0, a_1, \dots, a_p)(x, y)^n,$$

$$u_2 \equiv (b_0, b_1, \dots, b_p)(x, y)^n,$$

$$\&c., \quad \&c.,$$

are exactly akin to those detailed. We have only to write in the analytical formulæ $\Sigma\Omega$, ΣO , $\Sigma(ip)$ for Ω , O , ip respectively. The convention introduced to supplement necessities, corresponding to that introduced before (8) above, is that the only concomitants regarded as simple are those which, as well as being of one order and one class throughout, are of one degree i_1 throughout in $a_0, a_1, \dots, a_p$, the coefficients in the first quantic; one degree i_2 in $b_0, b_1, \dots, b_p$, those in the second quantic; and so on. More complex concomitants are linear functions with numerical coefficients of simple concomitants of different orders, classes, and partial degrees, for which

$$\Sigma(ip) + \varpi' - \varpi,$$

which is double the index of the power of the modulus in the expression of invariancy, has the same value.

As in § 4, every coefficient in every one of the quantics leads a concomitant, for which ϖ' is the weight of the coefficient in question, and $\varpi + \varpi'$ is the order of the quantic in which it occurs. Also there are as many linearly independent concomitants of partial degrees $i_1, i_2, \dots$, weight of leading coefficient w , order $\Sigma(ip) - w$, and class w , as there are products of weight w of i_1 out of $a_0, a_1, \dots, a_p$, i_2 out of $b_0, b_1, \dots, b_p$, &c. Moreover, as in § 5, every coefficient of weight w in every covariant of degree i leads a concomitant of *order + class* = *order of covariant*, which concomitant cannot be reduced to one of lower *order + class* by division by $\xi x + \eta y$; and these concomitants are a complete system for the type (w, i) of leaders, in terms of which

and $\xi x + \eta y$, the most general concomitant with the leader of type (w, i) can be rationally and integrally expressed. To supply the necessary modification of wording in conclusions (a) to (e) is so direct that a new statement is unnecessary.

II.

7. We have now before us an interesting aspect of the fact that the number of asyzygetic seminvariants (including invariants if $ip = 2w$) of degree i in the coefficients of a binary p -ic, and of weight not exceeding the smaller of w and $ip - w$, namely

$$(w; i, p),$$

is also the number of terms with arbitrary multipliers in the most general gradient of type w, i ; and in particular that the whole number of seminvariants of degree i (including invariants if there be any), namely

$$\binom{ip}{2}; i, p \quad \text{or} \quad \binom{ip-1}{2}; i, p,$$

according as ip is even or odd, is equal to the number of terms in the most general gradient of type $\binom{ip}{2}, i$ or $\binom{ip-1}{2}, i$, according to this evenness or oddness. The explanation has been seen to be that every seminvariant of degree i leads a covariant, that every coefficient C_w of weight w which occurs in any such covariant leads a concomitant of a fundamental character, that all these coefficients C_w , for the same w , are linearly independent, and that there are as many of them as there are terms in the most general gradient of degree i and weight w , namely $(w; i, p)$. Now, if $ip \geq 2w$, the seminvariants which lead covariants involving coefficients of weight w are those whose weight does not exceed w . [If, on the other hand, $ip < 2w$, they are those whose weight does not exceed $ip - w$, and they are $(ip - w; i, p)$, or still correctly $(w; i, p)$, in number.] We will confine attention for the present to cases of $ip \geq 2w$.

A problem which presents itself for solution is the separation of the general gradient of degree i and weight w ($\leq \frac{1}{2}ip$), containing $(w; i, p)$ arbitraries, into its parts containing respectively

$$\begin{aligned} & (w; i, p) - (w-1; i, p), \\ & (w-1; i, p) - (w-2; i, p), \\ & \text{\&c.}, \quad \text{\&c.}, \end{aligned}$$

arbitraries which are leaders, second coefficients, &c., in covariants, *i.e.*, which are seminvariants leading covariants of order $ip-2w$, leaders of concomitants of order $ip-2w+1$ and class 1, those of concomitants of order $ip-2w+2$ and class 2, &c.

A method is afforded by Clebsch's process (*Binäre Formen*, § 7) for the expansion of a form, in two pairs of variables, in powers of the determinant of those pairs, with coefficients in the form of polars of forms in one of the two pairs which are themselves polars. Thus by one of (11) to (15) write down the concomitant K , of order $ip-w$ and class w , which is led by the most general gradient of degree i and weight w . This by Clebsch's method can be expanded in the form

$$\sum_{m=0}^{m=w} \mu_m (\xi x + \eta y)^m \left(\eta \frac{\partial}{\partial x} - \xi \frac{\partial}{\partial y} \right)^{w-m} \left(x \frac{\partial}{\partial \eta} - y \frac{\partial}{\partial \xi} \right)^{w-m} \left(\frac{\partial^2}{\partial x \partial \xi} + \frac{\partial^2}{\partial y \partial \eta} \right)^m K.$$

The leading coefficients of the concomitants which are here multiplied by the various powers $(\xi x + \eta y)^m$, for $m=0, 1, 2, \dots, w$, are the linearly independent parts required of the general gradient which leads K .

But what we seek is to separate the general gradient of given type into its critical parts by means of differential operations with regard to $a_0, a_1, a_2, \dots, a_p$, and to construct the covariants in which the parts are coefficients.

8. Take G the general gradient of degree i and weight w ($\leq \frac{1}{2}ip$), a sum of $(w; i, p)$ arbitrary multiples of coefficients. It is known (*cf. Algebra of Quantics*, § 128) that, because $ip-2w > -2$, $\Omega G, \Omega^2 G, \dots, \Omega^w G$ are all general gradients of their types, and so contain respectively $(w-1; i, p), (w-2; i, p), \dots, (0; i, p) = 1$, arbitraries. There are then numbers

$$\begin{aligned} & (w; i, p) - (w-1; i, p), \\ & (w-1; i, p) - (w-2; i, p), \\ & \dots \quad \dots \quad \dots \quad \dots \\ & (1; i, p) - 1 = 0, \\ & 1, \end{aligned}$$

of arbitrary multipliers in parts together constituting G , which parts are respectively annihilated by Ω , by Ω^2 but not by Ω , ..., by Ω^w but not by Ω^{w-1} , and by Ω^{w+1} but not by Ω^w ; *i.e.*, G is the sum of the

above numbers of arbitrary multiples of sets of terms, and so altogether of $(w; i, p)$ sets of terms, which when operated on repeatedly by Ω give eventually seminvariants (each at the stage before the one which effects annihilation). The whole number of sets is the number of terms in G .

Now (*Algebra of Quantics*, §§ 180, 181) the operator

$$1 - \frac{O\Omega}{1^2} + \frac{\Omega O^2\Omega}{1^2 \cdot 2^2} - \frac{\Omega^2 O^3\Omega}{1^2 \cdot 2^2 \cdot 3^2} + \dots \quad (25)$$

acting on G produces a sum of arbitrary multiples of all the seminvariants of degree i and weight w ; the same operator on ΩG , the most general gradient of type $(w-1, i)$, produces a sum of arbitrary multiples of all the seminvariants of degree i and weight $w-1$; the same on $\Omega^2 G$ those of degree i and weight $w-2$, and so on. The orders of covariants led by seminvariants of the types of $G, \Omega G, \Omega^2 G, \dots$ are $ip-2w, ip-2w+2, ip-2w+4, \dots$, respectively. Thus, if $\lambda_0, \lambda_1, \lambda_2, \dots$ be any (assigned or arbitrary) numerical quantities,

$$\left(1 - \frac{O\Omega}{1^2} + \frac{\Omega O^2\Omega}{1^2 \cdot 2^2} - \frac{\Omega^2 O^3\Omega}{1^2 \cdot 2^2 \cdot 3^2} + \dots\right) (\lambda_0 + \lambda_1 x^2 \Omega + \lambda_2 x^4 \Omega^2 + \dots) x^{ip-2w} G \quad (26)$$

is a sum of arbitrary multiples of the leading terms of all those covariants (including invariants, if $ip = 2w$) of degree i in which there are coefficients of weight w .

Again, $e^{y/x} O$ operating on a leading term produces the whole covariant led. Thus

$$e^{y/x} O \left(1 - \frac{O\Omega}{1^2} + \frac{\Omega O^2\Omega}{1^2 \cdot 2^2} - \frac{\Omega^2 O^3\Omega}{1^2 \cdot 2^2 \cdot 3^2} + \dots\right) (\lambda_0 + \lambda_1 x^2 \Omega + \lambda_2 x^4 \Omega^2 + \dots) x^{ip-2w} G \quad (27)$$

is, whatever assigned numbers $\lambda_0, \lambda_1, \lambda_2, \dots$ be, a sum of arbitrary numerical multiples of all the $(w; i, p)$ covariants described.

Once more, the term whose coefficient is of weight w in a covariant

$$e^{y/x} O^r x^{ip-2w+2r} S,$$

where S is a seminvariant of weight $w-r$, is

$$x^{ip-2w+r} y^r \frac{1}{r!} O^r S.$$

Thus

$$\sum_{r=0}^{r=ip} \frac{\lambda_r}{r!} O^r \left(1 - \frac{O\Omega}{1^2} + \frac{\Omega O^2\Omega}{1^2 \cdot 2^2} - \dots\right) \Omega^r G \quad (28)$$

is, for any assigned numerical $\lambda_0, \lambda_1, \dots, \lambda_w$, a general gradient of type (w, i) expressed as a sum of w parts of which the $(r+1)$ th is the coefficient of weight w which occurs in the most general covariant of degree i and order $ip-2w+2r$.

G and (28) are both general gradients of the same type (w, i) . It occurs as desirable so to assign the values of $\lambda_0, \lambda_1, \dots, \lambda_w$ that they may be identical. In order so to assign them we notice that, if

$$G \equiv H_0 + H_1 + \dots + H_w,$$

where H_r ($r = 0, 1, 2, \dots, w$) is the coefficient of degree i and weight w in a covariant of order $ip-2w+2r$ led by S_r , then

$$S_r = \frac{1}{(ip-2w+2r)(ip-2w+2r-1)\dots(ip-2w+r+1)} \Omega^r H_r,$$

and that the operator (25) acting on S_r reproduces it exactly. Also

$$\Omega^r (H_0 + H_1 + \dots + H_{r-1}) = 0$$

and

$$\Omega^r (H_{r+1} + H_{r+2} + \dots + H_w)$$

is annihilated by (25), because it is of the form OF (cf. *Algebra of Quantics*, loc. cit.). Thus we secure what is desired by taking, after $\lambda_0 = 1$,

$$\lambda_r = \frac{1}{(ip-2w+2r)\dots(ip-2w+r+1)} = \frac{(ip-2w+r)!}{(ip-2w+2r)!}$$

Accordingly,

$$G \equiv \sum_{r=0}^{r=w} \frac{(ip-2w+r)!}{r!(ip-2w+2r)!} \Omega^r \left(1 - \frac{O\Omega}{1^2} + \frac{\Omega O^2 \Omega}{1^3 \cdot 2^2} - \dots \right) \Omega^r G \quad (29)$$

is what exhibits G as separated into the derived sum of coefficients in covariants.

Perhaps, however, it is best so to choose the λ 's that G may be split up into a sum

$$G \equiv G_0 + G_1 + \dots + G_w,$$

in such a way that G_r multiplied by its appropriate binomial coefficient $\frac{(ip-2w+2r)!}{r!(ip-2w+r)!}$ is the coefficient of weight w , i.e., the multiplier of $x^{ip-2w+r}y^r$, in the general covariant of order $ip-2w+2r$ occurring in the sum (27). The partial G 's are thus C_w 's of § 5. In this notation

$$S_r = \frac{1}{r!} \Omega^r G_r,$$

so that we have to take $\lambda_r = \frac{1}{r!}$;

we thus obtain $G \equiv \sum_{r=0}^{r=w} \frac{O^r}{1^3 \cdot 2^3 \dots r^3} \left(1 - \frac{O\Omega}{1^2} + \frac{\Omega O^2 \Omega}{1^3 \cdot 2^3} - \dots\right) \Omega^r G$, (30)

as the separation of G into parts as required.

The sum of the covariants with the multiples of parts

$$\frac{(ip - 2w + 2r)!}{r! (ip - 2w + r)!} G_r$$

for coefficients of weight w in them is thus from (27)

$$e^{y/z} O \left(1 - \frac{O\Omega}{1^2} + \frac{\Omega O^2 \Omega}{1^3 \cdot 2^3} - \dots\right) \left(1 + \frac{x^2 \Omega}{1} + \frac{x^4 \Omega^3}{1 \cdot 2} + \dots\right) x^{ip-2w} G. \quad (31)$$

The sum of the covariants with the actual parts of G for coefficients of weight w in them is in like manner written down from (27) by substituting

$$\lambda_r = \frac{(ip - 2w + 2r)!}{(ip - 2w + r)!}, \text{ for } r = 0, 1, 2, \dots, w.$$

9. The consideration of the extreme case when w is whichever is integral of $\frac{1}{2}ip$ and $\frac{1}{2}(ip-1)$ is of special interest. It in fact gives us all covariants (including invariants) of degree i in one formula.

First, if ip is even, so that there may be invariants. There are altogether $\left(\frac{ip}{2}; i, p\right)$ covariants and invariants of degree i in the coefficients. Take G the most general gradient of degree i and weight $\frac{ip}{2}$. It involves just that same number $\left(\frac{ip}{2}; i, p\right)$ of arbitraries. The sum

$$e^{y/z} O \left(1 - \frac{O\Omega}{1^2} + \frac{\Omega O^2 \Omega}{1^3 \cdot 2^3} - \dots\right) \sum_{r=0}^{r=\frac{1}{2}ip} \lambda_r x^{2r} \Omega^r G, \quad (32)$$

contains, for any chosen non-vanishing numerical values of the λ 's, that same number of arbitraries, and is the sum of arbitrary multiples of all the covariants and invariants. In particular, by taking

$$\lambda_r = \frac{r!}{(2r)!} \quad (r = 0, 1, 2, \dots, w),$$

we thus write down from G the general sum of the covariants and invariants in such a way that G is actually separated into the sum of the coefficients of weight $\frac{1}{2}ip$, i.e., of the middle coefficients in the

separate covariants, &c., composing the sum; the separation of G in question being

$$\sum_{r=0}^{r=\frac{1}{2}ip} O^r \left(1 - \frac{O\Omega}{1^2} + \frac{\Omega O^2 \Omega}{1^2 \cdot 2^2} - \dots \right) \frac{\Omega^r}{(2r)!} G. \quad (33)$$

And again by taking $\lambda_r = \frac{1}{r!}$ the separation of G is into the middle coefficients of the separate covariants of the sum, each divided by the binomial coefficient $\frac{(2r)!}{(r!)^2}$ which is appropriate to its position in its own covariant; i.e., into

$$\sum_{r=0}^{r=\frac{1}{2}ip} O^r \left(1 - \frac{O\Omega}{1^2} + \frac{\Omega O^2 \Omega}{1^2 \cdot 2^2} - \dots \right) \frac{\Omega^r}{1^2 \cdot 2^2 \dots r^2} G. \quad (34)$$

In precisely the same way if ip is odd, so that there are no invariants of degree i , but only covariants of the first and higher odd orders, the full system of covariants of degree i is

$$e^{v/z} O \left(1 - \frac{O\Omega}{1^2} + \frac{\Omega O^2 \Omega}{1^2 \cdot 2^2} - \dots \right) \sum_{r=0}^{r=\frac{1}{2}(ip-1)} \lambda_r x^{2r+1} \Omega^r G \quad (35)$$

for any assigned non-vanishing values of the λ 's; and the system of coefficients of $x^{r+1}y^r$ in such covariants, i.e., of coefficients of weight $\frac{ip-1}{2}$ in covariants of degree i , is

$$\sum_{r=0}^{r=\frac{1}{2}(ip-1)} \frac{\lambda_r}{r!} O^r \left(1 - \frac{O\Omega}{1^2} + \dots \right) \Omega^r G. \quad (36)$$

The general gradient G of type $\left(\frac{ip-1}{2}, i \right)$ is exhibited as the sum of parts which are the actual coefficients in question if we take each

$$\lambda_r = \frac{1}{(2r+1) 2r \dots (r+2)},$$

and as the sum of parts equal to the actual coefficients each divided by its appropriate binomial factor if we take each

$$\lambda_r = \frac{1}{r!}.$$

10. We have so far in this section confined ourselves to weights w which do not exceed $\frac{1}{2}ip$. When, on the other hand, $ip < 2w$, the reasoning in § 8 has only to be so slightly modified that repetition is hardly necessary. The general gradient G of weight w and degree i

can still be separated into parts, containing in the aggregate the same number of arbitraries as does G , which are coefficients in linearly independent covariants led by seminvariants of degree i . These seminvariants, instead of ranging in weight from 0 to w inclusive as before, now range from 0 to $ip-w$; and the orders of the covariants, consequently, instead of ranging from ip to $ip-2w$ inclusive, range from ip to $ip-2(ip-w)$, i.e., $2w-ip$. In (26) and (27) we have to omit the terms in

$$\lambda_0 + \lambda_1 x^2 \Omega + \lambda_2 x^4 \Omega^2 + \dots,$$

which precede

$$\lambda_{2w-ip} x^{2(2w-ip)} \Omega^{2w-ip};$$

in (31) to omit the corresponding terms; and in (28), (29) and (30) to replace the range of summation by

$$\sum_{r=2w-ip}^{r=ip}$$

With these modifications the conclusions are as before.

11. The reasoning and results of this section apply equally to covariants and invariants of a set of binary quantics $u_1, u_2, u_3, \dots$ of orders $p_1, p_2, p_3, \dots$. We have merely to write $\sum \Omega$, $\sum O$ and $i_1 p_1 + i_2 p_2 + i_3 p_3 + \dots$ for Ω , O and ip , where $i_1, i_2, i_3, \dots$ are partial degrees in the coefficients of the quantics separately. We thus find, for instance, as in § 9, from the most general gradient of partial degrees $i_1, i_2, i_3, \dots$, and of weight $\frac{1}{2} \sum ip$, in cases where this is integral, an expression for the sum of arbitrary multiples of all the covariants, including invariants, whose partial degrees are $i_1, i_2, i_3, \dots$, and effect the separation of the general gradient, of the given partial degrees and whole weight, into parts which are the general invariants, and the middle coefficients of the general covariants, of these partial degrees and of the different possible orders from O to $\sum ip$ inclusive.

III.

12. We have exhibited compact analytical expressions for the separate parts of the general gradient of weight w and degree i of a binary p -ic which are coefficients in the general covariants of degree i and the various different orders. A desideratum which still presents itself is the algebraic performance of the separation without often-repeated application of the operators Ω , O . We shall see that one

operation with each, followed by the solution of algebraic equations, suffices.

The fact which suffices to effect the separation is that, if F is a coefficient in a covariant, $O\Omega F$ is a numerical multiple of F . We shall see, in fact, that if, for a numerical or vanishing k , such an equation as

$$O\Omega F = kF \quad (37)$$

has gradient solutions, then any such solution F is a coefficient in a covariant, and conversely that all coefficients in covariants are given and can be discriminated by means of equations of the form (37).

Suppose that F is a gradient (rational, integral, homogeneous, isobaric function) of degree i and weight w which satisfies an equation (37). We will first see that for this to be possible k must be either a positive integer or zero. Write, as does Sylvester, η for $ip - 2w$, which is zero or integral, but is now allowed to be either positive or negative.

It is a known fact (*Algebra of Quantics*, § 125) that, F being any gradient for which $ip - 2w = \eta$, and r being a positive integer,

$$O^r \Omega^r F \equiv O\Omega (O\Omega - 1.\overline{\eta+2})(O\Omega - 2.\overline{\eta+3}) \dots (O\Omega - r-1.\overline{\eta+r}) F.$$

If then F satisfies (37), where k is numerical, it follows that

$$O^r \Omega^r F \equiv k(k-1.\overline{\eta+2})(k-2.\overline{\eta+3}) \dots (k-r-1.\overline{\eta+r}) F. \quad (38)$$

But for some value of r , and all greater values, $\Omega^r F$, and therefore $O^r \Omega^r F$, vanishes. This must at any rate be the case for $r = w + 1$. One at least of the numerical factors on the right must then vanish when $r = w + 1$. Let the first which vanishes be the $(t+1)$ th, where $w \geq t \geq 0$. Then

$$k = t(\eta + t + 1), \quad (39)$$

a product of two integers, of which the first may be zero, but the second, as we shall see, cannot be. To be sure that k (if not zero) is positive we must assure ourselves that $\eta + t + 1$ is positive even when η is not.

To establish this, notice that $O\Omega F = kF$ together with the known identity

$$(\Omega O - O\Omega) F = \eta F$$

necessitates

$$\Omega O F = (\eta + k) F.$$

Thus there is a companion identity to (38) above; viz.,

$$\Omega' O' F = (\eta + k)(\eta + k - 1. \overline{2 - \eta})(\eta + k - 2. \overline{3 - \eta}) \dots (\eta + k - r - 1. \overline{r - \eta}) F', \quad (40)$$

and it is necessitated as above that there is a positive or zero t' , such that

$$\eta + k = t'(-\eta + t' + 1). \quad (41)$$

From this and (39) we obtain, by elimination of k ,

$$(t' + t + 1)(\eta + t - t') = 0.$$

Here $t' + t + 1$ is not zero. Consequently

$$\eta + t = t',$$

which is not negative. Accordingly $\eta + t + 1$ is a positive integer; and therefore by (39) so is k , unless $t = 0$, in which case $k = 0$.

Having thus limited the possible values of k in (37), we have now to see that any gradient solution of the equation, with a proper k , is a coefficient in a covariant of our binary quantic.

First suppose that η is positive or zero. We have

$$O^{t+1} \Omega^{t+1} F = 0.$$

Now, for $O^t \Omega^{t+1} F$, $ip - 2w'$ is $\eta + 2(t+1) - 2\lambda$, which is positive if $\lambda < t+1$. Hence, since O cannot annihilate any gradient for which $ip - 2w' > 0$, it follows in succession from

$$O^{t+1} \Omega^{t+1} F = 0,$$

$$\text{i.e.,} \quad O \cdot O^t \Omega^{t+1} F = 0,$$

$$\text{that} \quad O^t \Omega^{t+1} F = 0, \quad O^{t-1} \Omega^{t+1} F = 0, \quad \dots, \quad \Omega^{t+1} F = 0,$$

of which facts the last tells us that $\Omega^t F$ is a seminvariant. It is not zero, for $O^t \Omega^t F$ is not. The covariant which it leads is of order

$$ip - 2(w - t) = \eta + 2t.$$

$$\begin{aligned} \text{Also, since } k - (s-1)(\eta + s) &= t(\eta + t + 1) - (s-1)(\eta + s) \\ &= (t - s + 1)(\eta + t + s), \end{aligned}$$

we have from (38), by taking $s = 1, 2, 3, \dots, t$,

$$O' \Omega^t F \equiv t(t-1)(t-2) \dots 1 (\eta + t + 1)(\eta + t + 2) \dots (\eta + 2t) F,$$

$$\text{i.e.,} \quad F \equiv \frac{O^t}{t!} \frac{\Omega^t F}{(\eta + t + 1) \dots (\eta + 2t)},$$

so that F is the coefficient of $x^{\eta+t}y^t$ in the covariant whose leading term is

$$\frac{\Omega' F}{(\eta+t+1) \dots (\eta+2t)} x^{\eta+2t}. \quad (42)$$

If t is zero, this does not apply verbally, but F is then annihilated by Ω , *i.e.*, is a seminvariant, or first coefficient in a covariant led by Fx^η .

We have still to consider cases of η negative. It is best then to reason from (40). We have

$$\Omega'^{\lambda'+1} O'^{\lambda'+1} F = 0,$$

where

$$t' = \eta + t.$$

Now for $\Omega' O'^{\lambda'+1} F$, with $\lambda' < t' + 1$, we have

$$ip - 2w' = \eta - 2(t' + 1 - \lambda') < \eta < 0.$$

Hence, by reasoning as before,

$$O'^{\lambda'+1} F = 0,$$

i.e., $O' F$ is an antiseminvariant, the last coefficient in a covariant whose order is $-\eta + 2t'$, *i.e.*, $\eta + 2t$; and we obtain

$$\begin{aligned} \eta + k - (s-1)(s-\eta) &= t'(-\eta + t' + 1) - (s-1)(s-\eta) \\ &= (t' - s - 1)(-\eta + t' + s), \end{aligned}$$

and

$$F = \frac{\Omega''}{t'!} \frac{O'' F}{(-\eta + t' + 1) \dots (-\eta + 2t')},$$

so that F is the coefficient of $x'' y^{-\eta+t''}$, *i.e.*, as before of $x^{\eta+t} y^t$, in a covariant whose last term is

$$\frac{O'' F}{(-\eta + t' + 1) \dots (-\eta + 2t')} y^{-\eta+2t'}, \quad (43)$$

where the order $-\eta + 2t' = \eta + 2t$ as before.

We notice that the two factors

$$t \quad \text{and} \quad \eta + t + 1,$$

as the product of which k is exhibited, have for their sum

$$\eta + 2t + 1,$$

i.e., one more than the order π of the covariant in which F , determined by

$$O\Omega F = kF,$$

is a coefficient, and for their difference

$$\eta + 1 \quad \text{or} \quad -\eta - 1,$$

according as η is not or is negative.

13. Having now found the forms of k for some or all of which the equation

$$O\Omega F = kF \quad (37)$$

has gradient solutions, and having shown that all solutions which exist are coefficients in covariants of determinate orders, we still have to see that every coefficient in every covariant is thus given. This will be established by showing that every such coefficient satisfies an equation (37).

$$\text{If} \quad Sx^{\varpi} + \dots + Px^{\varpi-t+1}y^{t-1} + Qx^{\varpi-t}y^t + \dots + S'y^{\varpi}$$

be any covariant, P, Q may be written as

$$\frac{\Omega^{\varpi-t+1}}{(\varpi-t+1)!} S', \quad \frac{\Omega^{\varpi-t}}{(\varpi-t)!} S',$$

so that

$$\Omega Q = (\varpi-t+1) P,$$

and also as

$$\frac{O^{t-1}}{(t-1)!} S, \quad \frac{O^t}{t!} S,$$

so that

$$OP = tQ.$$

Therefore

$$O\Omega Q = t(\varpi-t+1) Q,$$

where

$$t + (\varpi-t+1) = \varpi+1,$$

and $t \sim (\varpi-t+1) = \varpi+1 \sim 2t$

$$= ip-2w'+1 \sim 2t, \text{ where } w' \text{ refers to } S,$$

$$= 1 \sim (2w-ip), \text{ where } w \text{ refers to } Q.$$

This difference is $\eta+1$ or $-\eta-1$ according as $\eta = ip-2w$ is not or is negative; which is in full accordance with the last article.

The method used does not apply to first coefficients S , i.e., to seminvariants. But, as $O\Omega S = 0$, there is, with $t=0$, still accordance as before.

14. We are in position, then, to find all coefficients in all covariants. We seek those of which the degree is i and the weight w , a number which for any gradient lies between 0 and ip inclusive. Write down the most general gradient F of weight w and degree i . It involves $(w; i, p)$ arbitraries λ . Seek to identify

$$O\Omega F \text{ with } kF.$$

The identification requires the satisfaction of $(w; i, p)$ linear homogeneous equations in the $(w; i, p)$ λ 's, k being involved to the first degree in one coefficient in each. To make them consistent we have to choose k so as to satisfy an equation of degree $(w; i, p)$ which presents itself in the characteristic determinant form

$$\begin{vmatrix} a_{11}-k, & a_{12}, & a_{13}, & \dots \\ a_{21}, & a_{22}-k, & a_{23}, & \dots \\ a_{31}, & a_{32}, & a_{33}-k, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0. \quad (44)$$

The solution of this equation of, it may be, high degree is facilitated by our knowledge, from theory adduced above, that its roots must be all real and not fractional or negative. In fact, we know numbers to one or other of which every root must be equal. Thus, if η , i.e., $ip-2w$, is positive (or zero), the roots must consist of

$(w; i, p) - (w-1; i, p)$ which are zero, equal to 0 ($\eta+1$) say,

$(w-1; i, p) - (w-2; i, p)$ which are equal to 1 ($\eta+2$),

$(w-2; i, p) - (w-3; i, p)$ „ „ 2 ($\eta+3$),

&c., &c.; and, lastly,

$(0; i, p)$, i.e., one, which is equal to $w(\eta+w+1)$.

If on the other hand η is negative (or zero), they consist of

$(w; i, p) - (w+1; i, p)$ which are equal to $-\eta \cdot 1$,

$(w+1; i, p) - (w+2; i, p)$ „ „ $(1-\eta) 2$,

&c., &c.; and, lastly,

$(ip; i, p)$, i.e., one, which is equal to $(ip-w-\eta)(ip-w+1)$,

i.e., to $w(\eta+w+1)$.

Some of these numbers of roots of particular values may well be zero.

Take any root of the solved equation for k , say one which occurs r times. Substitute it in the equations for the λ 's which it makes consistent. These then suffice to determine all the λ 's in terms of r of them; and these, on substitution in the gradient F , determine the gradient of greatest generality of the type (w, i) which is a coefficient in a covariant of the order related to the root k . If η is positive (or

zero), and the root in question is $t(\eta+t+1)$, so that

$$r = (w-t; i, p) - (w-t-1; i, p),$$

this order ϖ is $\eta+2t$, *i.e.*, is less by one than the sum of the two factors of k . If on the other hand η is negative, and the root in question is $(t'-\eta)(t'+1)$, so that

$$r = (w+t'; i, p) - (w+t'+1; i, p),$$

the order is $2t'-\eta$, *i.e.*, is still less by one than the sum of the two factors of k .

15. As an example, let us discriminate all covariants (and invariants) of degree $i=3$ of the quartic, $p=4$, by separating from one another the five gradients, included in

$$\lambda ace + \mu bcd + \nu au^3 + \rho b^3e + \sigma c^3,$$

which are their middle coefficients.

Identification of ONF with kF gives

$$\begin{aligned} \frac{10\lambda + \mu + 6\nu + 6\rho}{\lambda} &= \frac{16\lambda + 28\mu + 24\nu + 24\rho + 24\sigma}{\mu} = \frac{8\lambda + 2\mu + 12\nu}{\nu} \\ &= \frac{8\lambda + 2\mu + 12\rho}{\rho} = \frac{9\mu + 18\sigma}{\sigma} = k. \end{aligned}$$

Hence comes the equation for k ,

$$\begin{vmatrix} 10-k & 1 & 6 & 6 & . \\ 16 & 28-k & 24 & 24 & 24 \\ 8 & 2 & 12-k & . & . \\ 8 & 2 & . & 12-k & . \\ . & 9 & . & . & 18-k \end{vmatrix} = 0,$$

whose roots must occur among

$$0, \quad 1.2, \quad 2.3, \quad 3.4, \quad 4.5, \quad 5.6, \quad 6.7,$$

$$\text{i.e.,} \quad 0, \quad 2, \quad 6, \quad 12, \quad 20, \quad 30, \quad 42,$$

$$\text{and prove to be} \quad 0, \quad 6, \quad 12, \quad 20, \quad 42$$

each once.

These values of k in turn inserted give the ratios $\lambda : \mu : \nu : \rho : \sigma$ for

(1) an invariant (covariant of order $\eta = 0$)

$$ace + 2bcd - ad^2 - b^2e - c^3,$$

(2) the mid-coefficient in a covariant of order $2 + 3 - 1 = 4$,

(3) " " " $3 + 4 - 1 = 6$,

(4) " " " $4 + 5 - 1 = 8$,

(5) " " " $6 + 7 - 1 = 12$;

the last covariant (5) being the cube of the quartic itself.

16. The other aspect of what is effected, in accordance with Section I. of these notes, by the determination of gradient solutions of equations $O\Omega u = ku$, needs only brief mention. To have determined all the $(w; i, p)$ linearly independent gradients of degree i and weight w which are coefficients in covariants, is to have determined the leading coefficients of the $(w; i, p)$ concomitants of lowest order + class which have leading coefficients of the type. Thus a found coefficient C_w for which $k = t(\eta + t + 1)$, as well as being the coefficient of $x^{i+t}y^t$ in a covariant of order $\eta + 2t$, is the leading coefficient in a reduced concomitant of order and class ϖ, ϖ' , where

$$\varpi - \varpi' = \eta,$$

$$\varpi + \varpi' = \eta + 2t,$$

related to the covariant as a polar in the way exhibited in § 5.

The general concomitant whose leader is the general gradient of type (w, i) is the sum of arbitrary multiples of the concomitants thus obtained, each multiplied by a power of $\xi x + \eta y$ sufficient to raise it to the order $ip - w$ and class w , i.e., $(ip - w) - \eta$, and the whole multiplied by an arbitrary power of $\xi x + \eta y$.