XXVI. On the application of legendre's functions to the theory of the Jacobian elliptic integrals

J.W. Nicholson B.Sc.

To cite this article: J.W. Nicholson B.Sc. (1905) XXVI. On the application of legendre's functions to the theory of the Jacobian elliptic integrals , Philosophical Magazine Series 6, 9:50, 264-284, DOI: [10.1080/14786440509463278](https://doi.org/10.1080/14786440509463278)

To link to this article: <http://dx.doi.org/10.1080/14786440509463278>

Published online: 08 Jun 2010.

Submit your article to this journal [↗](#)

Article views: 5

View related articles [↗](#)

Copper wire, No. 27 S.W.G. $l = 26\cdot50$ cms., mean diameter $\cdot0412$ cm. $\therefore$ area a of cross-section $= \cdot00133$ sq. cm.,

and $\frac{gl^3}{a} = 1\cdot37 \times 10^{10}$.

Initial stretching force in dynes.	M. gr.	y mean. cms.	$\frac{M}{y}$	y^2 .	$\frac{M_1}{y_1} - \frac{M}{y}$	$y_1^2 - y^2$.	$\frac{M_1 - M}{y_1^2 - y^2}$	ϵ . dynes/sq.cm.
650 <i>g</i>	15	2749	54.56	0756	7.47	0863	86.6	$1\cdot19 \times 10^{12}$
	25	4024	62.03	1619	7.50	0915	82.0	1.12
	35	5034	69.53	2534	14.97	1778	84.2	1.15
380 <i>g</i>	15	3630	41.32	1318	9.44	1108	85.2	1.17
	25	4925	50.76	2426	8.80	1027	85.7	1.17
	35	5876	59.55	3453	18.24	2135	85.4	1.17
630 <i>g</i>	15	2774	54.07	0769	7.75	0866	89.5	1.23
	25	4044	61.82	1635	7.83	0890	88.0	1.21
	35	5025	69.65	2525	15.58	1756	88.7	1.22

Mean $\epsilon = 1\cdot18 \times 10^{12}$.

TABLE OF RESULTS.

	YOUNG'S MODULUS at 16° C.
	dynes per sq. cm.
Iron wire, No. 34 A.	$1\cdot95 \times 10^{12}$
„ No. 34 B.	2.01
„ No. 29	2.00
Steel pianoforte-wire, No. 29 A.	2.70
„ No. 29 B.	2.78
Nickel wire, No. 30	2.04
Copper wire, No. 27	1.18

XXVI. *On the Application of Legendre's Functions to the Theory of the Jacobian Elliptic Integrals.* By J. W. NICHOLSON, B.Sc.(*Lond. & Vict.*), Trinity College, Cambridge*.

THE object of this paper is to indicate a method of some generality, of expanding (1) functions expressible in terms of the complete elliptic integrals of Jacobi; and (2) definite integrals containing incomplete elliptic integrals in their integrand, in series of Legendre coefficients of increasing

* Communicated by the Author.

order, and to deduce from the theory certain integrals which are otherwise difficult to evaluate. Most of the results of this latter class, also, are new.

We shall employ the two expressions for $P_n(\mu)$, where $\mu = \cos \theta$, which are due to Mehler, viz. :

$$P_n(\mu) = \frac{2}{\pi} \int_0^\theta \frac{\cos(n + \frac{1}{2})\phi \, d\phi}{\sqrt{2(\cos \phi - \cos \theta)}}, \quad . \quad . \quad . \quad (1)$$

$$P_n(\mu) = \frac{2}{\pi} \int_\theta^\pi \frac{\sin(n + \frac{1}{2})\phi \, d\phi}{\sqrt{2(\cos \theta - \cos \phi)}}, \quad . \quad . \quad . \quad (2)$$

and in all future work θ is considered to be a positive angle, lying between 0 and π .

Let $F(z)$ be a function of z , satisfying the conditions for a Fourier expansion between $z = z_1$, $z = z_2$, the expansion having one of the two forms :—

$$F(z) = \sum_{n=0}^{\infty} a_n \cos(2n+1)z,$$

$$F(z) = \sum_{n=0}^{\infty} a_n \sin(2n+1)z.$$

Then between the limits $2z_1$, $2z_2$, $F\left(\frac{\phi}{2}\right)$ has one of the forms

$$F\left(\frac{\phi}{2}\right) = \sum_0^\infty a_n \cos(n + \frac{1}{2})\phi, \quad . \quad . \quad . \quad (3a)$$

$$F\left(\frac{\phi}{2}\right) = \sum_0^\infty a_n \sin(n + \frac{1}{2})\phi. \quad . \quad . \quad . \quad (3b)$$

Multiplying the first form by $\frac{2}{\pi} \cdot \frac{1}{\sqrt{2(\cos \phi - \cos \theta)}}$ and integrating from 0 to θ , supposed not to include $2z_1$, or $2z_2$,

$$\therefore \frac{2}{\pi} \int_0^\theta \frac{F\left(\frac{\phi}{2}\right) d\phi}{\sqrt{2(\cos \phi - \cos \theta)}} = \sum_0^\infty a_n P_n(\mu). \quad . \quad . \quad (4)$$

Similarly, the second form gives

$$\frac{2}{\pi} \int_\theta^\pi \frac{F\left(\frac{\phi}{2}\right) d\phi}{\sqrt{2(\cos \theta - \cos \phi)}} = \sum_0^\infty a_n P_n(\mu). \quad . \quad . \quad (5)$$

These formulæ will be applied to several Fourier expansions. The latter will be freely quoted, and are all to be found in

266 Mr. Nicholson on *Legendre's Functions and the*
Whittaker, 'Modern Analysis,' Chap. vii., or a similar
work.

The simplest case is the series

$$\cos \frac{\phi}{2} - \cos \frac{3\phi}{2} + \cos \frac{5\phi}{2} \dots\dots$$

which we shall not dwell upon.

The expansion produced is

$$\frac{1}{2 \cos \frac{\theta}{2}} = P_0(\mu) - P_1(\mu) + P_2(\mu) \dots + (-)^n P_n(\mu) + \dots \quad (6)$$

where θ is not zero or π , but may have any intermediate value. Therefore, by the integral properties of Legendre's functions, the integral on left being finite at both limits,

$$\frac{1}{2} \int_{-1}^1 \frac{P_n(\mu) d\mu}{\cos \frac{\theta}{2}} = (-)^n \frac{2}{2n+1}$$

or

$$\int_0^\pi P_n(\cos \theta) \sin \frac{\theta}{2} d\theta = (-)^n \frac{2}{2n+1} \dots \dots \quad (7)$$

The next case is the result:

$$\frac{\pi}{4} = \sin z + \frac{1}{3} \sin 3z + \frac{1}{5} \sin 5z + \dots\dots$$

holding from $z=0$ to π , both exclusive.

Therefore between $\phi=0$ and 2π ,

$$\frac{\pi}{4} = \sum_0^\infty \frac{\sin (2n+1) \frac{\phi}{2}}{2n+1}.$$

Hence

$$\begin{aligned} \sum_0^\infty \frac{P_n(\mu)}{2n+1} &= \frac{2}{\pi} \sum_0^\infty \frac{1}{2n+1} \int_\theta^\pi \frac{\sin (2n+1) \frac{\phi}{2}}{\sqrt{2(\cos \theta - \cos \phi)}} \\ &= \frac{1}{4} \int_\theta^\pi \frac{d\phi}{\sqrt{\cos^2 \frac{\theta}{2} - \cos^2 \frac{\phi}{2}}}; \end{aligned}$$

the excluded values of ϕ being outside the range of integration.

Put $\tan \frac{\phi}{2} = \tan \frac{\theta}{2} \operatorname{cosec} \lambda,$

$$\therefore d\phi = - \frac{2 \tan \frac{\theta}{2} \cdot \operatorname{cosec} \lambda \cot \lambda d\lambda}{1 + \tan^2 \frac{\theta}{2} \operatorname{cosec}^2 \lambda}$$

$$\cos^2 \frac{\theta}{2} - \cos^2 \frac{\phi}{2} = \frac{\sin^2 \frac{\theta}{2} \cot^2 \lambda}{1 + \tan^2 \frac{\theta}{2} \operatorname{cosec}^2 \lambda}$$

$$\begin{aligned} \therefore \int_{\theta}^{\pi} \frac{d\phi}{\sqrt{\cos^2 \frac{\theta}{2} - \cos^2 \frac{\phi}{2}}} &= 2 \int_0^{\frac{\pi}{2}} \frac{\sec \frac{\theta}{2} \operatorname{cosec} \lambda d\lambda}{\sqrt{1 + \tan^2 \frac{\theta}{2} \operatorname{cosec}^2 \lambda}} \\ &= 2K\left(\cos \frac{\theta}{2}\right), \end{aligned}$$

where $K(k)$ is Jacobi's complete integral $\int_0^{\frac{\pi}{2}} \frac{d\lambda}{\sqrt{1-k^2 \sin^2 \lambda}}.$

Hence

$$K\left(\cos \frac{\theta}{2}\right) = 2 \sum_0^{\infty} \frac{P_n(\mu)}{2n+1} \dots \dots \dots (8)$$

where $\mu = \cos \theta.$

The formula

$$\frac{\pi}{4} = \cos z - \frac{1}{3} \cos 3z + \frac{1}{5} \cos 5z \dots \dots$$

holds from $z = -\frac{\pi}{2}$ to $z = +\frac{\pi}{2}$, both exclusive.

$$\therefore \frac{2}{\pi} \int_0^{\theta} \frac{\pi}{4} \cdot \frac{d\phi}{\sqrt{2(\cos \phi - \cos \theta)}} = \sum_0^{\infty} \frac{(-)^n}{2n+1} P_n(\mu).$$

To transform this integral, put $\tan \frac{\phi}{2} = \tan \frac{\theta}{2} \cos \lambda.$

$$\therefore d\phi = \frac{-2 \tan \frac{\theta}{2} \cdot \sin \lambda d\lambda}{1 + \tan^2 \frac{\theta}{2} \cos^2 \lambda}$$

$$\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2} = \frac{\sin^2 \frac{\theta}{2} \sin^2 \lambda}{1 + \tan^2 \frac{\theta}{2} \cos^2 \lambda}$$

$$\therefore \int_0^\theta \frac{d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} = 2 \int_0^{\frac{\pi}{2}} \frac{d\lambda}{\sqrt{1 - \sin^2 \frac{\theta}{2} \sin^2 \lambda}} = 2K\left(\sin \frac{\theta}{2}\right).$$

Hence

$$K\left(\cos \frac{\theta}{2}\right) = K\left(\sin \frac{\theta}{2}\right) = 2 \sum_0^\infty \frac{(-)^n}{2n+1} P_n(\mu). \quad (9)$$

It follows from (8) that

$$\int_{-1}^1 P_n(\mu) K\left(\sqrt{\frac{1+\mu}{2}}\right) d\mu = \frac{4}{(2n+1)^2} \quad (10)$$

or

$$\int_0^1 k K P_n(2k^2-1) dk = \frac{1}{(2n+1)^2} \quad (11)$$

Now when n is large,

$$P_n(\mu) = \sqrt{\frac{2}{n\pi \sin \theta}} \cdot \cos \left\{ \left(n + \frac{1}{2}\right)\theta + \frac{\pi}{4} \right\} \quad (12)$$

and the series (8) and (9) are accordingly absolutely convergent between $\theta=0$ and π , but at these limits the theorems are not true. The integrals further deduced, however, may be taken over the entire range, the integrals being finite at each limit.

Squaring (8) and integrating with respect to μ ,

$$\begin{aligned} \int_{-1}^1 K^2(k) d\mu &= 4 \sum_0^\infty \int_{-1}^1 \frac{P_n^2(\mu) d\mu}{(2n+1)^2} + 4 \sum_0^\infty \int_{-1}^1 \frac{P_n(\mu) P_m(\mu) d\mu}{(2n+1)(2m+1)} \\ &= 8 \sum_0^\infty \frac{1}{(2n+1)^3}. \end{aligned}$$

Let

$$\sigma_{2m+1} = \frac{1}{1^{2m+1}} + \frac{1}{2^{2m+1}} + \frac{1}{3^{2m+1}} + \dots \quad (13)$$

$$\therefore \int_0^1 k K^2 dk = \frac{7}{4} \sigma_3. \quad (14)$$

The formulæ arising in a similar way from (9) are

$$\int_{-1}^1 P_n(\mu) \cdot K'\left(\sqrt{\frac{1+\mu}{2}}\right) d\mu = (-)^n \frac{4}{(2n+1)^2} \quad (15)$$

$$\int_0^1 k K' P_n(2k^2-1) dk = \frac{(-)^n}{(2n+1)^2} \quad (16)$$

and

$$\int_0^1 k K'^2 dk = \frac{7}{4} \sigma_3 \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and we note that if n is even,

$$\int_0^1 k(K - K') P_n(2k^2 - 1) dk = 0 \quad . \quad . \quad . \quad (18)$$

and also that

$$\int_0^1 k(K^2 - K'^2) dk = 0 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Combining (7) and (8)

$$\begin{aligned} \int_0^\pi K \left(\cos \frac{\theta}{2} \right) \sin \frac{\theta}{2} d\theta &= 2 \sum_0^\infty \frac{1}{2n+1} \int_0^\pi P_n(\cos \theta) \sin \frac{\theta}{2} d\theta \\ &= 4 \sum_0^\infty \frac{(-)^n}{(2n+1)^2} \\ \therefore \int_0^1 K dk &= 2 \left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \dots \right) \quad . \quad . \quad (20) \end{aligned}$$

while

$$\begin{aligned} \int_0^1 K' dk &= \frac{1}{2} \sum_0^\infty \frac{4}{(2n+1)^2}, \\ \therefore \int_0^1 K' dk &= \frac{\pi^2}{4} \quad . \quad . \quad . \quad . \quad . \quad (21) \end{aligned}$$

Again,

$$\begin{aligned} \int_{-1}^1 K K' d\mu &= \sum_0^\infty \frac{2}{2n+1} \int_{-1}^1 P_n(\mu) K' \left(\sqrt{\frac{1+\mu}{2}} \right) d\mu \\ &= 8 \left\{ 1 - \frac{1}{3^3} + \frac{1}{5^3} \dots \right\} \\ &= 8 \left(\frac{\pi}{2} \right)^3 \frac{E_1}{2 \cdot 2!} \text{ by the use of Euler's numbers} \\ &= \frac{\pi^3}{4}; \\ \therefore \int_0^1 k K K' dk &= \frac{\pi^3}{16} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22) \end{aligned}$$

Expressions may be found for integrals containing higher powers of K , K' , but cannot be concisely exhibited.

A theorem will be used, which will be stated as follows, and is easy to prove.

If m is equal to, or less than n ,

$$P_m(z)P_n(z) = \sum_{r=0}^m \frac{A_r A_{m-r} A_{n-r}}{A_{m+n-r}} \binom{2n+2m-4r+1}{2n+2m-2r+1} P_{n+m-2r}(z) \quad (23)$$

where

$$A_r = \frac{2m!}{2^m(m!)^2} \cdot \cdot \cdot \cdot \cdot \quad (24)$$

Now

$$\int_{-1}^1 K^2\left(\cos \frac{\theta}{2}\right) P_n(\mu) d\mu = \sum_{s=0}^{\infty} \frac{2}{2s+1} \int_{-1}^1 K\left(\cos \frac{\theta}{2}\right) P_s(\mu) P_n(\mu) d\mu.$$

Write (23) in the form

$$P_m(z)P_n(z) = \sum_{r=0}^m \left(\lambda_r\right)_n^m P_{n+m-2r}(z) \cdot \cdot \cdot \quad (25)$$

where $m \leq n$;

$$\begin{aligned} \therefore \int_{-1}^1 K^2\left(\cos \frac{\theta}{2}\right) P_n(\mu) d\mu &= \sum_{s=0}^n \frac{2}{2s+1} \int_{-1}^1 K\left(\cos \frac{\theta}{2}\right) \sum_{r=0}^s \left(\lambda_r\right)_n^s P_{n+s-2r}(\cos \theta) d\mu \\ &\quad + \sum_{s=n+1}^{\infty} \frac{2}{2s+1} \int_{-1}^1 K\left(\cos \frac{\theta}{2}\right) \sum_{r=0}^n \left(\lambda_r\right)_s^n P_{n+s-2r}(\cos \theta) d\mu \\ &= \sum_{s=0}^n \sum_{r=0}^s \frac{2}{2s+1} \left(\lambda_r\right)_n^s \cdot \frac{2}{2n+2s-4r+1} \\ &\quad + \sum_{s=n+1}^{\infty} \sum_{r=0}^n \frac{2}{2s+1} \left(\lambda_r\right)_s^n \cdot \frac{2}{2n+2s-4r+1}. \\ \therefore \int_0^1 k K^2 P_n(2k^2-1) dk &= \sum_{s=0}^n \sum_{r=0}^s \frac{(\lambda_r)_n^s}{(2s+1)(2n+2s-4r+1)} \\ &\quad + \sum_{s=n+1}^{\infty} \sum_{r=0}^n \frac{(\lambda_r)_s^n}{(2s+1)(2n+2s-4r+1)} \cdot \cdot \cdot \quad (26) \end{aligned}$$

and

$$\begin{aligned} \int_0^1 k K^3 dk &= \frac{1}{4} \int_{-1}^1 K^2(k) \cdot \sum_0^{\infty} \frac{2}{2n+1} P_n(\mu) d\mu. \\ \therefore \int_0^1 k K^3 dk &= 2 \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \sum_{r=0}^s \frac{(\lambda_r)_n^s}{(2s+1)(2n+1)(2n+2s-4r+1)} \\ &\quad + 2 \sum_{n=0}^{\infty} \sum_{s=n+1}^{\infty} \sum_{r=0}^n \frac{(\lambda_r)_s^n}{(2s+1)(2n+1)(2n+2s-4r+1)} \cdot \quad (27) \end{aligned}$$

where $(\lambda_r)_n^s = (\lambda_r)_s^n$, and is defined in (25).

This process is general, and we can theoretically find a general expression for $\int_0^1 kK^r K'^s dk$, which, however, is of no importance.

Results will now be obtained involving the complete elliptic integral

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \phi} d\phi \quad . \quad . \quad . \quad (28)$$

The formulæ connecting E and K are

$$k \frac{dE}{dk} = E - K, \quad . \quad . \quad . \quad . \quad (29 a)$$

$$\frac{E}{1 - k^2} = K + k \frac{dK}{dk} \quad . \quad . \quad . \quad . \quad (29 b)$$

Integrating (10) by parts,

$$\begin{aligned} & \int_{-1}^1 P_n(\mu) K(k) d\mu \\ &= \left[-\frac{K(k)}{n(n+1)} (1 - \mu^2) \frac{dP_n}{d\mu} \right]_{-1}^1 + \frac{1}{n(n+1)} \int_{-1}^1 (1 - \mu^2) \frac{dP_n}{d\mu} \cdot \frac{dK(k)}{dk} d\mu \\ &= \frac{4}{n(n+1)} \int_0^1 P_n'(2k^2 - 1) \cdot k^2 k'^2 \frac{dK}{dk} dk \\ &= \frac{4}{n(n+1)} \int_0^1 P_n(2k^2 - 1) \{E - k'^2 K\} k dk. \\ &\therefore \int_0^1 k \{E - k'^2 K\} P_n'(2k^2 - 1) dk = \frac{n(n+1)}{(2n+1)^2} \quad . \quad . \quad (30) \end{aligned}$$

Therefore changing the variables k, k' ,

$$\int_0^1 k \{E' - k^2 K'\} P_n'(2k^2 - 1) dk = (-)^{n-1} \frac{n(n+1)}{(2n+1)^2} \quad . \quad (31)$$

An integration of (30) by parts gives

$$\frac{4n(n+1)}{(2n+1)^2} = \left[(E - k'^2 K) P_n(\mu) \right]_{-1}^1 - \int_{-1}^1 P_n(\mu) \frac{d}{4k dk} \{E - K k'^2\} d\mu.$$

The integrated term is zero, for $\lim_{k' \rightarrow 0} k'^2 \log \frac{4}{k'} = 0$,

and, again, $\frac{d}{k dk} \{E - K k'^2\} = K$ by (29).

$$\therefore \frac{4n(n+1)}{(2n+1)^2} = E(1) P_n(1) - \int_0^1 k K P_n(2k^2 - 1) dk,$$

272 Mr. Nicholson on *Legendre's Functions and the*
giving

$$\int_0^1 k K P_n(2k^2-1) dk = \frac{1}{(2n+1)^2}$$

which is equation (10) again.

We now proceed to treat (15) in the same way as (10).
The formulæ connecting E' and K' are

$$\frac{dE'}{dk} = -\frac{k}{k'^2} (E' - K') \quad . \quad . \quad . \quad (32 a)$$

$$\frac{dK'}{dk} = -\frac{1}{kk'^2} \{ E' - k^2 K' \} \quad . \quad . \quad . \quad (32 b)$$

(15) gives

$$(-)^n \frac{4}{(2n+1)^2} = \left[\frac{-K'(k)}{n(n+1)} (1-\mu^2) \frac{dP_n}{d\mu} \right]_{-1}^1 + \int_{-1}^1 \frac{(1-\mu^2)}{n(n+1)} \frac{dP_n}{d\mu} \frac{d}{d\mu} (K') d\mu.$$

$$\therefore \int_0^1 (1-\mu^2) \frac{dP_n}{d\mu} \frac{dK'}{dk} dk = (-)^n \frac{4n(n+1)}{(2n+1)^2},$$

or, by (32 b),

$$\int_0^1 k \{ k^2 K' - E' \} P_n'(2k^2-1) dk = (-)^n \frac{n(n+1)}{(2n+1)^2},$$

a repetition of (31) by a different method.

We now consider another expansion in a series of Legendre functions, commencing with the equation

$$1 + \cos 2z = \frac{4}{\pi} \left\{ \frac{\cos z}{1.3} + \frac{\cos 3z}{1.3.5} - \frac{\cos 5z}{3.5.7} \dots + \frac{(-)^{n-1} \cos (2n+1)z}{(2n-1)(2n+1)(2n+3)} + \dots \right\} \quad (33)$$

It leads to

$$\frac{1}{2\sqrt{2}} \int_0^\theta \frac{(1+\cos \phi) d\phi}{\sqrt{\cos \phi - \cos \theta}} = \frac{1}{3} P_0(\mu) + \sum_1^\infty \frac{(-)^{n-1} P_n(\mu)}{(2n-1)(2n+1)(2n+3)}.$$

But

$$\frac{1+\cos \phi}{\sqrt{\cos \phi - \cos \theta}} = \sqrt{\cos \phi - \cos \theta} + \frac{1+\cos \theta}{\sqrt{\cos \phi - \cos \theta}},$$

$$\int_0^\theta \frac{d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} = 2K' \left(\cos \frac{\theta}{2} \right),$$

$$\int_0^\theta \sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}} d\phi = 2 \int_0^{\frac{\pi}{2}} \frac{\sin \frac{\theta}{2} \cdot \tan \frac{\theta}{2} \cdot \sin^2 \lambda d\lambda}{\left(1 + \tan^2 \frac{\theta}{2} \cos^2 \lambda \right)^{\frac{3}{2}}},$$

with the usual substitution.

If $\cos \frac{\theta}{2} = k$, this becomes

$$2 \int_0^{\frac{\pi}{2}} \frac{k^2 k'^2 \sin^2 \lambda \, d\lambda}{(1 - k'^2 \sin^2 \lambda)^{\frac{3}{2}}} = -4k^2 k'^2 \frac{dK'}{dk^2} \\ = 2(E' - k^2 K') \text{ by (32 } b \text{).}$$

$$\therefore \frac{1}{3} P_0(\mu) + \sum_1^{\infty} \frac{(-)^{n-1} P_n(\mu)}{(2n-1)(2n+1)(2n+3)} \\ = \frac{1}{2\sqrt{2}} \cdot \left\{ 2\sqrt{2} (E' - k^2 K') + \frac{(1 + \cos \theta)}{\sqrt{2}} \dots 2K' \right\} \\ = E' - k^2 K' + \frac{2k^2 \cdot \sqrt{2}}{2\sqrt{2}} K' \\ = E'.$$

$$\therefore E' \left(\cos \frac{\theta}{2} \right) = \sum_0^{\infty} \frac{(-)^{n-1} P_n(\mu)}{(2n-1)(2n+1)(2n+3)} \text{ if } \mu = \cos \theta. \quad (34)$$

The series being absolutely convergent.
It follows that

$$\int_{-1}^1 E'(k) P_n(\mu) d\mu = \frac{(-)^{n-1} 2}{(2n-1)(2n+1)^2(2n+3)} \quad \dots \quad (35)$$

$$\therefore \int_0^{\pi} E' \left(\cos \frac{\theta}{2} \right) P_n(\cos \theta) \sin \theta \, d\theta = \frac{(-)^{n-1} 2}{(2n-1)(2n+3)(2n+1)^2}. \quad (36)$$

Combining (7) and (34)

$$\int_0^{\pi} E' \left(\cos \frac{\theta}{2} \right) \sin \frac{\theta}{2} \, d\theta = -2 \sum_0^{\infty} \frac{1}{(2n-1)(2n+3)(2n+1)^2} \\ = 2 \sum_0^{\infty} \frac{1}{(2n+1)^2} \text{ on reduction} \\ = \frac{\pi^2}{4}.$$

$$\therefore \int_0^1 E'(k) dk = \frac{\pi^2}{8}. \quad \dots \quad (37)$$

Squaring (34) and integrating,

$$\int_{-1}^1 E'^2(k) d\mu = 2 \sum_0^{\infty} \frac{1}{(2n-1)^2(2n+3)^2(2n+1)^3}.$$

Now

$$\begin{aligned} \left(\frac{1}{2n-1} - \frac{2}{2n+1} + \frac{1}{2n+3} \right)^2 &= \frac{64}{(2n-1)^2(2n+1)^2(2n+3)^2} \\ \therefore \int_0^1 (E'(k))^2 k dk &= \frac{1}{128} \sum_0^\infty \frac{1}{2n+1} \left(\frac{1}{2n-1} + \frac{1}{2n+3} - \frac{2}{2n+1} \right)^2 \\ &= \frac{1}{128} \sum_0^\infty \left[\frac{1}{2n+1} \left(\frac{1}{(2n-1)^2} + \frac{1}{(2n+3)^2} + \frac{4}{(2n+1)^2} \right) + \frac{3}{2} \right] \\ &= \frac{1}{32} \sum_0^\infty \frac{1}{(2n+1)^3}, \end{aligned}$$

and if $\sigma_3 = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots$

$$\therefore \int_0^1 (E'(k))^2 k dk = \frac{7}{256} \sigma_3. \quad \dots \quad (38)$$

We now integrate (35) by parts,

$$\begin{aligned} \therefore \frac{(-)^{n-1} 2}{(2n-1)(2n+3)(2n+1)^2} &= \left[\frac{-E'(k)}{n(n+1)} (1-\mu^2) \frac{dP_n}{d\mu} \right]_{-1}^1 \\ &\quad + \frac{1}{n(n+1)} \int_{-1}^1 (1-\mu^2) \frac{dP_n}{d\mu} \frac{dE'(k)}{d\mu} d\mu, \\ \therefore \int_0^1 (1-\mu^2) \frac{dP_n}{d\mu} \frac{dE'}{dk} dk &= \frac{(-)^{n-1} 2n(n+1)}{(2n-1)(2n+3)(2n+1)^2}, \end{aligned}$$

or

$$\begin{aligned} \int_0^1 k^3 (E' - K') P_n' (2k^2 - 1) dk &= \frac{(-)^n}{2} \cdot \frac{n(n+1)}{(2n-1)(2n+3)(2n+1)^2}. \quad (39) \end{aligned}$$

$\therefore$ Changing the variable from k to k' ,

$$\begin{aligned} \int_0^1 k(1-k^2)(E-K)P_n'(2k^2-1)dk &= \frac{1}{2} \cdot \frac{n(n+1)}{(2n-1)(2n+3)(2n+1)^2}. \quad (40) \end{aligned}$$

Combining (40) with (30), and (39) with (31) respectively, we find

$$\begin{aligned} \int_0^1 k^3 E P_n' (2k^2 - 1) dk &= \frac{n(n+1)}{(2n+1)^2} \left\{ 1 - \frac{1}{2(2n+3)(2n-1)} \right\} \\ &= \frac{n(n+1)(8n^2+8n-7)}{2(2n+3)(2n-1)(2n+1)^2}, \quad (41) \end{aligned}$$

and

$$\int_0^1 k k'^2 E' P'_n(2k^2 - 1) dk = \frac{(-)^{n-1} n(n+1)}{2} \cdot \frac{8n^2 + 8n - 5}{(2n+1)^2} \cdot \frac{8n^2 + 8n - 5}{(2n-1)(2n+3)}. \quad (42)$$

Now in (34), write $(\pi - \theta)$ for θ .

$$\therefore E\left(\cos \frac{\theta}{2}\right) = -\sum_0^\infty \frac{P_n(\mu)}{(2n+1)(2n-1)(2n+3)}, \quad \dots \quad (43)$$

whence

$$\int_0^\pi E\left(\cos \frac{\theta}{2}\right) P_n(\cos \theta) \sin \theta d\theta = \frac{2}{(2n-1)(2n+3)(2n+1)^2}. \quad (44)$$

Combining (7) and (43)

$$\begin{aligned} \int_0^\pi E\left(\cos \frac{\theta}{2}\right) \sin \frac{\theta}{2} d\theta &= -2 \sum_0^\infty \frac{(-)^n}{(2n-1)(2n+3)(2n+1)^2} \\ &= \left\{ 3 - \frac{2}{3^2} + \frac{2}{5^2} \dots \right\}; \end{aligned}$$

$$\therefore \int_0^1 E(k) dk = \left\{ 1 - \frac{1}{3^2} + \frac{1}{5^2} \dots \right\} + \frac{1}{2}. \quad \dots \quad (45)$$

We note that

$$\int_0^1 E dk = \frac{1}{2} \int_0^1 K dk + \frac{1}{2}, \quad \dots \quad (46)$$

while

$$\int_0^1 E' dk = \frac{1}{2} \int_0^1 K' dk \quad \dots \quad (47)$$

These might be obtained by integration by parts.

By (43)

$$\begin{aligned} \int_{-1}^1 E^2\left(\cos \frac{\theta}{2}\right) d\mu &= 2 \sum_0^\infty \frac{1}{(2n+1)^3 (2n-1)^2 (2n+3)^2} \\ &= \int_{-1}^1 E'^2\left(\cos \frac{\theta}{2}\right) d\mu, \end{aligned}$$

$\therefore$ by previous result

$$\int_0^1 (E(k))^2 k dk = \frac{7}{256} \sigma_3. \quad \dots \quad (48)$$

Again

$$\int_{-1}^1 E E' d\mu = \sum_0^\infty \frac{2}{2n+1} \cdot \frac{(-)^n}{(2n-1)^2 (2n+1)^2 (2n+3)^2}$$

$$\begin{aligned}\therefore \int_0^1 kEE'dk &= \frac{1}{2} \sum_0^\infty \frac{(-)^n}{(2n-1)^2 (2n+1)^3 (2n+3)^2} \\ &= \frac{1}{32} \sum_0^\infty \frac{(-)^n}{(2n+1)^3} \\ &= \frac{\pi^3}{1024}.\end{aligned}$$

Hence $\int_0^1 kEE'dk = \frac{\pi^3}{1024} = \frac{1}{64} \int_0^1 kKK'dk. \quad . \quad . \quad (49)$

Again, by (43) and (9),

$$\begin{aligned}\int_0^1 kEK'dk &= -\sum_0^\infty \frac{1}{(2n+1)^2 (2n+1) (2n-1) (2n+3)} \frac{(-1)^n}{(2n-1) (2n+3) (2n+1)^3}, \\ &= \sum_0^\infty \frac{(-1)^{n-1}}{(2n-1) (2n+3) (2n+1)^3},\end{aligned}$$

and

$$\begin{aligned}\int_0^1 kE'Kdk &= \sum_0^\infty \frac{(-)^{n-1}}{(2n-1) (2n+3) (2n+1)^3} \\ &= \int_0^1 kEK'dk \quad . \quad . \quad . \quad . \quad . \quad (50)\end{aligned}$$

Now

$$EK' + E'K = KK' + \frac{\pi}{2}$$

(Cayley, *Elliptic Functions*, p. 48)

$$\begin{aligned}\int_0^1 kEK'dk &= \frac{1}{2} \int_0^1 k \left\{ KK' + \frac{\pi}{2} \right\} dk \\ &= \frac{1}{2} \left(\frac{\pi^2}{16} + \frac{\pi}{4} \right),\end{aligned}$$

$$\therefore \int_0^1 kEK'dk = \int_0^1 kE'Kdk = \frac{\pi^3}{32} + \frac{\pi}{8}. \quad . \quad . \quad (51)$$

Again,

$$\int_0^1 kKEdk = -\sum_0^\infty \frac{1}{(2n-1) (2n+3) (2n+1)^3},$$

and

$$\int_0^1 kK'E'dk = -\sum_0^\infty \frac{1}{(2n-1) (2n+3) (2n+1)^3};$$

$$\therefore \int_0^1 kKEdk = \int_0^1 kK'E'dk \quad . \quad . \quad . \quad (52)$$

The value of each

$$\begin{aligned}
 &= -\frac{1}{4} \sum_0^\infty \left(\frac{1}{2n-1} - \frac{1}{2n+3} \right) \cdot \frac{1}{(2n+1)^3} \\
 &= -\frac{1}{4} \sum_0^\infty \frac{1}{8} \left(\frac{1}{2n-1} + \frac{1}{2n+3} \right) - \frac{1}{8} \left(\frac{4}{(2n+1)^3} + \frac{2}{(2n+1)^2} + \frac{1}{(2n+1)} \right) \\
 &\quad - \frac{1}{8} \left\{ \frac{4}{(2n+1)^3} - \frac{2}{(2n+1)^2} + \frac{1}{2n+1} \right\} \\
 &= \frac{1}{4} \sum_0^\infty \frac{1}{(2n+1)^3} + \frac{1}{16} \\
 &= \frac{7}{32} \sigma_3 + \frac{1}{16};
 \end{aligned}$$

$$\therefore \int_0^1 k K E d k = \int_0^1 k K' E' d k = \frac{7}{32} \sigma_3 + \frac{1}{16} \quad \dots (53)$$

We may employ the result

$$\int_{-1}^1 (1-\mu^2)^r \frac{d^r P_m(\mu)}{d\mu^r} \frac{d^r P_n(\mu)}{d\mu^r} d\mu = 0, \quad \text{or} \quad \frac{2}{2n+1} \frac{n+r!}{n-r!}, \quad (54)$$

according as n and m are unequal or equal, provided r be a small enough integer. *E. g.*, we deduce, if $r=1$,

$$\int_{-1}^1 (1-\mu^2) \frac{d}{d\mu} K(k) \cdot \frac{d P_n}{d\mu} d\mu = \frac{2n(n+1)}{(2n+1)^2},$$

which coincides with a previous result,

and also

$$\int_{-1}^1 (1-\mu^2) \frac{dE}{d\mu} \frac{d P_n}{d\mu} d\mu = - \frac{2n(n+1)}{(2n-1)(2n+1)^2(2n+3)},$$

also a previous result.

If $r=2$,

$$\int_{-1}^1 (1-\mu^2)^2 \frac{d^2 E}{d\mu^2} \frac{d^2 P_n}{d\mu^2} d\mu = - \frac{2(n-1)(n)(n+1)(n+2)}{(2n-1)(2n+1)^2(2n+3)},$$

or

$$\int_0^1 k^3 k'^2 E \cdot P_n''(2k^2-1) dk = - \frac{1}{4} \cdot \frac{(n-1)(n)(n+1)(n+2)}{(2n-1)(2n+1)^2(2n+3)} \quad (55)$$

and further particular theorems follow in a similar way, the expansion for E' giving rise to an equal number.

Some applications to incomplete elliptic integrals will now be treated. The elliptic integrals in question occur in the

integrands of other integrals, and the integration is now with respect to their amplitude, and not their modulus as in the above treatment. $E(\lambda)$ now will denote

$$\int_0^\lambda \sqrt{1-k^2 \sin^2 \lambda} d\lambda, \quad \text{and } F(\lambda) = \int_0^\lambda \frac{d\lambda}{\sqrt{1-k^2 \sin^2 \lambda}}.$$

In all cases, the modulus is $k = \cos \frac{\theta}{2}$ for E , F , and $\sin \frac{\theta}{2} = k'$ for E' , F' .

If z is between 0 and π ,

$$\frac{\pi}{8}(\pi - 2z) = \cos z + \frac{1}{3^2} \cos 3z + \dots;$$

$$\therefore \sum_0^\infty \frac{P_n(\mu)}{(2n+1)^2} = \frac{1}{8} \int_0^\theta \frac{(\pi - \phi) d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}}.$$

But

$$\begin{aligned} \int_0^\theta \frac{\pi d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} &= 2\pi K' \left(\cos \frac{\theta}{2} \right) \\ &= 4\pi \sum_0^\infty \frac{(-)^n}{(2n+1)} P_n(\mu) \quad \text{by (9);} \end{aligned}$$

$$\therefore \int_0^\theta \frac{\phi d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} = 4\pi \sum_0^\infty \frac{(-)^n}{2n+1} P_n(\mu) - 8 \sum_0^\infty \frac{P_n(\mu)}{(2n+1)^2} \quad (56)$$

Put

$$\tan \frac{\phi}{2} = \tan \frac{\theta}{2} \cdot \cos \lambda;$$

$$\therefore \int_0^\theta \frac{d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} = 2F'(\lambda, k);$$

$$\begin{aligned} \therefore \int_0^\theta \frac{\phi d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} &= \left[2\phi F'(\lambda) \right]_0^\theta - 2 \int_0^{\frac{\pi}{2}} \frac{d\phi}{d\lambda} \cdot F'(\lambda) d\lambda \\ &= 4 \int_0^{\frac{\pi}{2}} \frac{F'(\lambda) \sin \lambda d\lambda}{1 + \tan^2 \frac{\theta}{2} \cos^2 \lambda} \cdot \tan \frac{\theta}{2}, \end{aligned}$$

since

$$\frac{d\phi}{d\lambda} = \frac{-2 \sin \lambda \cdot \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2} \cos^2 \lambda};$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{F'(\lambda) \sin \lambda d\lambda}{k^2 + k'^2 \cos^2 \lambda} = \frac{\pi}{kk'} \sum_0^{\infty} \frac{(-)^n}{2n+1} P_n(\mu) - \frac{2}{kk'} \sum_0^{\infty} \frac{P_n(\mu)}{(2n+1)^2}, \quad (57)$$

if $k = \cos \frac{\theta}{2}$, $\mu = \cos \theta$.

It follows that

$$\begin{aligned} \int_0^1 k^2 k' dk \int_0^{\frac{\pi}{2}} \frac{F'(\lambda) \sin \lambda d\lambda}{k^2 + k'^2 \cos^2 \lambda} &= \int_{-1}^1 \frac{\pi - 2}{4} d\mu \\ &= \frac{\pi}{2} - 1. \quad \dots \quad (58) \end{aligned}$$

Also

$$\int_0^1 k^2 k' P_n(2k^2 - 1) dk \int_0^{\frac{\pi}{2}} \frac{F'(\lambda) \sin \lambda d\lambda}{k^2 + k'^2 \cos^2 \lambda} = \frac{(-)^n \pi}{2(2n+1)^2} - \frac{1}{(2n+1)^3}. \quad (59)$$

While squaring (57), and calling the integral on the left, I,

$$\begin{aligned} \int_{-1}^1 I k^2 k'^2 d\mu &= \sum_0^{\infty} \left(\frac{\pi(-)^n}{2n+1} - \frac{2}{(2n+1)^2} \right)^2 \frac{2}{2n+1} \\ &= 2\pi^2 \cdot \frac{7}{8} \sigma_3 + \frac{31}{4} \sigma_5 - 8\pi \left(1 - \frac{1}{3^4} + \frac{1}{5^4} \dots \right); \\ \therefore \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{k^3 k'^2 F'(\lambda) F'(\mu) \sin \lambda \sin \mu dk d\lambda d\mu}{(k^2 + k'^2 \cos^2 \lambda)(k^2 + k'^2 \cos^2 \mu)} \\ &= \frac{7}{16} \pi^2 \sigma_3 + \frac{31}{8} \sigma_5 - 2\pi \left(1 - \frac{1}{3^4} + \frac{1}{5^4} \dots \right). \quad (60) \end{aligned}$$

It may be noted that if

$$G(\theta) = \int_0^{\theta} \frac{(\pi - \phi) d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}};$$

then

$$\begin{aligned} \int_0^{\pi} G^2(\theta) \sin \theta d\theta &= 64 \sum_0^{\infty} \frac{2}{(2n+1)^5} \\ &= 128 \left(1 - \frac{1}{2^5} \right) \sigma_5; \end{aligned}$$

$$\therefore \int_0^{\pi} G^2(\theta) \sin \theta d\theta = 124 \sigma_5 \quad \dots \quad (61)$$

Similar results follow from the series

$$\cos z - \frac{1}{3^3} \cos 3z + \frac{1}{5^3} \cos 5z \dots = \frac{\pi}{8} (\pi^2 - z^2) \quad z \text{ between } \pm \frac{\pi}{2};$$

for

$$\begin{aligned}\sum_0^{\infty} \frac{(-)^n}{(2n+1)^3} P_n(\mu) &= \frac{1}{32} \int_0^{\theta} \frac{(4\pi^2 - \phi^2) d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} \\ &= \frac{\pi^2}{8} \cdot 2K' - \frac{1}{32} \int_0^{\theta} \frac{\phi^2 d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}}\end{aligned}$$

But by previous substitution

$$\begin{aligned}\int_0^{\theta} \frac{\phi^2 d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} &= \left[2\phi^2 F' \left(\lambda, \cos \frac{\theta}{2} \right) \right]_{\phi=0}^{\theta} \\ &\quad + 16 \int_0^{\frac{\pi}{2}} \frac{F'(\lambda) \sin \lambda \tan \frac{\theta}{2} \tan^{-1} \left(\frac{k'}{k} \cos \lambda \right) d\lambda}{1 + \tan^2 \frac{\theta}{2} \cos^2 \lambda} \\ &= 16kk' \int_0^{\frac{\pi}{2}} \frac{F'(\lambda) \sin \lambda \tan^{-1} \left(\frac{k'}{k} \cos \lambda \right) d\lambda}{k^2 + k'^2 \cos^2 \lambda}; \\ \therefore \int_0^{\frac{\pi}{2}} \frac{F'(\lambda) \tan^{-1} \left(\frac{k'}{k} \cos \lambda \right) \sin \lambda d\lambda}{k^2 + k'^2 \cos^2 \lambda} &= \frac{\pi^2}{kk'} \sum_0^{\infty} \frac{(-)^n}{(2n+1)} P_n(\mu) \\ &\quad + \frac{2}{kk'} \sum_0^{\infty} \frac{(-)^{n-1}}{(2n+1)^3} P_n(\mu), \quad (62)\end{aligned}$$

and if

$$\begin{aligned}G_1(\theta) &= \int_0^{\theta} \frac{(4\pi^2 - \phi^2) d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}}; \\ \therefore \int_0^{\pi} G_1^2(\theta) \sin \theta d\theta &= (32)^2 \sum_0^{\infty} \frac{2}{(2n+1)^7} \\ &= 2 \times (32)^2 \cdot \sigma_7 \left(1 - \frac{1}{2^7} \right) \\ &= 2032 \sigma_7, \quad . \quad . \quad . \quad . \quad (63)\end{aligned}$$

and

$$\begin{aligned}\int_0^1 k^2 k' P_n(2k^2 - 1) dk \int_0^{\frac{\pi}{2}} \frac{F'(\lambda) \sin \lambda \tan^{-1} \left(\frac{k'}{k} \cos \lambda \right) d\lambda}{k^2 + k'^2 \cos^2 \lambda} \\ = \frac{\pi (-)^n}{2(2n+1)^2} + \frac{(-)^{n-1}}{(2n+1)^4}; \quad . \quad (64)\end{aligned}$$

also

$$\int_0^1 k^2 k' dk \int_0^{\frac{\pi}{2}} \frac{F'(\lambda) \sin \lambda \tan^{-1} \left(\frac{k'}{k} \cos \lambda \right) d\lambda}{k^2 + k'^2 \cos^2 \lambda} = \frac{\pi}{2} - 1; \quad (65)$$

and squaring (62) and integrating to k from 0 to 1,

$$\begin{aligned} \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{k^3 k'^2 F'(\lambda) F'(\mu) \tan^{-1} \left(\frac{k'}{k} \cos \lambda \right) \tan^{-1} \left(\frac{k'}{k} \cos \mu \right) \sin \lambda \sin \mu dk d\lambda d\mu}{(k^2 + k'^2 \cos^2 \lambda) (k^2 + k'^2 \cos^2 \mu)} \\ = \frac{1}{4} \sum_0^\infty \frac{2}{2n+1} \left(\frac{(-)^n \pi^2}{2n+1} + \frac{2(-)^{n-1}}{(2n+1)^3} \right)^2 \\ = \frac{\pi^4}{2} \left(1 - \frac{1}{2^3} \right) \sigma_3 + 2 \left(1 - \frac{1}{2^7} \right) \sigma_7 - 2\pi^2 \left(1 - \frac{1}{2^5} \right) \sigma_5 \\ = \frac{7}{16} \pi^4 \sigma_3 + \frac{127}{64} \sigma_7 - \frac{31}{16} \sigma_5 \cdot \cdot \cdot \cdot \cdot \quad (66) \end{aligned}$$

It may be remarked that formulæ of these kinds can be successively obtained, and the expansion of the integral

$$\int_0^{\frac{\pi}{2}} F'(\lambda) \left[\tan^{-1} \left(\frac{k'}{k} \cos \lambda \right) \right]^n \frac{\sin \lambda d\lambda}{k^2 + k'^2 \cos^2 \lambda}$$

in terms of Legendre's functions, when n is any positive integer, by taking a succession of Fourier series of the kinds already considered in the last two cases.

We may write (56) in a form involving only complete elliptic integrals outside, for

$$\begin{aligned} \int_0^\theta \frac{\phi d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} &= \left[\phi \cdot 2K' \cos \frac{\phi}{2} \right]_0^\theta - 2 \int_0^\theta K' \left(\cos \frac{\phi}{2} \right) d\phi \\ \therefore \int_0^\theta \frac{\phi d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} &= 4 \left\{ K' \cdot \cos^{-1} k - \frac{1}{2} \int_0^\theta K' \left(\cos \frac{\phi}{2} \right) d\phi \right\} \end{aligned}$$

$\therefore$ by (56),

$$2K' \cos^{-1} k - \int_0^\theta K' \left(\cos \frac{\phi}{2} \right) d\phi = 2\pi \sum_0^\infty \frac{(-)^n}{2n+1} P_n(\mu) - 4 \sum_0^\infty \frac{P_n(\mu)}{(2n+1)^2};$$

$\therefore$ by (9),

$$K' \cdot (2\pi - \cos^{-1} k) + \frac{1}{2} \int_0^\theta K' \left(\cos \frac{\phi}{2} \right) d\phi = 2 \sum_0^\infty \frac{P_n(\mu)}{(2n+1)^2}. \quad (67)$$

By changing θ to $\pi - \theta$,

$$K\left(\cos^{-1}k - \frac{\pi}{2}\right) - \frac{1}{2} \int_0^{\pi-\theta} K'\left(\cos \frac{\phi}{2}\right) d\phi = 2 \sum_0^{\infty} \frac{(-)^n}{(2n+1)^2} P_n(\mu). \quad (68)$$

These with (8) and (9) give the curious results

$$\left. \begin{aligned} & \left\{ 2\pi - \theta + \frac{1}{2} \int_0^{\theta} \frac{K'\left(\cos \frac{\phi}{2}\right)}{K'\left(\cos \frac{\theta}{2}\right)} d\phi \right\} \sum_0^{\infty} \frac{(-)^n}{2n+1} P_n(\cos \theta) \\ & \qquad \qquad \qquad = 4 \sum_0^{\infty} \frac{P_n(\cos \theta)}{(2n+1)^2}, \quad \dots \quad (69) \\ \text{and} & \left\{ \theta - \frac{\pi}{2} + \frac{1}{2} \int_{\theta}^{\pi} \frac{K\left(\cos \frac{\phi}{2}\right)}{K\left(\cos \frac{\theta}{2}\right)} d\phi \right\} \sum_0^{\infty} \frac{P_n(\cos \theta)}{2n+1}, \\ & \qquad \qquad \qquad = 4 \sum_0^{\infty} \frac{(-)^n P_n(\cos \theta)}{(2n+1)^2}, \quad (70) \end{aligned} \right\}$$

which hold from $\theta=0$ to 2π except at $\frac{\pi}{2}, \frac{3\pi}{2}$.

If

$$I_1 = \int_0^{\theta} \frac{K'\left(\cos \frac{\phi}{2}\right)}{K'\left(\cos \frac{\theta}{2}\right)} d\phi, \quad I_2 = \int_{\theta}^{\pi} \frac{K\left(\cos \frac{\phi}{2}\right)}{K\left(\cos \frac{\theta}{2}\right)} d\phi.$$

We may at once deduce certain integrals as follows:

$$\int_0^1 k(4\pi - 2 \cos^{-1} k + I_1) \cdot K' \cdot dk = 2 \quad \dots \quad (71)$$

$$\int_0^1 k(I_2 - \pi + 2 \cos^{-1} k) \cdot K \cdot dk = 2 \quad \dots \quad (72)$$

$$\int_0^1 k(4\pi - 2 \cos^{-1} k + I_1) \cdot K' \cdot P_n(2k^2 - 1) \cdot dk = \frac{7}{4} \sigma_3 \quad \dots \quad (73)$$

$$\int_0^1 k(I_2 - \pi + 2 \cos^{-1} k) \cdot K \cdot P_n(2k^2 - 1) \cdot dk = \frac{\pi^3}{16} \quad \dots \quad (74)$$

$$\int_0^1 k(4\pi - 2 \cos^{-1} k + I_1)^2 K'^2 dk = \frac{31}{4} \sigma_5 \quad \dots \quad (75)$$

$$\int_0^1 k(I_2 - \pi + 2 \cos^{-1} k)^2 \cdot K^2 \cdot dk = \frac{31}{4} \sigma_5 \quad \dots \quad (76)$$

From (67) and (7),

$$\int_0^1 (4\pi - 2 \cos^{-1} k + I_1) \cdot K' \cdot dk = 4 \left(1 - \frac{1}{3^2} + \frac{1}{5^2} \dots \right) = \frac{\pi^3}{8};$$

∴ by (21),

$$\int_0^1 K' \cdot (I_1 - 2 \cos^{-1} k) \cdot dk = \frac{7}{8} \pi^3. \quad \dots \quad (77)$$

By (68) and (7),

$$\int_0^1 (2 \cos^{-1} k - \pi + I_2) \cdot K \cdot dk = \frac{7}{2} \sigma_3. \quad \dots \quad (78)$$

Integrating all the above by parts, a large number of integrals involving E and K, E' and K' may be deduced.

By combinations of (67) and (68) with previous series of Legendre functions, we may deduce a large number of definite integrals of functions of k from 0 to 1. The following are examples :

$$(1) \ kPQ \left\{ \cos^{-1} k + \frac{1}{2} (I_1, I_2) \right\} \quad \text{where } P=K, K', E, E'$$

$$(2) \ kPQ \left\{ \sin^{-1} k + \frac{1}{2} (I_1, I_2) \right\} \quad ,, \quad Q=K, K', E, E'$$

and these integrals, when found, will on integration by parts give rise to great numbers of elliptic integrals, integrated with respect to their modulus.

It may here be noted, in connexion with the result (62) and similar results, that by a similar investigation with series of the type

$$\frac{\pi y}{4} = \sin y - \frac{1}{3^2} \sin 3y + \frac{1}{5^2} \sin 5y \dots$$

when y is between $\pm \frac{\pi}{2}$, we may in all cases expand, in a series of Legendre functions, the integral

$$\int_0^{\frac{\pi}{2}} \frac{F(\lambda) \left(\tan^{-1} \left(\frac{k'}{k} \sin \lambda \right) \right)^n \cos \lambda d\lambda}{k'^2 + k^2 \sin^2 \lambda},$$

where n is a positive integer, and $F(\lambda)$ is the incomplete elliptic integral

$$\int_0^\lambda \frac{d\lambda}{\sqrt{1-k^2 \sin^2 \lambda}}.$$

Considering now the equation (62), which may be written

$$\int_0^\theta \frac{\phi^2 d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} = 8\pi^2 K' - 32 \sum_0^\infty \frac{(-)^n}{(2n+1)^3} P_n(\mu),$$

we have

$$\int_0^\theta \frac{\phi^2 d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} = \left[2\phi^2 K' \left(\cos \frac{\phi}{2} \right) \right]_0^\theta - 4 \int_0^\theta \phi K' \left(\cos \frac{\phi}{2} \right) d\phi.$$

In this and above, (66)–(70), $K' \left(\cos \frac{\phi}{2} \right)$ denotes an incomplete integral equal to K' when $\phi = \theta$.

$$\therefore (\pi - \cos^{-1} k)(\pi + \cos^{-1} k) K' + \int_0^\theta \phi K' \left(\cos \frac{\phi}{2} \right) d\phi = 4 \sum_0^\infty \frac{(-)^n}{(2n+1)^3} (P_n \mu), \quad (79)$$

and putting $\pi - \theta$ for θ in the above,

$$\begin{aligned} \therefore \left[2\phi^2 K' \left(\cos \frac{\phi}{2} \right) \right]_0^{\pi-\theta} - 4 \int_0^{\pi-\theta} \phi K' \left(\cos \frac{\phi}{2} \right) d\phi &= 8\pi^2 K - 32 \sum_0^\infty \frac{P_n(\mu)}{(2n+1)^3} \\ \therefore \sin^{-1} k \{ \pi + \sin^{-1} k \} K + \int_0^{\pi-\theta} \phi K' \left(\cos \frac{\phi}{2} \right) d\phi &= 4 \sum_0^\infty \frac{P_n(\mu)}{(2n+1)^3} \quad \dots \quad (80) \end{aligned}$$

Proceeding in this manner, it follows that

$$\sum_0^\infty \frac{P_n(\mu)}{(2n+1)^r}, \quad \sum_0^\infty \frac{(-)^n P_n(\mu)}{(2n+1)^r},$$

where r is a positive integer, can be expressed in a similar manner to the above in terms of Jacobian elliptic functions, and their integrals with respect to a function of their amplitude.

The sum of all such series can therefore be concisely expressed.