XI.—*The Differentiation of a Continuant.*

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(1) The continuant

$$\begin{vmatrix} a_1 & b_1 & . & . \\ -1 & a_2 & b_2 & . \\ . & -1 & a_3 & b_3 \\ . & . & -1 & a_4 \end{vmatrix}$$

is conveniently denoted by

$$K \begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix};$$

and, when the elements b_1, b_2, b_3 of the variable minor diagonal are each equal to 1, a further simplification is obtained by leaving them out;—thus,

$$K(a, b, c, d) \quad \text{or} \quad (a, b, c, d)$$

stands for

$$abcd + ab + ad + cd + 1.$$

(2) The general continuant, K , of the n^{th} order is a function of not more than $2n - 1$ independent variables,— n in the principal diagonal and $n - 1$ in the variable minor diagonal. We thus may be expected to know the differential-quotient of the continuant with respect to an element in either of these diagonals.

Taking first the case where the element is in the minor diagonal, b_h say, we use the known identity

$$\begin{pmatrix} b_1 & \dots & b_{h-1} & b_h & \dots & b_{n-1} \\ a_1 & a_2 & \dots & a_{h-1} & a_h & a_{h+1} & \dots & a_{n-1} & a_n \end{pmatrix} = \begin{pmatrix} b_1 & \dots & b_{h-1} \\ a_1 & a_2 & \dots & a_{h-1} & a_h \end{pmatrix} \cdot \begin{pmatrix} b_{h+1} & \dots & b_{n-1} \\ a_{h+1} & \dots & a_{n-1} & a_n \end{pmatrix} \\ + b_h \begin{pmatrix} b_1 & \dots & b_{h-1} \\ a_1 & a_2 & \dots & a_{h-1} \end{pmatrix} \cdot \begin{pmatrix} b_{h+1} & \dots & b_{n-1} \\ a_{h+2} & \dots & a_{n-1} & a_n \end{pmatrix}$$

to separate out the element in question, and thus see at a glance that

$$\frac{\partial K}{\partial b_h} = \begin{pmatrix} b_1 & \dots & b_{h-1} \\ a_1 & a_2 & \dots & a_{h-1} \end{pmatrix} \begin{pmatrix} b_{h+1} & \dots & b_{n-1} \\ a_{h+2} & \dots & a_{n-1} & a_n \end{pmatrix}.$$

In other words, the differential-quotient of K with respect to b_h is obtained mechanically by taking K , deleting $b_{h-1}, a_h, b_h, a_{h+1}, b_{h+1}$ from it, and inserting the brackets $()$ instead.

Secondly, taking the case where the element is in the principal diagonal, a_h say, we use the same identity at the outset, but go a step further and alter

$$\begin{pmatrix} b_1 & \dots & b_{h-1} \\ a_1 & a_2 & \dots & a_{h-1} & a_h \end{pmatrix} \quad \text{into} \quad \begin{pmatrix} b_1 & \dots & b_{h-1} \\ a_1 & a_2 & \dots & a_{h-2} \end{pmatrix} b_{h-1} + \begin{pmatrix} b_1 & \dots & b_{h-1} \\ a_1 & a_2 & \dots & a_{h-1} \end{pmatrix} a_h$$

for the purpose, as before, of separating out the element concerned, thus obtaining

$$\frac{\partial K}{\partial a_h} = \begin{pmatrix} b_1 & \dots & b_{n-1} \\ a_1 & a_2 & \dots & a_{h-1} \end{pmatrix} \begin{pmatrix} \dots & b_{n-1} \\ a_{h+1} & \dots & a_{n-1} & a_n \end{pmatrix}.$$

In other words, the differential-quotient of K with respect to a_h is obtained mechanically by taking K , deleting b_{h-1} , a_h , b_h , and inserting the brackets $\begin{pmatrix} \dots & b_{n-1} \\ a_{h+1} & \dots & a_{n-1} & a_n \end{pmatrix}$ instead.

When one of the end elements is used as independent variable the result is of course simpler: thus

$$\begin{aligned} \frac{\partial K}{\partial a_1} &= \begin{pmatrix} b_2 & \dots & b_{n-1} \\ a_2 & a_3 & \dots & a_{n-1} & a_n \end{pmatrix}, \\ \frac{\partial K}{\partial b_{n-1}} &= \begin{pmatrix} b_1 & \dots & b_{n-3} \\ a_1 & a_2 & \dots & a_{n-3} & a_{n-2} \end{pmatrix}. \end{aligned}$$

(3) It is thus seen that in whatever diagonal the independent variable is situated the differential-quotient of K is in general the product of two coaxial minors of K , and that when the independent variables are in the same row the differential-quotients have a factor in common. As a result of this

$$\begin{aligned} \frac{\partial K}{\partial a_h} \div \frac{\partial K}{\partial b_h} &= \frac{\begin{pmatrix} \dots & b_{n-1} \\ a_{h+1} & \dots & a_{n-1} & a_n \end{pmatrix}}{\begin{pmatrix} \dots & b_{n-1} \\ a_{h+2} & \dots & a_{n-1} & a_n \end{pmatrix}}, \\ &= a_{h+1} + \frac{b_{h+1}}{a_{h+2}} + \frac{b_{h+2}}{a_{h+3}} + \dots + \frac{b_{n-1}}{a_n}. \end{aligned}$$

(4) If we differentiate $\partial K / \partial a_h$ with respect to another element of the main diagonal, that element must be in one of the two factors of $\partial K / \partial a_h$,

$$\begin{pmatrix} b_1 & \dots & b_{n-1} \\ a_1 & a_2 & \dots & a_{h-1} \end{pmatrix}, \quad \begin{pmatrix} \dots & b_{n-1} \\ a_{h+1} & \dots & a_{n-1} & a_n \end{pmatrix},$$

and not in the other, so that the result of the second differentiation is in general to give an expression of three factors. Thus if

$$K = \begin{pmatrix} a & b & c & d & e & f & g \\ \alpha & \beta & \gamma & \delta & \epsilon & \zeta & \eta \end{pmatrix}$$

we have

$$\frac{\partial^2 K}{\partial c \partial f} = \begin{pmatrix} a \\ \alpha & b \end{pmatrix} \cdot \begin{pmatrix} \delta \\ d & e \end{pmatrix} \cdot g.$$

When the two elements used as independent variables belong to the minor diagonal, the like consequences ensue: it has to be noted however that if they be consecutive the result is 0, because when we differentiate with respect to b_h , neither b_{h-1} nor b_{h+1} appears in the result.

(5) As each term of the continuant is linear in each of the elements occurring in it,

the differential-quotient with respect to any element is the same as the cofactor of that element. Hence instead of

$$\frac{\partial K}{\partial a_h}, \quad \frac{\partial^2 K}{\partial a_h \partial a_{h+1}},$$

we may write

$$\text{cof } a_h, \quad \text{cof } a_h a_{h+1}.$$

(6) When the two elements used as independent variables are consecutive in the main diagonal, the result is the same as is got by one differentiation with respect to an element of the minor diagonal. For, taking $\partial K / \partial a_h$ as above, we have

$$\begin{aligned} \frac{\partial^2 K}{\partial a_h \partial a_{h+1}} &= \begin{pmatrix} b_1 & \dots & \dots \\ a_1 & a_2 & \dots & a_{h-1} \end{pmatrix} \begin{pmatrix} \dots & b_{h-1} \\ a_{h+2} & \dots & a_{h-1} & a_h \end{pmatrix}, \\ &= \frac{\partial K}{\partial b_h}. \end{aligned}$$

Corroboration of this is found in the fact that one of the two-line minors of K is

$$\begin{vmatrix} a_h & b_h \\ -1 & a_{h+1} \end{vmatrix} \quad \text{or} \quad a_h a_{h+1} + b_h,$$

thus ensuring that the cofactor of $a_h a_{h+1}$ is the same as the cofactor of b_h .

(7) When the continuant concerned has unit elements in the minor diagonal the rule for mechanically obtaining the differential-quotient is still simpler, the only element to be deleted being that which is taken as the independent variable. Thus if

$$K = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$$

we have

$$\frac{\partial K}{\partial a_5} = (a_1, a_2, a_3, a_4) (a_6, a_7),$$

and

$$\frac{\partial^2 K}{\partial a_h \partial a_3} = (a_1, a_2) \cdot (a_4) \cdot (a_6, a_7).$$

(8) The foregoing considerations have been suggested by a curious theorem which has turned up recently in the course of an investigation connected with Hessians.

The Hessian, it will be remembered, is always axisymmetric: and, if the originating function be linear in each of the variables, its second differential-quotient with respect to any variable will be 0,—that is to say, the Hessian will be *zero-axial* as well as axisymmetric. In this case, consequently, there arises a semi-quadrant array which seems worthy of study.

(9) Taking the case where the originating function is $K(a, b, c, d)$, i.e. $abcd + ad + cd + ab + 1$ we have

$$\begin{vmatrix} \frac{\partial^2 K}{\partial a \partial b} & \frac{\partial^2 K}{\partial a \partial c} & \frac{\partial^2 K}{\partial a \partial d} \\ \frac{\partial^2 K}{\partial b \partial c} & \frac{\partial^2 K}{\partial b \partial d} & \frac{\partial^2 K}{\partial c \partial d} \end{vmatrix} = \begin{vmatrix} cd+1 & bd & bc+1 \\ & ad & ac \\ & & ab+1 \end{vmatrix},$$

$$\begin{aligned}
&= (cd+1)(ab+1) - bd.ac + (bc+1)ad, \\
&= abcd + ab + cd + 1 - abcd + abcd + ad, \\
&= K(a,b,c,d).
\end{aligned}$$

Again, taking the case where the originating function is

$$K(a,b,c,d,e,f)$$

we have our Hessian-like Pfaffian

$$\begin{aligned}
&\begin{vmatrix} \frac{\partial^2 K}{\partial a \partial b} & \frac{\partial^2 K}{\partial a \partial c} & \frac{\partial^2 K}{\partial a \partial d} & \frac{\partial^2 K}{\partial a \partial e} & \frac{\partial^2 K}{\partial a \partial f} \\ & \frac{\partial^2 K}{\partial b \partial c} & \frac{\partial^2 K}{\partial b \partial d} & \frac{\partial^2 K}{\partial b \partial e} & \frac{\partial^2 K}{\partial b \partial f} \\ & & \frac{\partial^2 K}{\partial c \partial d} & \frac{\partial^2 K}{\partial c \partial e} & \frac{\partial^2 K}{\partial c \partial f} \\ & & & \frac{\partial^2 K}{\partial d \partial e} & \frac{\partial^2 K}{\partial d \partial f} \\ & & & & \frac{\partial^2 K}{\partial e \partial f} \end{vmatrix} \\
&= \begin{vmatrix} (c,d,e,f) & b(d,e,f) & (b,c)(e,f) & (b,c,d)f & (b,c,d,e) \\ & a(d,e,f) & ac(e,f) & a(c,d)f & a(c,d,e) \\ & & (a,b)(e,f) & (a,b)d f & (a,b)(d,e) \\ & & & (a,b,c)f & (a,b,c)e \\ & & & & (a,b,c,d) \end{vmatrix} \\
&= (c,d,e,f) \begin{vmatrix} (a,b)(e,f) & (a,b)d f & (a,b)(d,e) \\ & (a,b,c)f & (a,b,c)e \\ & & (a,b,c,d) \end{vmatrix} \\
&- b(d,e,f) \begin{vmatrix} ac(e,f) & a(c,d)f & a(c,d,e) \\ & (a,b,c)f & (a,b,c)e \\ & & (a,b,c,d) \end{vmatrix} \\
&+ \dots \\
&+ (b,c,d,e) \begin{vmatrix} a(d,e,f) & ac(e,f) & a(c,d)f \\ & (a,b)(e,f) & (a,b)d f \\ & & (a,b,c)f \end{vmatrix}.
\end{aligned}$$

Now it is easily verified that the five minor Pfaffians here are equal to

$$(a,b) \cdot K, \quad ac \cdot K, \quad ad \cdot K, \quad ae \cdot K, \quad af \cdot K;$$

consequently the original Pfaffian

$$\begin{aligned}
&= K \cdot [(a,b)(c,d,e,f) - abc(d,e,f) + a(b,c)d(e,f) - a(b,c,d)ef + a(b,c,d,e)f], \\
&= K \cdot [(a,b)(c,d,e,f) - \{abcd(e,f) + abcf\} + \{abcd(e,f) + ad(e,f)\} - a(b,c,d)ef + \{a(b,c,d)ef + a(b,c)f\}], \\
&= K \cdot [(a,b)(c,d,e,f) - abcf + ad(e,f) + a(b,c)f], \\
&= K \cdot [(a,b)(c,d,e,f) + ad(e,f) + af], \\
&= K \cdot (a,b,c,d,e,f), \\
&= K^2.
\end{aligned}$$

(10) These two cases raise the presumption that when the originating function is

$$K(a_1, a_2, a_3, \dots, a_{2n})$$

the semi-Hessian of K will be

$$K^{n-1}.$$

It is clear however that before we can test this the auxiliary theorems which we have used in proving the first two cases must be generalised.

(11) The first of these auxiliary theorems is that which involves the fact that all our minor Pfaffians of the second order contain K as a factor. What the theorem itself actually is becomes more apparent if we adopt the 'cofactor' notation instead of the notation of differential-quotients: for then the instances are—

$$\begin{vmatrix} \text{cof } cd & \text{cof } ce & \text{cof } cf \\ & \text{cof } de & \text{cof } df \\ & & \text{cof } ef \end{vmatrix} = K \cdot (a, b),$$

$$\begin{vmatrix} \text{cof } bd & \text{cof } be & \text{cof } bf \\ & \text{cof } de & \text{cof } df \\ & & \text{cof } ef \end{vmatrix} = K \cdot ac,$$

$$\dots \dots \dots ;$$

or

$$\begin{aligned} \text{cof } cd \cdot \text{cof } ef &- \text{cof } ce \cdot \text{cof } df + \text{cof } cf \cdot \text{cof } de = K \cdot (a, b), \\ \text{cof } bd \cdot \text{cof } ef &- \text{cof } be \cdot \text{cof } df + \text{cof } bf \cdot \text{cof } de = K \cdot ac, \\ &\dots \dots \dots \end{aligned}$$

(12) The general theorem here exemplified is—

If K be a simple continuant of any number of elements $a_1, a_2, \dots, a_n$, and a, β, γ, δ be the suffixes of any four elements taken in the order in which they occur, then

$$\begin{aligned} \text{cof } a_a a_\beta \cdot \text{cof } a_\gamma a_\delta &- \text{cof } a_a a_\gamma \cdot \text{cof } a_\beta a_\delta + \text{cof } a_a a_\delta \cdot \text{cof } a_\beta a_\gamma \\ &= K \cdot \text{cof } a_a a_\beta a_\gamma a_\delta. \end{aligned}$$

In proof of this we note in the first place that the expression on the left

$$\begin{aligned} &= (a_1, \dots, a_{a-1}) (a_{a+1}, \dots, a_{\beta-1}) (a_{\beta+1}, \dots, a_n) \cdot (a_1, \dots, a_{\gamma-1}) (a_{\gamma+1}, \dots, a_{\delta-1}) (a_{\delta+1}, \dots, a_n) \\ &- (a_1, \dots, a_{a-1}) (a_{a+1}, \dots, a_{\gamma-1}) (a_{\gamma+1}, \dots, a_n) \cdot (a_1, \dots, a_{\beta-1}) (a_{\beta+1}, \dots, a_{\delta-1}) (a_{\delta+1}, \dots, a_n) \\ &+ (a_1, \dots, a_{a-1}) (a_{a+1}, \dots, a_{\delta-1}) (a_{\delta+1}, \dots, a_n) \cdot (a_1, \dots, a_{\beta-1}) (a_{\beta+1}, \dots, a_{\gamma-1}) (a_{\gamma+1}, \dots, a_n), \end{aligned}$$

which, on account of there being two factors common to all the terms,

$$\begin{aligned} &= (a_1, \dots, a_{a-1}) (a_{\delta+1}, \dots, a_n) [(a_{a+1}, \dots, a_{\beta-1}) (a_{\beta+1}, \dots, a_n) (a_1, \dots, a_{\gamma-1}) (a_{\gamma+1}, \dots, a_{\delta-1}) \\ &- (a_{a+1}, \dots, a_{\gamma-1}) (a_{\gamma+1}, \dots, a_n) (a_1, \dots, a_{\beta-1}) (a_{\beta+1}, \dots, a_{\delta-1}) \\ &+ (a_{a+1}, \dots, a_{\delta-1}) (a_1, \dots, a_{\beta-1}) (a_{\beta+1}, \dots, a_{\gamma-1}) (a_{\gamma+1}, \dots, a_n)]. \end{aligned}$$

Again, the last two terms inside the rectangular brackets here have two factors in common, their aggregate thus being

$$= (a_1, \dots, a_{\beta-1}) (a_{\gamma+1}, \dots, a_n) \{ - (a_{a+1}, \dots, a_{\gamma-1}) (a_{\beta+1}, \dots, a_{\delta-1}) + (a_{a+1}, \dots, a_{\delta-1}) (a_{\beta+1}, \dots, a_{\gamma-1}) \},$$

which, by an extension of the general theorem used in § 2

$$= (a_1, \dots, a_{\beta-1}) (a_{\gamma+1}, \dots, a_n) \cdot (-1)^{\gamma-\beta+1} (a_{a+1}, \dots, a_{\beta-1}) (a_{\gamma+1}, \dots, a_{\delta-1});$$

and as this has two factors in common with the first term inside the rectangular brackets, the original expression takes the form

$$(a_1, \dots, a_{\alpha-1})(a_{\delta+1}, \dots, a_n)(a_{\alpha+1}, \dots, a_{\beta-1})(a_{\gamma+1}, \dots, a_{\delta-1}) \\ \cdot \{(a_{\beta+1}, \dots, a_n)(a_1, \dots, a_{\gamma-1}) + (-1)^{\gamma-\beta+1}(a_1, \dots, a_{\beta-1})(a_{\gamma+1}, \dots, a_n)\}.$$

Making a second use of the above-mentioned general theorem we can substitute

$$(a_1, \dots, a_n)(a_{\beta+1}, \dots, a_{\gamma-1})$$

for the expression inside the double-curved brackets. The original expression is thus resolved into six factors, one of which is K, and the others

$$(a_1, \dots, a_{\alpha-1}), (a_{\alpha+1}, \dots, a_{\beta-1}), (a_{\beta+1}, \dots, a_{\gamma-1}), (a_{\gamma+1}, \dots, a_{\delta-1}), (a_{\delta+1}, \dots, a_n),$$

the product of which, from §§ 4, 5, we know to be

$$\text{cof } a_{\alpha} a_{\beta} a_{\gamma} a_{\delta}.$$

(13) The next auxiliary theorem used in § 9 is that which affirms that

$$(a, b)(c, d) - abcd + a(b, c)d = (a, b, c, d)^*$$

and

$$(a, b)(c, d, e, f) - abc(d, e, f) + a(b, c)d(e, f) - a(b, c, d)ef + a(b, c, d, e)f = (a, b, c, d, e, f);$$

or that

$$\text{cof } ab \cdot \text{cof } cd - \text{cof } ac \cdot \text{cof } bd + \text{cof } ad \cdot \text{cof } bc = (a, b, c, d)$$

and

$$\left. \begin{aligned} & \text{cof } ab \cdot \text{cof } cdef \\ & - \text{cof } ac \cdot \text{cof } bdef \\ & + \text{cof } ad \cdot \text{cof } becf \\ & - \text{cof } ae \cdot \text{cof } bcdf \\ & + \text{cof } af \cdot \text{cof } bcde \end{aligned} \right\} = (a, b, c, d, e, f).$$

The general theorem which includes these concerns $(a_1, a_2, \dots, a_{2n})$, and is

$$\left. \begin{aligned} & \text{cof } a_1 a_2 \cdot \text{cof } a_3 a_4 a_5 \dots a_{2n} \\ & - \text{cof } a_1 a_3 \cdot \text{cof } a_2 a_4 a_5 \dots a_{2n} \\ & + \text{cof } a_1 a_4 \cdot \text{cof } a_2 a_3 a_5 \dots a_{2n} \\ & - \dots \dots \dots \end{aligned} \right\} = (a_1, a_2, \dots, a_{2n});$$

and there is a corresponding theorem for the case where the number of elements is odd, viz.,

$$\left. \begin{aligned} & \text{cof } a_1 a_2 \cdot \text{cof } a_3 a_4 a_5 \dots a_{2n+1} \\ & - \text{cof } a_1 a_3 \cdot \text{cof } a_2 a_4 a_5 \dots a_{2n+1} \\ & + \dots \dots \dots \end{aligned} \right\} = (a_1, a_2, \dots, a_{2n+1}).$$

(14) The two theorems may therefore be enunciated together thus:—

If $\text{cof } x$ denote the cofactor of x in the continuant $K(a_1, a_2, \dots, a_n)$, then

$$\sum (-1)^r \text{cof } a_1 a_r \cdot \text{cof } a_1 a_2 \dots a_n / a_1 a_r = K, \\ \text{or} = \text{cof } a_1 a_2,$$

according as n is even or odd.

* It should be noted that this is also a case of the first auxiliary theorem.

By way of proof we note that the first of the $n-1$ terms on the left is

$$(a_1, a_2)(a_3, a_4, \dots, a_n),$$

and that therefore we may write instead of it,

$$K - a_1(a_2, \dots, a_n).$$

Then, as the second of the $n-1$ terms is

$$-a_1 a_2 a_3(a_4, \dots, a_n)$$

the aggregate of the first two is

$$K - a_1(a_2, a_3)(a_4, \dots, a_n).$$

Similarly, the third term having the factor (a_2, a_3) in common with this aggregate, the aggregate of the first three is found to be

$$K - a_1(a_2, a_3)(a_4, \dots, a_n).$$

This process being continued it is seen that the aggregates of odd numbers of terms follow one law, and the aggregates of even numbers another law, viz.,

No. of Terms.	Aggregate.	No. of Terms.	Aggregate.
1	$K - a_1(a_2, \dots, a_n),$	2	$K - a_1(a_2, a_3)(a_4, \dots, a_n),$
3	$K - a_1(a_2, a_3)(a_4, \dots, a_n),$	4	$K - a_1(a_2, \dots, a_5)(a_6, \dots, a_n),$
5	$K - a_1(a_2, \dots, a_5)(a_6, \dots, a_n),$	6	$K - a_1(a_2, \dots, a_7)(a_8, \dots, a_n),$
.			

From the first column we learn that when n is odd the aggregate of the $n-1$ terms is

$$K - a_1(a_2, \dots, a_n) \\ \text{i.e., } (a_3, \dots, a_n),$$

as was to be proved.

Also, from the second column it is seen that when n is even the aggregate of $n-2$ terms (i.e., all the terms except the last) is

$$K - a_1(a_2, \dots, a_{n-1})a_n$$

to which if we add the last, viz., $+a_1(a_2, \dots, a_{n-1})a_n$, we obtain the aggregate

$$K$$

as desired.

(15) There is however a more elementary theorem which leads easily up to this, and which can be more readily proved in the same way. It has to be noted too, that from some points of view it is of greater interest than that to which it leads. It is:—

If cof a_r stand for the cofactor of a_r in the continuant $K(a_1, a_2, \dots, a_n)$, then

$$a_1 \text{ cof } a_1 - a_2 \text{ cof } a_2 + a_3 \text{ cof } a_3 - \dots = K \text{ or } 0$$

according as n is odd or even.

To condense the proof we may take advantage of the fact that $(a_1, \dots, a_n) = (a_n, \dots, a_1)$, and sum the half of the terms from the beginning and the half from the end. Thus, when n is even, equal to $2m$, the aggregate of the first m terms is found to be

$$K - (a_1, \dots, a_m)(a_{m+1}, \dots, a_{2m}) \quad \text{or} \quad K - (a_1, \dots, a_{m-1})(a_{m+2}, \dots, a_{2m}),$$

(17) We are now prepared to show that

$$\begin{vmatrix} (a,b) & ac & ad & ae & af \\ & (b,c) & bd & be & bf \\ & & (c,d) & ce & cf \\ & & & (d,e) & df \\ & & & & (e,f) \end{vmatrix} = (a,b,c,d,e,f).$$

For, expanding the Pfaffian in terms of the elements of the first frame-line and their complementary minors, we have

$$\begin{aligned} (a,b) \begin{vmatrix} (c,d) & ce & cf \\ (d,e) & df \\ (e,f) \end{vmatrix} &- ac \begin{vmatrix} bd & be & bf \\ (d,e) & df \\ (e,f) \end{vmatrix} + ad \begin{vmatrix} (b,c) & be & bf \\ & ce & cf \\ & (e,f) \end{vmatrix} \\ &- ae \begin{vmatrix} (b,c) & bd & bf \\ & (c,d) & cf \\ & & df \end{vmatrix} + af \begin{vmatrix} (b,c) & bd & be \\ & (c,d) & ce \\ & & (d,e) \end{vmatrix}, \end{aligned}$$

which by § 16 is equal to

$$(a,b)(c,d,e,f) - ac \cdot b(d,e,f) + ad \cdot (b,c)(e,f) - ae \cdot (b,c,d)f + af(b,c,d,e),$$

and therefore by § 14 equal to

$$(a,b,c,d,e,f).$$

(18) The effect of changing (a,b) into ab on the left of the preceding identity is to subtract

$$\begin{vmatrix} (c,d) & ce & cf \\ (d,e) & df \\ (e,f) \end{vmatrix} \quad \text{i.e., } (c,d,e,f),$$

and as the right-hand side equals

$$a(b,c,d,e,f) + (c,d,e,f),$$

the result is

$$a(b,c,d,e,f).$$

The other changes taken in order lead to

$$\begin{aligned} &(a,b)(c,d,e,f), \\ &(a,b,c)(d,e,f), \\ &(a,b,c,d)(e,f), \\ &(a,b,c,d,e)f. \end{aligned}$$

(19) In this way we reach the perfectly general theorem*

$$\begin{vmatrix} (a_1, a_2) & a_1 a_3 & a_1 a_4 & \dots & a_1 a_{2n} \\ & (a_2, a_3) & a_2 a_4 & \dots & a_2 a_{2n} \\ & & (a_3, a_4) & \dots & a_3 a_{2n} \\ & & & \dots & \dots \\ & & & & (a_{2n-1}, a_{2n}) \end{vmatrix} = (a_1, a_2, \dots, a_{2n}),$$

with which is connected the series of identities resulting from the change of any one of

* It is worth noting that the elements of the Pfaffian are the two-line coaxial minors of the equivalent continuant.

the hypotenuse elements from (a_r, a_{r+1}) into $a_r a_{r+1}$, the corresponding change on the right-hand being from $(a_1, a_2, \dots, a_r, a_{r+1}, \dots, a_{2n})$ into

$$(a_1, a_2, \dots, a_r)(a_{r+1}, \dots, a_{2n}).$$

(20) Now as the Law of Complementaries holds in regard to Pfaffians as well as in regard to determinants, let us apply it to find the Complementary of the general theorem just reached. Instead of each element of the Pfaffian we have to substitute its complementary minor; and as the value of the Pfaffian is $(a_1, a_2, \dots, a_{2n})$ the complementary minor of any element is nothing else than its cofactor in $(a_1, a_2, \dots, a_{2n})$. On the right-hand side the complementary minor of $(a_1, a_2, \dots, a_{2n})$ is 1, and therefore we have to annex to it a power of $(a_1, a_2, \dots, a_{2n})$ of the same degree as the altered Pfaffian on the left, viz., the degree $n(2n-2)$. We thus have the theorem

$$\begin{vmatrix} \text{cof}(a_1, a_2) & \text{cof } a_1 a_3 & \text{cof } a_1 a_4 & \dots & \text{cof } a_1 a_{2n} \\ & \text{cof}(a_2, a_3) & \text{cof } a_2 a_4 & \dots & \text{cof } a_2 a_{2n} \\ & & \text{cof}(a_3, a_4) & \dots & \text{cof } a_3 a_{2n} \\ & & & \dots & \\ & & & & \text{cof}(a_{2n-1}, a_{2n}) \end{vmatrix} = (a_1, a_2, \dots, a_{2n})^{n-1},$$

which is the generalisation surmised in § 10 to exist by reason of the two cases established in § 9.

(21) Now, however, that we have accomplished the purpose with which we started, it is important to notice that the fundamental theorem of the whole is that given in § 19. Any other mode of establishing it is thus of interest; and the known quantitative relation between Pfaffians and determinants suggests that by substituting determinants for squares of Pfaffians, the theorem in determinants thus resulting may be easy of proof,—in other words, that instead of proving, for example, that

$$\begin{vmatrix} (a,b) & ac & ad \\ (b,c) & bd \\ (c,d) \end{vmatrix} = (a,b,c,d),$$

we may seek to prove the identity resulting from this by squaring both sides, viz., the identity

$$\begin{vmatrix} . & (a,b) & ac & ad \\ -(a,b) & . & (b,c) & bd \\ -ac & -(b,c) & . & (c,d) \\ -ad & bd & -(c,d) & . \end{vmatrix} = \begin{vmatrix} a & 1 & . & . \\ -1 & b & 1 & . \\ . & -1 & c & 1 \\ . & . & -1 & d \end{vmatrix}^2.$$

(22) Beginning then on the left-hand side, but taking a determinant of higher order so as to leave no doubt of the generality of the process, viz., the determinant

$$\begin{vmatrix} . & (a,b) & ac & ad & ae & af \\ -(a,b) & . & (b,c) & bd & be & bf \\ -ac & -(b,c) & . & (c,d) & ce & cf \\ -ad & -bd & -(c,d) & . & (d,e) & df \\ -ae & -be & -ce & -(d,e) & . & (e,f) \\ -af & -bf & -cf & -df & -(e,f) & . \end{vmatrix}$$

let us first perform on it the operations

$$\begin{aligned}(\text{row})_1 &= \frac{a}{b} (\text{row})_2, \\(\text{row})_2 &= \frac{b}{c} (\text{row})_3, \\&\dots \dots \dots \\(\text{row})_5 &= \frac{e}{f} (\text{row})_6,\end{aligned}$$

and then, to preserve the skewness, the corresponding operations

$$\begin{aligned}(\text{col})_1 &= \frac{a}{b} (\text{col})_2, \\(\text{col})_2 &= \frac{b}{c} (\text{col})_3, \\&\dots \dots \dots \\(\text{col})_5 &= \frac{e}{f} (\text{col})_6.\end{aligned}$$

The result of this is a zero-axial skew determinant which is the square of

$$\begin{vmatrix} \frac{(a,b,c)}{c} & -\frac{a}{b} & . & . & . \\ & \frac{(b,c,d)}{d} & -\frac{b}{c} & . & . \\ & & \frac{(c,d,e)}{e} & -\frac{c}{d} & . \\ & & & \frac{(d,e,f)}{f} & -\frac{d}{e} \\ & & & & \frac{(e,f)}{1} \end{vmatrix}.$$

Developing the Pfaffian in terms of the elements of the first frame-line and their complementary minors, we see that it

$$\begin{aligned}&= \frac{(a,b,c)}{c} \left\{ \frac{(c,d,e)}{e} \cdot (e,f) - \frac{c}{e} \right\} - \frac{a}{b} \cdot \frac{b}{c} \cdot \frac{(e,f)}{1} \\&= \frac{(a,b,c)}{c} \cdot (c,d,e,f) - \frac{a}{c} (e,f) \\&= (a,b,c,d,e,f),\end{aligned}$$

as was to be proved.

(23) If, on the other hand, we begin with $(a, b, c, d, e, f)^2$ we must try to transform its equal factors in such a way that the subsequent application of the multiplication-theorem may produce the left-hand member.

Writing the factors in the form

$$\begin{vmatrix} a & 1 & . & . & . & . \\ -1 & b & 1 & . & . & . \\ . & -1 & c & 1 & . & . \\ . & . & -1 & d & 1 & . \\ . & . & . & -1 & e & 1 \\ . & . & . & . & -1 & f \end{vmatrix}, \begin{vmatrix} 1 & -a & . & . & . & . \\ b & 1 & . & -1 & . & . \\ -1 & . & 1 & -c & . & . \\ . & . & d & 1 & . & -1 \\ . & . & -1 & . & 1 & -e \\ . & . & . & . & f & 1 \end{vmatrix},$$

the second being got from the first by changing the signs of the 1st, 3rd, 5th columns and then exchanging the latter with the 2nd, 4th, 6th columns respectively, we find the product to be

$$\begin{vmatrix} . & ab+1 & -a & . & . & . \\ -ab-1 & . & 2 & d & -1 & . \\ a & -2 & . & cd+1 & -c & . \\ . & -d & -cd-1 & . & 2 & f \\ . & 1 & c & -2 & . & ef+1 \\ . & . & . & -f & -ef-1 & . \end{vmatrix}.$$

This is zero-axial skew, and a fair approximation to the form required; but the further changes necessary are too many in number to be worth noting.