The first and last of these being satisfied, the second and third are satisfied identically. Divide the first and last by $\alpha^3\Omega$, and make

$$\frac{\kappa}{\Omega} = \Omega \quad \text{and} \quad \frac{\beta}{\alpha} = \alpha; \quad \text{whence}$$

$$\alpha^3 = \Omega,$$

$$\kappa^3(2\alpha + 1) = \Omega(\alpha^3 + 2\alpha).$$

Therefore equation (II.) is again divisible by

$$u^4 - v^4 + 2uv(1 - u^2v^2).$$

But we observe that equation (II.) is symmetrical in u and v , and therefore, as it is divisible by

$$\{u^4 - v^4 + 2uv(1 - u^2v^2)\}^2,$$

it must also be divisible by

$$\{v^4 - u^4 + 2uv(1 - u^2v^2)\}^2.$$

Then, actually performing the division, we find equation (II.) to be

$$\begin{aligned} &\{u^4 - v^4 + 2uv(1 - u^2v^2)\}^2 \{v^4 - u^4 + 2uv(1 - u^2v^2)\}^2 \\ &\times \{(1 - u^8)(1 - v^8) - (1 - uv)^8\} = 0. \end{aligned}$$

This is the product of five factors, and, as the first four belong to cubic transformations, the last only can belong to the septic transformation. We have thus, as the modular equation for the septic transformation,

$$(1 - u^8)(1 - v^8) = (1 - uv)^8,$$

a result which I believe has never been obtained before algebraically.

On a Symmetric Determinant connected with Lagrange's Interpolation Problem. By THOMAS MUIR, M.A., F.R.S.E.

[Read April 6th, 1882.]

1. In Lagrange's interpolation-problem it is required to find the values of $A_1, A_2, \dots, A_n$, so that

$$A_1 \sin x + A_2 \sin 2x + \dots + A_n \sin nx$$

may coincide in value with an assigned function when x has the values $\pi/(n+1), 2\pi/(n+1), \dots, n\pi/(n+1)$. The data lead to a set of n equa-

tions, the determinant of which is

$$\begin{vmatrix} \sin \frac{\pi}{n+1}, & \sin \frac{2\pi}{n+1}, & \sin \frac{3\pi}{n+1}, & \dots, & \sin \frac{n\pi}{n+1} \\ \sin \frac{2\pi}{n+1}, & \sin \frac{4\pi}{n+1}, & \sin \frac{6\pi}{n+1}, & \dots, & \sin \frac{2n\pi}{n+1} \\ \sin \frac{3\pi}{n+1}, & \sin \frac{6\pi}{n+1}, & \sin \frac{9\pi}{n+1}, & \dots, & \sin \frac{3n\pi}{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ \sin \frac{n\pi}{n+1}, & \sin \frac{2n\pi}{n+1}, & \sin \frac{3n\pi}{n+1}, & \dots, & \sin \frac{n^2\pi}{n+1} \end{vmatrix}$$

$$\text{or} \quad \begin{vmatrix} \sin \frac{1 \cdot 1\pi}{n+1}, & \sin \frac{2 \cdot 2\pi}{n+1}, & \sin \frac{3 \cdot 3\pi}{n+1}, & \dots, & \sin \frac{n \cdot n\pi}{n+1} \end{vmatrix}$$

Certain of the properties of this symmetric determinant, L_n say, seem well worthy of attention.

2. It is readily made apparent that the number of different elements in the determinant—different, that is to say, in magnitude—is $\frac{1}{2}n$ or $\frac{1}{2}(n+3)$, according as n is even or odd; also, that with respect to the secondary diagonal the determinant is alternately symmetric and skew; and that with respect to the bisecting line parallel to the columns it has the same property.

3. Multiplying each element of the s^{th} column by $\sin \frac{rs\pi}{n+1}$, and increasing the product by $\sin \frac{r(s-1)\pi}{n+1}$ times the corresponding element of the $(s-1)^{\text{th}}$ column, by $\sin \frac{r(s-2)\pi}{n+1}$ times the corresponding element of the $(s-2)^{\text{th}}$ column, and so on with the corresponding elements of all the other columns, we obtain a new determinant equal to

$$L_n \times \sin \frac{rs\pi}{n+1}.$$

This new determinant, however, has all the elements of its s^{th} column equal to zero, except the element in the r^{th} row, which equals $\frac{1}{2}(n+1)$, in virtue of the trigonometrical theorem that

$$\sin \frac{r\pi}{n+1} \sin \frac{s\pi}{n+1} + \sin \frac{2r\pi}{n+1} \sin \frac{2s\pi}{n+1} + \dots + \sin \frac{nr\pi}{n+1} \sin \frac{ns\pi}{n+1}$$

has the value 0 or $\frac{1}{2}(n+1)$ according as r is unequal or equal to s . It thus results that

$$L_n \times \sin \frac{rs\pi}{n+1} = \frac{1}{2}(n+1) \times \text{cofactor of } \sin \frac{rs\pi}{n+1},$$

and therefore that
$$\frac{\sin \frac{rs\pi}{n+1}}{\text{cofactor of } \sin \frac{rs\pi}{n+1}} = \frac{\frac{1}{2}(n+1)}{L_n} \dots\dots\dots (A),$$

that is to say, is equal to a constant with respect to r and s . Consequently we have the theorem—

Any two elements of

$$\left| \sin \frac{1.1\pi}{n+1}, \sin \frac{2.2\pi}{n+1}, \dots, \sin \frac{n.n\pi}{n+1} \right|$$

are proportional to their cofactors in the determinant (A').

4. Multiplying L_n by itself, and using the trigonometrical theorem before mentioned, we obtain a determinant having each element in the principal diagonal equal to $\frac{1}{2}(n+1)$, and every other element equal to 0. Hence

$$(L_n)^2 = \left\{ \frac{1}{2}(n+1) \right\}^n,$$

and therefore

$$\left| \sin \frac{1.1\pi}{n+1}, \sin \frac{2.2\pi}{n+1}, \dots, \sin \frac{n.n\pi}{n+1} \right| = \left\{ \frac{1}{2}(n+1) \right\}^{1/2} \dots (B).$$

The sign-factor to be taken on the right, when the square root is extracted, will be seen from what follows to be $(-1)^{1/2 n(n-1)}$.

From this and (A) we have the evident deduction that the complementary minor of any element, $\sin \frac{rs\pi}{n+1}$, in L_n is equal to

$$(-1)^{1/2 n(n-1)+r+s} \sin \frac{rs\pi}{n+1} \cdot \left\{ \frac{1}{2}(n+1) \right\}^{1/2 n-1}.$$

5. Taking now a determinant of the $2m^{\text{th}}$ order, and increasing each element of the first row by the corresponding element of the last row, each element of the second row by the corresponding element of the second row from the end, and so on, we have

$$L_{2m} = \begin{vmatrix} 2 \sin \frac{\pi}{2m+1}, & 0, & 2 \sin \frac{3\pi}{2m+1}, & \dots, & 0 \\ 2 \sin \frac{2\pi}{2m+1}, & 0, & 2 \sin \frac{6\pi}{2m+1}, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 2 \sin \frac{m\pi}{2m+1}, & 0, & 2 \sin \frac{3m\pi}{2m+1}, & \dots, & 0 \\ \sin \frac{(m+1)\pi}{2m+1}, & \sin \frac{(m+1)2\pi}{2m+1}, & \sin \frac{(m+1)3\pi}{2m+1}, & \dots, & \sin \frac{(m+1)2m\pi}{2m+1} \\ \dots & \dots & \dots & \dots & \dots \\ \sin \frac{2m\pi}{2m+1}, & \sin \frac{2m.2\pi}{2m+1}, & \sin \frac{2m.3\pi}{2m+1}, & \dots, & \sin \frac{2m.2m\pi}{2m+1} \end{vmatrix},$$

$$\begin{aligned}
 & \text{for } \sin \frac{rs\pi}{2m+1} + \sin \frac{(2m-r+1)s\pi}{2m+1} \\
 &= \sin \frac{rs\pi}{2m+1} + (-1)^{r-1} \sin \left(s\pi - \frac{2m-r+1}{2m+1} s\pi \right) \\
 &= \sin \frac{rs\pi}{2m+1} + (-1)^{r-1} \sin \frac{rs\pi}{2m+1} \\
 &= 2 \sin \frac{rs\pi}{2m+1} \text{ or } 0.
 \end{aligned}$$

according as s is odd or even. This determinant, however, is expressible as the product of two complementary minors, viz.,

$$\begin{aligned}
 L_{2m} &= (-1)^{im(3m+1)} 2^m \begin{vmatrix} \sin \frac{\pi}{2m+1}, & \sin \frac{3\pi}{2m+1}, & \dots, & \sin \frac{(2m-1)\pi}{2m+1} \\ \sin \frac{2\pi}{2m+1}, & \sin \frac{6\pi}{2m+1}, & \dots, & \sin \frac{(2m-1)2\pi}{2m+1} \\ \dots & \dots & \dots & \dots \\ \sin \frac{m\pi}{2m+1}, & \sin \frac{3m\pi}{2m+1}, & \dots, & \sin \frac{(2m-1)m\pi}{2m+1} \end{vmatrix} \\
 &\times \begin{vmatrix} \sin \frac{(m+1)2\pi}{2m+1}, & \sin \frac{(m+1)4\pi}{2m+1}, & \dots, & \sin \frac{(m+1)2m\pi}{2m+1} \\ \sin \frac{(m+2)2\pi}{2m+1}, & \sin \frac{(m+2)4\pi}{2m+1}, & \dots, & \sin \frac{(m+2)2m\pi}{2m+1} \\ \dots & \dots & \dots & \dots \\ \sin \frac{2m \cdot 2\pi}{2m+1}, & \sin \frac{2m \cdot 4\pi}{2m+1}, & \dots, & \sin \frac{2m \cdot 2m\pi}{2m+1} \end{vmatrix}.
 \end{aligned}$$

But the element in the s^{th} row and r^{th} column of the second determinant

$$\begin{aligned}
 \text{here} \quad &= \sin \frac{(m+s)2r\pi}{2m+1} \\
 &= (-1)^{r-1} \sin \left(r\pi - \frac{m+s}{2m+1} 2r\pi \right) \\
 &= (-1)^{r-1} \sin \frac{(2s-1)r\pi}{2m+1},
 \end{aligned}$$

that is to say, is equal to the element in the s^{th} column and r^{th} row of the first determinant when r is even, and differs from the said element only in sign when r is odd. Altering, therefore, the signs of all the elements in the odd-numbered columns of the second determinant, that

is, multiplying it by $(-1)^{im(m+1)}$, we have

$$L_{2m} = (-1)^m 2^m \begin{vmatrix} \sin \frac{\pi}{2m+1}, & \sin \frac{3\pi}{2m+1}, & \dots, & \sin \frac{(2m-1)\pi}{2m+1} \\ \sin \frac{2\pi}{2m+1}, & \sin \frac{6\pi}{2m+1}, & \dots, & \sin \frac{(2m-1)2\pi}{2m+1} \\ \dots & \dots & \dots & \dots \\ \sin \frac{m\pi}{2m+1}, & \sin \frac{3m\pi}{2m+1}, & \dots, & \sin \frac{(2m-1)m\pi}{2m+1} \end{vmatrix}^2$$

$$\text{or} \quad \begin{vmatrix} \sin \frac{1.1\pi}{2m+1}, & \sin \frac{2.2\pi}{2m+1}, & \dots, & \sin \frac{2m.2m\pi}{2m+1} \end{vmatrix} \\ = (-1)^m 2^m \begin{vmatrix} \sin \frac{1.1\pi}{2m+1}, & \sin \frac{2.3\pi}{2m+1}, & \dots, & \sin \frac{m(2m-1)\pi}{2m+1} \end{vmatrix}^2 \dots (C).$$

From this result and (B) we obtain another, viz.,

$$\begin{vmatrix} \sin \frac{1.1\pi}{2m+1}, & \sin \frac{2.3\pi}{2m+1}, & \dots, & \sin \frac{m(2m-1)\pi}{2m+1} \end{vmatrix} \\ = \left\{ \frac{1}{2}(2m+1) \right\}^{im} \dots \dots \dots (D).$$

6. If now we take the determinant in the right-hand member of (C), and reverse the order of the columns,—a process equivalent to multiplying the determinant by $(-1)^{im(m-1)}$,—the element in the r^{th} row and s^{th} column will be

$$\sin \frac{r(2m-2s+1)\pi}{2m+1}.$$

But this is equal to

$$(-1)^{r-1} \sin \left\{ r\pi - \frac{r(2m-2s+1)\pi}{2m+1} \right\} \\ = (-1)^{r-1} \sin \frac{rs.2\pi}{2m+1}.$$

Substituting the latter form in place of the original, and altering the signs of all the elements in the even-numbered rows, that is, multiplying again by $(-1)^{im(m-1)}$, we obtain the curious identity

$$\begin{vmatrix} \sin \frac{1.1\pi}{2m+1}, & \sin \frac{2.3\pi}{2m+1}, & \dots, & \sin \frac{m(2m-1)\pi}{2m+1} \end{vmatrix} \\ = \begin{vmatrix} \sin \frac{1.1.2\pi}{2m+1}, & \sin \frac{2.2.2\pi}{2m+1}, & \dots, & \sin \frac{m.m.2\pi}{2m+1} \end{vmatrix} \dots \dots (E).$$

The value of the new determinant on the right is given by (D), and (C) assumes a more elegant form by the substitution which is now possible.

7. The determinant L_n is resolvable into two determinant factors in the case also when n is odd. Little interest, however, attaches to this case in comparison with the other.

8. The investigation of the determinant which differs from L_n in having *cosines* everywhere instead of *sines* leads to results more or less closely analogous with the foregoing. Thus we have

$$\left\{ \begin{array}{l} \left| \cos \frac{1 \cdot 1 \pi}{n+1}, \cos \frac{2 \cdot 2 \pi}{n+1}, \dots, \cos \frac{n \cdot n \pi}{n+1} \right| \\ = (-1)^{\frac{n}{2}} \left\{ \frac{1}{2} (n+1) \right\}^{\frac{n-1}{2}} \quad (n \text{ even}) \\ = 0 \quad (n \text{ odd}) \end{array} \right\} \dots (F),$$

$$\left| \cos \frac{1 \cdot 1 \pi}{2m+1}, \cos \frac{2 \cdot 2 \pi}{2m+1}, \dots, \cos \frac{2m \cdot 2m \pi}{2m+1} \right| \\ = (-1)^m 2^m \left| \cos \frac{1 \cdot 1 \pi}{2m+1}, \cos \frac{2 \cdot 3 \pi}{2m+1}, \dots, \cos \frac{m(2m-1) \pi}{2m+1} \right|^2 \dots (G),$$

$$\left| \cos \frac{1 \cdot 1 \pi}{2m+1}, \cos \frac{2 \cdot 3 \pi}{2m+1}, \dots, \cos \frac{m(2m-1) \pi}{2m+1} \right| \\ = \frac{1}{2} \left\{ \frac{1}{2} (2m+1) \right\}^{\frac{m-1}{2}} \dots (H),$$

$$\text{and} \quad \left| \cos \frac{1 \cdot 1 \pi}{2m+1}, \cos \frac{2 \cdot 3 \pi}{2m+1}, \dots, \cos \frac{m(2m-1) \pi}{2m+1} \right| \\ = (-1)^m \left| \cos \frac{1 \cdot 1 \cdot 2 \pi}{2m+1}, \cos \frac{2 \cdot 2 \cdot 2 \pi}{2m+1}, \dots, \cos \frac{m \cdot m \cdot 2 \pi}{2m+1} \right| \dots (I).$$

Note on the Condensation of Skew Determinants, which are partially zero-axial. By THOMAS MUIR, M.A., F.R.S.E.

[Read April 6th, 1882.]

1. Taking, in the first place, a skew determinant having a zero only in the last place of the principal diagonal, *e.g.*,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ -a_2 & a_1 & b_3 & b_4 & b_5 & b_6 \\ -a_3 & -b_3 & a_2 & c_4 & c_5 & c_6 \\ -a_4 & -b_4 & -c_4 & a_4 & d_5 & d_6 \\ -a_5 & -b_5 & -c_5 & -d_5 & a_5 & e_6 \\ -a_6 & -b_6 & -c_6 & -d_6 & -e_6 & 0 \end{vmatrix}, \text{ or } D_1,$$