VII. On the calculus of operations

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at which Professor Challis has arrived with reference to ray vibrations. I have done so partly because the subject has been taken up by the Astronomer Royal, partly because I have not leisure for the discussion at present, partly because the points which would have to be noticed, and which would be likely to arise in the course of the discussion, are so numerous, that I think it hardly fair to take up the pages of a Magazine like the present with the controversy. I cannot however conclude without recording my protest, first against equation (8.) (vol. xxxii. p. 282), and secondly against equations (B.) and (C.) (vol. xxxiii. pp. 99 and 100), by which an attempt is made to satisfy equation (A.).

Pembroke College, Cambridge,
Dec. 23rd, 1848.

VII. *On the Calculus of Operations.* By the Rev. CHARLES GRAVES, M.A., *Professor of Mathematics in Trinity College, Dublin**.

PROFESSOR YOUNG, objecting to the method by which the theorem of Leibnitz is usually extended to successive integration, has lately proved its applicability in that case by means of repeated "integrations by parts:" and he has shown how to obtain in this way a series of supplementary integrals, without the addition of which the theorem is not generally true, though they are commonly suppressed in the statement of it. Professor Young seems to impute this omission to the nature of the Calculus of Operations, by means of which the theorem is usually treated; as though that method necessarily gave the theorem in the imperfect form, and made no provision for the correction which he suggests.

It is my purpose here to show that the omission of the supplementary integrals has been caused by the use of an incomplete form of the binomial theorem, rather than by any inherent deficiency in the Calculus of Operations, which, if applied to this problem with proper caution, will furnish the desired result in a direct and elegant manner.

If we take the identity

$$(1+x)^{-1} = 1 - x + x^2 - \&c. \dots + (-1)^{m-1} x^{m-1} \\ + (-1)^m (1+x)^{-1} x^m,$$

and differentiate it $(n-1)$ times; we shall obtain a development, which coincides for its first m terms with that obtained by the use of the binomial theorem; and furnishes moreover

* Communicated by the Author.

the supplemental terms necessary to make the equation identically true, viz.

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{1.2} - \&c. \dots + (-1)^{m-1} A_n x^{m-1} \\ + (-1)^m \{ A_n (1+x)^{-1} + A_{n-1} (1+x)^{-2} + \&c. \dots \\ + \frac{m(m+1)}{1.2} (1+x)^{-n+2} + m(1+x)^{-n+1} + (1+x)^{-n} \} x^m.$$

A_n being used to stand for

$$\frac{n(n+1) \dots (n+m-2)}{1.2 \dots (m-1)},$$

or its equal

$$\frac{m(m+1) \dots (m+n-2)}{1.2 \dots (n-1)};$$

and so on for A_{n-1} , A_{n-2} , &c.

And, if we write $\frac{D'}{D''}$ in place of x , we have the identity

$$\{D'' + D'\}^{-n} = D''^{-n} - n D''^{-n-1} D' + \frac{n(n+1)}{1.2} D''^{-n-2} D'^2 - \&c. \dots \\ + (-1)^{m-1} A_n D''^{-n-m+1} D'^{m-1} + (-1)^m \{ A_n (D'' + D')^{-1} \\ D''^{-n-m+1} + A_{n-1} (D'' + D')^{-2} D''^{-n-m+2} + \&c. \dots \\ + \frac{m(m+1)}{1.2} (D'' + D')^{-n+2} D''^{-m-2} + m(D'' + D')^{-n+1} \\ D''^{-m-1} + (D'' + D')^{-n} D''^{-m} \} D'^m;$$

which continues to hold good when D' and D'' are any two symbols of commutative and distributive operation, just as much as if they denoted quantities.

Using D to denote the operation of taking the differential coefficient with respect to x , we have

$$D(vu) = vDu + uDv;$$

from which we derive the symbolical equation

$$D = D'' + D';$$

understanding that D'' shall operate exclusively on u , and D exclusively on v . The symbol D being thus absolutely equivalent to $D'' + D'$, we are entitled to operate on uv with the symbol $(D'' + D')^{-n}$, or its expansion as given above, for the purpose of effecting an n -fold integration. Thus we obtain the complete formula of which we are in search:

$$\int^n vudx^n = vu_n - n \frac{dv}{dx} u_{n+1} + \frac{n(n+1)}{1.2} \frac{d^2v}{dx^2} u_{n+2} - \&c. \dots$$

$$\begin{aligned}
& + (-1)^{m-1} A_n \frac{d^{m-1}v}{dx^{m-1}} u_{m+n-1} + (-1)^m \left\{ A_n \int \frac{d^m v}{dx^m} u_{m+n-1} dx \right. \\
& + A_{n-1} \int \frac{d^m v}{dx^m} u_{m+n-2} dx^2 + \dots + \frac{m(m+1)}{1.2} \\
& \int \frac{d^m v}{dx^m} u_{m+2} dx^{n-2} + m \int \frac{d^m v}{dx^m} u_{m+1} dx^{n-1} \\
& \left. + \int \frac{d^m v}{dx^m} u_m dx^n \right\},
\end{aligned}$$

u_r being used for brevity instead of $\int^r u dx^r$.

The series of supplemental integrals here given differs from that exhibited by Professor Young only as regards the signs of its terms, which by an oversight he has made alternately positive and negative.

The reasoning remaining the same, it is enough merely to indicate the corresponding mode of obtaining the general formula for the finite integration of a product. Sir John Herschel has given this formula in his excellent article on differences and series, appended to the Cambridge translation of Lacroix's treatise on the Differential and Integral Calculus: and it is to be observed that he has taken care to supply those supplementary integrals which are necessary to its correctness.

Since

$$\Delta(u_x v_x) = u_x \Delta v_x + v_{x+1} \Delta u_x = u_x \Delta v_x + e^D v_x \cdot \Delta u_x,$$

we have the symbolical equation

$$\Delta = \Delta'' + e^D \Delta';$$

it being understood that Δ'' and e^D operate only on v_x , and Δ' only upon u_x . And if we put Δ'' and $e^D \Delta'$ in place of D'' and D' in the formula given above for $(D' + D'')^{-n}$, we shall at once obtain Sir John Herschel's formula.

The examples here discussed by the Calculus of Operations are certainly instructive. Whilst they manifest the danger, noticed by Professor Young, of substituting symbols of operation for those of quantity in divergent infinite series, they indicate that, wherever we know how to express in a finite form the value of the remainder after any given number of terms of an infinite series, there is a safe way of effecting such a substitution. It must be made in the expression for the *remainder*, as well as in the terms of the series.

Dublin, November 30, 1848.