XXII.—*On the Partition of Energy between the Translatory and Rotational Motions of a Set of Non-Homogeneous Elastic Spheres.* By Professor W. BURNSIDE.

(Read July 18, 1887.)

At the suggestion of Prof. TAIT, an attempt has been made in this paper to apply the method used by him in § 21 of his paper on “The Foundations of the Kinetic Theory of Gases” to a case of the question of the distribution of energy in a system of non-homogeneous impinging spheres.

The problem may be stated as follows:—Given a very great number of smooth elastic spheres, equal and like in all respects, whose centres of figure and centres of inertia do not coincide, and the sum of whose volumes is but a small fraction of the space in which they move, it is required to find the ultimate distribution of energy among the various degrees of freedom when by collisions the system has attained a “special state.”

The received result for the general problem, of which this is a comparatively simple case, is that the energy is distributed equally among the various degrees of freedom. MAXWELL’S original proof of this (*Phil. Mag.*, 1860, ii. 37) is hardly more than a statement; while the reasoning given by WATSON, following BOLTZMANN, is on account of its vagueness difficult either to criticise or to verify. The result arrived at here is submitted with great diffidence, owing to its being directly opposed to the foregoing, but it is hoped that the nature of the reasoning is such that each step may be followed and accepted or rejected without doubt. As far as possible the notation of Prof. TAIT’S paper is adhered to.

To specify the nature of the spheres, A, B, C are taken as the principal moments of inertia at the centre of inertia; c as the distance of the centre of figure from the centre of inertia; and α, β, γ as the direction-cosines of the line joining these two points with respect to the principal axes.

In the special state of the system it is assumed—

(i.) That the distribution of the linear velocities of the spheres follows the same law as for a system of homogeneous spheres, viz., that the number of spheres whose speeds lie between v and $v + dv$ per unit volume is

$$4 \sqrt{\frac{h^3}{\pi}} n \cdot e^{-hv^2} v^2 dv,$$

and that the velocities are equally distributed as regards direction.

(ii.) That the number of spheres per unit volume whose angular velocities

about the A-, B-, and C-axes at the centre of inertia lie between ω_1 and $\omega_1 + d\omega_1$, ω_2 and $\omega_2 + d\omega_2$, and ω_3 and $\omega_3 + d\omega_3$ respectively is

$$\sqrt{\frac{k_1 k_2 k_3}{\pi^3}} n \cdot e^{-k_1 \omega_1^2 - k_2 \omega_2^2 - k_3 \omega_3^2} d\omega_1 d\omega_2 d\omega_3.$$

[This assumption is made by analogy from the form for the speeds, and can only be justified by results.]

(iii.) That for any sphere all directions of the velocity of the centre of inertia with regard to the principal axes are equally likely.

(iv.) That the distance c between the centre of figure and centre of inertia of a sphere is very small compared with the radius.

The last assumption is made for the following reason :—The “opacity,” or power of intercepting impinging particles, of a layer of such spheres as are being considered will depend both on the linear and angular velocities of the spheres, and the probability of a collision between different spheres will no longer be proportional to their relative speed, but to some function of their linear and angular velocities, which even if it could be expressed analytically would almost certainly be of a most intractable form. If, however, the distance c is assumed to be very small in comparison with the radius, the probabilities of a collision between different spheres, and the mean free path, will be sensibly independent of the angular velocities, and hence the same as for a system of homogeneous spheres, while there will still be an interchange between the energies of translation and rotation at each collision.

Let u, u' be the velocities of the centre of inertia of a sphere in the line of centres before and after an impact :—

$\omega_1, \omega_2, \omega_3, \omega'_1, \omega'_2, \omega'_3$ the angular velocities about the principal axes at the centre of inertia before and after an impact :—

l, m, n the direction-cosines of the line of centres with respect to the principal axes at an impact :—

and let large and small letters be used to distinguish the values of these quantities for the two impinging spheres.

Write also for brevity,

$$\begin{aligned} N\beta - M\gamma &= P, & n\beta - m\gamma &= p, \\ L\gamma - N\alpha &= Q, & l\gamma - n\alpha &= q, \\ M\alpha - L\beta &= R, & m\alpha - l\beta &= r. \end{aligned}$$

The spheres are said to be elastic in the sense that the energy of a pair of colliding spheres is unaltered by the impact; and MAXWELL shows, in the paper already referred to, that in the impact of two such spheres the relative velocity of the points of contact in the direction of the line of centres is simply reversed after impact.

Hence the dynamical equations may be written (the mass of a sphere being taken as unity)

$$\begin{aligned}
 U' + u' &= U + u \\
 U' - u' - c[P\Omega'_1 + p\omega'_1 + Q\Omega'_2 + q\omega'_2 + R\Omega'_3 + r\omega'_3] \\
 &= u - U + c[P\Omega_1 + p\omega_1 + Q\Omega_2 + q\omega_2 + R\Omega_3 + r\omega_3] \\
 A(\Omega'_1 - \Omega_1) &= cP(U - U') \\
 B(\Omega'_2 - \Omega_2) &= cQ(U - U') \\
 C(\Omega'_3 - \Omega_3) &= cR(U - U') \\
 A(\omega'_1 - \omega_1) &= cp(u' - u) \\
 B(\omega'_2 - \omega_2) &= cq(u' - u) \\
 C(\omega'_3 - \omega_3) &= cr(u' - u).
 \end{aligned}$$

The elimination of the angular velocities after impact from these equations leads to the following equations for U' and u' :—

$$\begin{aligned}
 U'(1 + c^2K) - u'(1 + c^2k) &= u(1 - c^2k) - U(1 - c^2K) + 2c\varpi \\
 U' + u' &= u + U,
 \end{aligned}$$

where again for brevity

$$\begin{aligned}
 \frac{P^2}{A} + \frac{Q^2}{B} + \frac{R^2}{C} &= K \\
 \frac{p^2}{A} + \frac{q^2}{B} + \frac{r^2}{C} &= k \\
 p\omega_1 + q\omega_2 + r\omega_3 + P\Omega_1 + Q\Omega_2 + R\Omega_3 &= \varpi.
 \end{aligned}$$

Hence the increase of energy of translation due to the impact (=T say)

$$\begin{aligned}
 &= \frac{1}{2} (U'^2 + u'^2 - U^2 - u^2) \\
 &= 2c \cdot \frac{(u - U + c\varpi)(2\varpi - c(k + K)(u - U))}{(2 + c^2(k + K))^2}.
 \end{aligned}$$

In the special state of the system the mean value of this quantity, as also of the increases in the energies of rotation about the three principal axes, must vanish; and the three independent results so obtained should give k_1, k_2, k_3 in terms of h .

In determining the mean value of any quantity connected with a collision of the spheres here considered, the fourth assumption made above permits the integrations involving the speeds and directions of motion of the colliding spheres and the direction of the line of centres to be performed just as in § 21 of Prof. TAIT's paper above referred to. The additional integrations to be performed in this case will be obtained as follows:—Suppose lines drawn from the centre of the unit sphere (in the figure of the paragraph referred to) parallel to the A- and B-axes of each sphere meet the unit sphere in a, A, b, B ; that ds, dS are elements of the surface of the sphere, surrounding a, A ; and

that ψ , Ψ are the angles made by ab , AB with arcs joining a , A to some fixed point. Then, to satisfy the third assumption,

$$\iint \dots ds dS d\psi d\Psi$$

must be taken between the limits 0 and 2π for ψ and Ψ , and over the whole sphere for each of the surface-integrals.

Finally, as regards the magnitudes of the angular velocities, the integrations are

$$\iint \dots e^{-k_1(\omega_1^2 + \Omega_1^2) - k_2(\omega_2^2 + \Omega_2^2) - k_3(\omega_3^2 + \Omega_3^2)} d\omega_1 d\Omega_1 d\omega_2 d\Omega_2 d\omega_3 d\Omega_3,$$

and the limits for ω_1 , Ω_1 , &c., are $\pm \infty$. It might appear that, having taken all possible positions for the principal axes, the limits for ω_1 , Ω_1 , &c., should be 0 and ∞ ; but a little consideration will make it clear that these latter limits would be correct only for bodies symmetrical with regard to three mutually rectangular planes through the centre of inertia.

In the process of finding $\bar{T}$ let the integrations with respect to ω_1 , Ω_1 , &c., be first performed. The only part of T which will contribute anything after integration is

$$2c^2 \frac{2(p^2\omega_1^2 + q^2\omega_2^2 + r^2\omega_3^2 + P^2\Omega_1^2 + Q^2\Omega_2^2 + R^2\Omega_3^2) - (k+K)(u-U)^2}{[2 + c^2(k+K)]^2}.$$

Performing on this expression the integrations indicated, and dividing by the corresponding part of the denominator of $\bar{T}$, the result is

$$2c^2 \frac{(p^2 + P^2)\left(\frac{1}{k_1} - \frac{(u-U)^2}{A}\right) + (q^2 + Q^2)\left(\frac{1}{k_2} - \frac{(u-U)^2}{B}\right) + (r^2 + R^2)\left(\frac{1}{k_3} - \frac{(u-U)^2}{C}\right)}{[2 + c^2(k+K)]^2}.$$

The value, at the same step in the process, of the average increase in ω_1 -energy at a collision is similarly found to be

$$2c^2 \frac{(p^2 + P^2)\left(\left(\frac{(u-U)^2}{A} - \frac{1}{k_1}\right) + \frac{1}{2}c^2\left\{(q^2 + Q^2)\left(\frac{1}{Ak_2} - \frac{1}{Bk_1}\right) + (r^2 + R^2)\left(\frac{1}{Ak_3} - \frac{1}{Ck_1}\right)\right\}\right)}{[2 + c^2(k+K)]^2},$$

and from this, by an interchange of symbols, the values of the corresponding quantities for the other two rotations may be written down.

In the four expressions so far obtained, write

$$\frac{A}{k_1} = \frac{B}{k_2} = \frac{C}{k_3} = \lambda;$$

they become respectively

$$\begin{aligned} & 2c^2 \frac{k+K}{[2+c^2(k+K)]} (\lambda - (u-U)^2) \\ & - 2c^2 \cdot \frac{p^2+P^2}{A} \frac{1}{[2+c^2(k+K)]^2} (\lambda - (u-U)^2) \\ & \quad \&c., \quad \&c. \end{aligned}$$

In completing the process of averaging on these new expressions, the integrations

$$\iint \dots ds dS d\psi d\Psi$$

affect only the first factors of each of them, and a consideration of the meaning of the factors shows that in each case the integral is a function of the constant quantities $A, B, C, \alpha, \beta, \gamma$ and c .

[Neglecting terms in c^4 as compared with those in c^2 , the approximate values of the averages of these factors are

$$\begin{aligned} & \frac{c^2}{3} \left(\frac{\beta^2 + \gamma^2}{A} + \frac{\gamma^2 + \alpha^2}{B} + \frac{\alpha^2 + \beta^2}{C} \right), \\ & -\frac{c^2}{3} \cdot \frac{\beta^2 + \gamma^2}{A}, \quad -\frac{c^2}{3} \cdot \frac{\gamma^2 + \alpha^2}{B}, \quad -\frac{c^2}{3} \cdot \frac{\alpha^2 + \beta^2}{C}. \end{aligned}$$

Hence the three equations

$$\frac{A}{k_1} = \frac{B}{k_2} = \frac{C}{k_3} = \overline{(u-U)^2}$$

are a solution, and therefore must be the solution, of the problem of the "special state."

The quantity $\overline{(u-U)^2}$ finally is affected only by the integrations of the § 21 of Prof. TAIT's paper already referred to; and indeed its value may be written down at once from the result of that article. For

$$\overline{(u-U)^2} = \overline{u^2} - \overline{uU} + \overline{U^2} - \overline{Uu} = \frac{2}{h}.$$

The required result then is

$$\frac{A}{k_1} = \frac{B}{k_2} = \frac{C}{k_3} = \frac{2}{h};$$

or, in words:—

The average energies of rotation of a sphere about each of the three principal axes are equal, and the whole average energy of rotation of a sphere is twice the average energy of translation.

The forms for the average changes in the rotation-energies at a collision indicate that, if at any time before the special state is attained the three rotation energies are equal, they will generally tend to become unequal again; and

therefore the problem of determining at what rate the system tends to reach the special state would be intractable even if it were legitimate to suppose the second assumption to hold throughout.

If specially constituted spheres, however, are taken in which

$$\frac{\beta^2 + \gamma^2}{A} = \frac{\gamma^2 + \alpha^2}{B} = \frac{\alpha^2 + \beta^2}{C} = \frac{2}{A+B+C},$$

an attempt may be taken to determine the rate in question, for the forms of the average changes in energy at a collision then show that if the equations

$$\frac{A}{k_1} = \frac{B}{k_2} = \frac{C}{k_3}$$

hold at any one instant, they will always hold.

Suppose, then, that in this case x, y are the whole energies of translation and rotation per unit volume; so that

$$x = \frac{3n}{4h} \quad y = \frac{3nA}{4k_1}.$$

By § 14 of Prof. TAIT's paper in conjunction with the forms found above

$$\begin{aligned} \dot{x} = -\dot{y} &= \sqrt{\frac{2\pi}{h}} n^2 s^2 \frac{2c^2}{A+B+C} \left(\frac{A}{k_1} - \frac{2}{h} \right) \\ &= \sqrt{\frac{8\pi n x}{3}} \frac{8c^2 s^2}{A+B+C} \frac{y-2x}{3}. \end{aligned}$$

If $3E$ be the whole energy per unit volume, and if

$$\begin{aligned} \sqrt{\frac{8\pi n}{3}} \cdot \frac{8c^2 s^2}{A+B+C} &= \frac{1}{T \sqrt{E}}, \\ \dot{x} &= \frac{1}{T} \sqrt{\frac{x}{E}} (E-x); \end{aligned}$$

the complete solution of which is

$$\frac{2 \sqrt{E}}{\sqrt{x} - \sqrt{E}} + 1 = \text{constant } e^{\frac{t}{T}},$$

on the supposition that the energy of translation is originally greater than one-third of the total energy.

The ratio of the quantity T found here with that in § 23 of Prof. TAIT's paper, supposing there the numbers and masses of the two sets equal, is

$$\frac{6c^2}{A+B+C}.$$

Hence it would seem that if c is of the order $s \times 10^{-3}$, and therefore this

fraction comparable with 10^{-6} , the rate at which the special state would be attained is still extremely rapid,

Note.—With regard to the form

$$\sqrt{\frac{k_1 k_2 k_3}{\pi^3}} n e^{-k_1 \omega_1^2 - k_2 \omega_2^2 - k_3 \omega_3^2} d\omega_1 d\omega_2 d\omega_3$$

for the number of particles per unit volume with angular velocities between given limits, the fact that $\omega_1, \omega_2, \omega_3$ are periodic functions while the particle is moving freely, suggests at first sight a difficulty. The period for each particle is a function of its energy of rotation and of its angular momentum. Suppose the particles whose energy of rotation lies between given close limits divided into sets, the angular momentum in each set also lying between given close limits. Then (compare KIRCHHOFF'S *Vorlesungen*, p. 64)

$$\omega_1 = p \operatorname{cn}(\lambda t + \mu), \quad \omega_2 = q \operatorname{sn}(\lambda t + \mu), \quad \omega_3 = r \operatorname{dn}(\lambda t + \mu),$$

where p, q, r, λ and the modulus of the elliptic functions are constants for any one set.

The individuals of each set are distinguished by the values of μ . If then the circumstances are such that μ may be regarded as a uniformly varying quantity between limits separated by a period for each set, the number of particles corresponding to the product $d\omega_1 d\omega_2 d\omega_3$ will be independent of the time.