On Aurifeuillians. By Lt.-Col. ALLAN CUNNINGHAM, R.E., Fellow of King's College, London. Part received and read February 10th, 1898. Part received and read March 10th, 1898.

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CHAPTER I., AURIFEUILLIANS.

CHAPTER II., BIN-AURIFEUILLIANS. CHAPTER III., TRIN-AURIFEUILLIANS.

Notation.—All symbols used in this paper are integer numbers: p denotes a prime, ω an odd number, ϵ an even number; λ, μ, ν denote odd numbers (except in Art. 17, 18, 31, 32.); A.P.F. denotes algebraic prime factor.

CHAPTER I., AURIFEUILLIANS.

1. Algebraic Prime Factors (A.P.F.).

Let n be an odd number; n_1, n_2, n_3 , &c., different odd primes.

Let $\phi(n, U, V)$, or (for shortness), $\phi(n) = (U^n \pm V^n)^*$ (1)

so that $\phi(1, U, V)$, or shortly, $\phi(1) = (U \pm V)^*$ (1a)

Here $\phi(n)$ may be used, for shortness, whenever the form of the functions U, V is not in question.

Then the algebraic prime factor, or, for shortness, A.P.F., of the function $\phi(n, U, V)$, or of $\phi(n)$, is defined by

$$\mathfrak{F}\phi(n_1) = \frac{\phi(n_1)}{\phi(1)}, \quad \mathfrak{F}\phi(n_1 n_2) = \frac{\phi(n_1 n_2) \phi(1)}{\phi(n_1) \phi(n_2)}, \quad (2a)$$

$$\mathfrak{F}\phi(n_1 n_2 n_3) = \frac{\phi(n_1 n_2 n_3) \phi(n_1) \phi(n_2) \phi(n_3)}{\phi(1) \phi(n_2 n_3) \phi(n_3 n_1) \phi(n_1 n_2)}, \quad \&c., \quad (2b)$$

and the law of formation is obvious.† When n_1, n_2, n_3 , &c., are not prime, the function $\mathfrak{F}$ above formed is simply an algebraic factor of $\phi(n)$, but not the algebraic prime factor thereof.

As above defined, the function $\mathfrak{F}\phi(n, U, V)$ is independent of the forms of U, V . When, however, $U = u^r, V = v^r$, and r is odd, a further reduction may be made, and the Reduced algebraic prime factor $\mathfrak{F}$ of the function $(u^r \pm v^r)$, i.e., the quantity $\mathfrak{F}\phi(nu, u, v)$, may be found

* There is no ambiguity of sign: the sign used in $\phi(n)$ being used in all its factors including the A.P.F., i.e. in $\mathfrak{F}\phi(n)$.

† In fact $\phi(n, U, V)$ is the Norm of the binomial complex integer $(U - V\rho)$, where ρ is a proper n^{th} root of 1.

by a similar procedure. Of course, this quantity is of a *lower order* than the algebraic prime factor $\mathfrak{F}\phi(n, u^r, v^r)$ previously found. In this way a function $(u^{nr} \pm v^{nr})$ with *composite* index (nr) may be said to have algebraic prime factors of *different orders*, viz.,

$$\mathfrak{F}\phi(nr, u, v), \quad \mathfrak{F}\phi(n, u^r, v^r), \quad \mathfrak{F}\phi(r, u^n, v^n),$$

according as it is considered as a function of (u, v) , or of (u^r, v^r) , or of (u^n, v^n) .

2. *Aurifeuillians*.—It is proposed to apply the name* *Aurifeuillian* of order $2n$ or n respectively to the following functions (both algebraic prime factors):—

$$N = \mathfrak{F}(X^{2n} + Y^{2n}), \quad \text{where } 2nXY = z^2 \text{ (a square),} \quad (3)$$

$$N = \mathfrak{F}(X^n + \epsilon^{n+1}Y^n), \quad \text{where } nXY = z^2 \text{ (a square) } \left. \vphantom{\begin{matrix} N = \mathfrak{F}(X^n + \epsilon^{n+1}Y^n), \\ \text{where } n \text{ is odd and } \epsilon = \sqrt{-1}, \text{ so that } \epsilon^{n+1} = \pm 1 \end{matrix}} \right\} \quad (4)$$

[Here n is supposed to contain *no square factors*; and X, Y are supposed relatively $\dagger$ prime.]

These functions will be styled, for shortness, *Bin-, Trin-, Quint-, &c.*, *Aurifeuillians* when the *order* = 2, 3, 5, &c.

The condition $2nXY = z^2$, or $nXY = z^2$, taken together with the working condition (X *prime* to Y) involves *one* of the following groups‡ of conditions:—

$$X = x^2, \quad Y = 2ny^2; \quad x \text{ odd, and prime to } n, y, \quad (3a)$$

$$X = n_1x^2, \quad Y = 2n_2y^2; \quad n = n_1n_2\S; \quad x \text{ odd; } n_1 \text{ and } x \text{ prime to } n_2 \text{ and } y, \quad (3b)$$

$$X = x^2, \quad Y = ny^2; \quad x \text{ prime to } n, y, \quad (4a)$$

$$X = n_1x^2, \quad Y = n_2y^2; \quad n = n_1n_2\S; \quad n_1 \text{ and } x \text{ prime to } n_2 \text{ and } y. \quad (4b)$$

Thus the most general forms are

$$N = \mathfrak{F}\{(x^2)^{2n} + (2ny^2)^{2n}\}, \quad (5a)$$

$$N = \mathfrak{F}\{(n_1x^2)^{2n} + (2n_2y^2)^{2n}\}, \quad \text{where } n_1n_2 = n, \quad (5b)$$

$$N = \mathfrak{F}\{(x^2)^n + \epsilon^{n+1}(ny^2)^n\}, \quad (6a)$$

$$N = \mathfrak{F}\{(n_1x^2)^n + \epsilon^{n+1}(n_2y^2)^n\}, \quad \text{where } n_1n_2 = n, \quad (6b)$$

* See Art. 3 for explanation of this term.

† These conditions greatly simplify the details, without any real loss of generality.

‡ Hereby N will be an *odd* number, free from unnecessary factors.

§ The working condition as to n above involves n_1, n_2 both free from square factors.

The forms (5a), (6a) are only simpler forms of the more general forms (5b), (6b); but, being the more important, are worth separate statement.

From (5b), (6b) it is seen that, when n is a *composite* odd number, there arise *different species* of Aurifeuillians all of the *same order* ($2n$ or n), the number of species being the same as the number of ways in which $2n$ or n can be resolved into a pair of (mutually prime) factors.

Ex. $\frac{x^{12} + 6^6 y^{12}}{x^4 + 6^2 y^4}$ and $\frac{3^6 x^{12} + 2^6 y^{12}}{3^2 x^4 + 2^2 y^4}$ are Sext-Aurifeuillians;

$\frac{x^{30} + 15^{15} y^{30}}{x^{10} + 15^3 y^{10}}$ and $\frac{x^2 + 15 y^2}{x^6 + 15^3 y^6}$ and $\frac{3^{15} x^{30} + 5^{15} y^{30}}{3^6 x^{10} + 5^6 y^{10}}$ and $\frac{3x^2 + 5y^2}{3^3 x^6 + 5^3 y^6}$ are Quindec-Aurifeuillians.

In the abbreviated notation of Art. 1, the Aurifeuillians (3), (4) would be written

$$N = \mathfrak{F}(n, X^2, Y^2), \text{ or simply } = \mathfrak{F}(n), \text{ when } 2nXY = \square, \quad (3, i)$$

$$N = \mathfrak{F}(n, X, Y), \text{ or simply } = \mathfrak{F}(n), \text{ when } nXY = \square \quad \left. \vphantom{\begin{matrix} N = \mathfrak{F}(n, X, Y), \\ \text{the sign } (\pm) \text{ between } x^n, y^n \text{ being fixed, so that } x^{n+1} = \pm 1 \end{matrix}} \right\} (4, i)$$

3. Aurifeuillian Factors (L, M).—The interest of these numbers lies chiefly in the fact of their being expressible* *algebraically* as the *difference of two squares* (say $P^2 - Q^2$), so that they are *algebraically resolvable* into two† factors ($P \pm Q$). These factors will be styled, for shortness, *Aurifeuillian Factors*, and will be denoted by L, M ; so that $N = L.M$: their salient property is

$$\text{If } L = f(x, -y), \text{ then } M = f(x, +y); \text{ and } L < M. \quad (7)$$

4. Ant-Aurifeuillians.—Changing Y into $-Y$ in (4), and therewith

* This property was discovered by M. Aurifeuille, of Toulouse. A table of the values of P and $Q + \sqrt{2nXY}$ or $Q + \sqrt{nXY}$ is given in Ed. Lucas's paper, "Sur la Série récurrente de Fermat," Rome, 1879, p. 6, for the following cases:—

$n = 3, 5, 7, 11, 13, 15, 17, 19, 21, 23, 29, 31, 33$; $2n = 2, 6, 10, 14, 22, 26, 30, 34$.

From these the factors L, M can be at once found, as ($P \mp Q$). The values of L, M are also given in recent papers by Mr. C. E. Bickmore, "On the Numerical Factors of $(a^n - 1)$," in the *Messenger of Mathematics*, Vol. xxv., 1896, pp. 1-44, and Vol. xxvi., 1897, pp. 1-37, for the following cases:—

$n = 3, 5, 7, 11, 15$; $2n = 2, 6, 10, 14, 30$.

† By this property, together with the use of "special divisors," it is possible to factorise many very large numbers *completely* (i.e., into their *prime* factors). Many examples will be given in the sequel of numbers of over 20 figures; and some of 30, 40, 50, and (over) 60 figures completely factorised, see Tables, Arts. 16, 18, 81, 82.

y^2 into $-y^2$ in $(4a, b)$, $(6a, b)$ a new number* (N') results, which may be styled *Ant-Aurifeuillian*, from its being *algebraically* separable into a sum of two squares (say $P^2 + Q^2$), in the same way as the Aurifeuillian itself is separable into a difference of squares, the expressions being in fact conjugate. Here

$$N' = \mathfrak{F} \{ (x^2)^n - \epsilon^{n+1} (ny^2)^n \}, \quad (8a)$$

$$N' = \mathfrak{F} \{ (n_1 x^2)^n - \epsilon^{n+1} (n_2 y^2)^n \}. \quad (8b)$$

5. *Quadratic Forms*.—In common with the A.P.F. of numbers of form $(X^n \pm Y^n)$ in general, all Aurifeuillians can be expressed *algebraically* in various quadratic forms of the type† $(n_1 X^2 + \epsilon^{n+1} n_2 H^2)$, where $n_1 n_2 = n$, and generally in more than one way in each type defined by a particular factorization $n_1 n_2 = n$ (n_1 prime to n_2).

Moreover the Aurifeuillian Factors L, M can themselves also be expressed *algebraically*‡ in quadratic forms *isomorphic* with each other, and *conformal* with those of N .

Thus, with Aurifeuillians of *odd* order (n),

$$\text{if} \quad L = n_1 \{ \phi_1(x, -y) \}^2 + \epsilon^{n+1} n_2 \{ \phi_2(x, -y) \}^2, \quad (9a)$$

$$\text{then, by (7), } M = n_1 \{ \phi_1(x, y) \}^2 + \epsilon^{n+1} n_2 \{ \phi_2(x, y) \}^2. \quad (9b)$$

Hence, by *conformal*§ multiplication,

$$N = \Phi_1^2 + \epsilon^{n+1} n \Phi_2^2, \quad (\text{since } n = n_1 n_2), \quad (10)$$

$$\text{where} \quad \Phi_1 = n_1 \phi_1(x, -y) \phi_1(x, y) \pm n_2 \phi_2(x, -y) \phi_2(x, y), \quad (10a)$$

$$\Phi_2 = \phi_1(x, -y) \phi_2(x, y) \mp \phi_2(x, -y) \phi_1(x, y). \quad (10b)$$

(the signs being *unlike*, or *like* in Φ_1, Φ_2 according as ϵ^{n+1} is + or -).

Results (9a, b), (10a, b) apply, with slight modifications, (e.g., omitting ϵ^{n+1} , &c.) to Aurifeuillians of *even* order ($2n$).

* This modification of the Aurifeuillian numbers (4) is due to Mr. Bickmore; see his papers quoted above, *passim*. Note that every Aurifeuillian of *even* order is obviously expressible in form $(a^2 + b^2)$ (being itself the A.P.F. of a sum of two squares).

† When $n > 10$, the form $\frac{1}{4} (n_1 X^2 + \epsilon^{n+1} n_2 H^2)$ has sometimes to be used instead of that given.

‡ This has been shown in the two papers by Mr. Bickmore, quoted above, for most of the values of n used by him; and, from the manner of composition of these numbers from their complex factors, it is seen to be generally true. This property greatly enhances the interest and utility of the Aurifeuillian numbers.

§ I.e., multiplication of two *conformals* $(n_1 \phi_1^2 + n_2 \phi_2^2) \cdot (n_1 \psi_1^2 + n_2 \psi_2^2)$ to form the *conformal* product (10). See the author's paper "On the Connexion of Quadratic Forms," in *Proc. Lond. Math. Soc.*, Vol. xxviii, 1897, Art. 6-13.

Hence every distinct partition of the factors L, M , defined by a particular set of values of n_1, n_2, ϕ_1, ϕ_2 , yields *two* isomorphic partitions of N , in consequence of the double sign ($\pm$) entering into ϕ_1, ϕ_2 .

But, further, each distinct pair of values of n_1, n_2 yields in many cases either two or three *alternative* forms of the functions ϕ_1, ϕ_2 , and therefore also of the pair L, M , corresponding to the sub-cases when one of x, y is even (and the other odd), or *both* of x, y odd; and each such alternative pair of L, M yields—by *conformal multiplication* as above (10, 10a, b)—*two* forms of the product (N) each conformal with L, M . This may be denoted as follows, where similar forms of L, M, N are denoted by like accents,

$$L', M' \text{ yielding } N' \text{ and } N^\circ, \text{ when } y \text{ is even,} \quad (11a)$$

$$L'', M'' \text{ yielding } N'' \text{ and } N^\circ, \text{ when } x \text{ is even,} \quad (11b)$$

$$L''', M''' \text{ yielding } N''' \text{ and } N^\circ, \text{ when } x, y \text{ are both odd;} \quad (11c)$$

or thus

$$L', M' \text{ yielding } N' \text{ and } N^\circ, \text{ when one of } x, y \text{ is even,} \quad (11d)$$

$$L'', M'' \text{ yielding } N'' \text{ and } N^\circ, \text{ when } x, y \text{ are both odd.} \quad (11e)$$

[The forms differently accented (as L', L'', L''' , &c.) are evidently *arithmetically distinct*: they differ, however, only by a unit-factor, the form of which depends on the value of n ; e.g., when $n = 3, 5$ the unit-factor is $\frac{1}{4}(1^2 + 3 \cdot 1^2), \frac{1}{4}(3^2 - 5 \cdot 1^2)$ respectively; from this it comes that the products of the similarly accented $L'M', L''M'', L'''M'''$ each yield *one* product form with the like accent N', N'', N''' , and each yields also the *same* (general) form N° .]

Ex.—See Art. 21 for examples of (11 a, b, c), and Art. 30a for examples of (11 d, e).

6. *Fermat's Divisors*.—This name may be applied to *prime* divisors (p) depending directly on Fermat's theorem. Let

$$\mathfrak{Y}_e = \{(n_1 x^2)^{2n} + (2n_2 y^2)^{2n}\}, \quad \text{where } n = n_1 n_2, \quad (12)$$

$$\mathfrak{Y}_w = \{(n_1 x^2)^n + i^{n+1} (n_2 y^2)^n\}, \quad \text{where } n = n_1 n_2. \quad (13)$$

The prime divisors to be considered are of *two* forms $(2\rho + 1)$ and $(4\rho + 1)$, where ρ denotes the Aurifeuillian order, i.e., $\rho = 2n$ or n in case of $\mathfrak{Y}_e, \mathfrak{Y}_w$ respectively.

6a. *Divisor* $p = (2\rho + 1)$. By Fermat's theorem applied to $\mathfrak{Y}_e$, for which $\rho = 2n$,

$$\mathfrak{Y}_e \equiv n_1^{1(p-1)} + (2n_2)^{1(p-1)}, \pmod{p}, \quad [\text{here } x, y \text{ both prime to } p]$$

$$\equiv \left(\frac{n_1}{p}\right) + \left(\frac{2}{p}\right) \left(\frac{n_2}{p}\right), \quad \text{in Legendre's notation.}$$

Here $n = 2n_1n_2$, $p = 2\rho + 1 = 4n + 1$.

Also, if n_1, n_2 be *primes*, then*

$$\left(\frac{n_1}{p}\right) = \left(\frac{n_2}{p}\right) = +1,$$

and, if n_1, n_2 be *composites*, viz.,

$$n_1 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \dots, \quad n_2 = \mu_1 \cdot \mu_2 \cdot \mu_3 \dots$$

Then, for every* factor λ_i, μ_i ,

$$\left(\frac{\lambda_i}{p}\right) = \left(\frac{\mu_i}{p}\right) = +1,$$

and

$$\left(\frac{2}{p}\right) = -1;$$

therefore

$$\mathfrak{Y}_e \equiv 0, \pmod{p}. \quad (14a)$$

Next, by Fermat's theorem applied to $\mathfrak{Y}_e$, for which $\rho = n$,

$$\begin{aligned} \mathfrak{Y} &= \frac{1}{n_1^n} \{ (n_1 x)^{2n} + \epsilon^{n+1} n^n y^{2n} \} \\ &\equiv \frac{1}{n_1^n} \{ 1 + \epsilon^{n+1} n^{1(p-1)} \}, \pmod{p}, \text{ [here } x, y \text{ both prime to } p] \\ &\equiv \frac{1}{n_1^n} \left\{ 1 + \epsilon^{n+1} \left(\frac{n}{p}\right) \right\}, \pmod{p}. \end{aligned}$$

Now suppose $n = ST$, where S contains only factors of form $(4s+1)$, and T contains only factors of form $(4t-1)$, so that

$$S = s_1 s_2 s_3 \dots s_a, \quad \text{where } s_z = 4s_z + 1, \quad (z = 1 \text{ to } z = a),$$

$$T = t_1 t_2 t_3 \dots t_\beta, \quad \text{where } t_z = 4t_z - 1, \quad (z = 1 \text{ to } z = \beta),$$

$$p = 2n + 1 = 4\varpi - 1, \quad (\text{since } n \text{ is odd}).$$

Therefore $\left(\frac{s_z}{p}\right) = +1$, for all† values of z ; and $\left(\frac{S}{p}\right) = +1$,

$$\left(\frac{t_z}{p}\right) = -1, \text{ for all† values of } z,$$

$$\left(\frac{T}{p}\right) = +1; \quad \text{and } n = 4\nu + 1, \epsilon^{n+1} = -1, \text{ when } \beta \text{ is even,}$$

$$\left(\frac{T}{p}\right) = -1; \quad \text{and } n = 4\nu - 1, \epsilon^{n+1} = +1, \text{ when } \beta \text{ is odd.}$$

* This follows from the law of quadratic reciprocity, noting that $n_1, n_2; \lambda_i, \mu_i$ are factors of $(p-1)$.

† This follows from the law of quadratic reciprocity, noting that s_z, t_z are factors of $(p-1)$.

Hence $\mathfrak{F}_\omega \equiv 0, \pmod{p}$, in all cases. (14b)

Hence, by (14a, b),

$$\mathfrak{F}_\epsilon \text{ and } \mathfrak{F}_\omega \text{ are always divisible by } p = 2\rho + 1, \quad (14c)$$

(when x, y prime to p),

and since, by definition, the Aurifeuillian (N) is the A.P.F. of $\mathfrak{F}_\epsilon$ or $\mathfrak{F}_\omega$, it follows* that—

“Either the Aurifeuillian (N), or some one of the other submultiples of $\mathfrak{F}_\epsilon, \mathfrak{F}_\omega$, is divisible by $p = 2\rho + 1$ (when x, y prime to p).” (14)

6b. *Divisor* $p = (4\rho + 1)$. Applying Fermat's theorem to $\mathfrak{F}_\epsilon, \mathfrak{F}_\omega$,

$$\begin{aligned} \mathfrak{F}_\epsilon &= \frac{1}{n_1^{2n}} \{ (n_1 x)^{4n} + (2n)^{2n} y^{4n} \}, \text{ where } \rho = 2n, p = 8n + 1 \\ &\equiv \frac{1}{n_1^{2n}} \left\{ \left(\frac{n_1}{p} \right) \left(\frac{x}{p} \right) + (2n)^{2n} \left(\frac{y}{p} \right) \right\}, \pmod{p}, \\ &\quad [\text{here } x, y \text{ both prime to } p]; \end{aligned}$$

$$\begin{aligned} \mathfrak{F}_\omega &\equiv \frac{1}{n_1^n} \{ (n_1 x)^{2n} + \epsilon^{n+1} \cdot n^n \cdot y^{2n} \}, \text{ where } \rho = n, p = 4n + 1 \\ &\equiv \frac{1}{n_1^n} \left\{ \left(\frac{n_1}{p} \right) \left(\frac{x}{p} \right) + \epsilon^{n+1} \cdot n^n \left(\frac{y}{p} \right) \right\}, \pmod{p}, \\ &\quad [\text{here } x, y \text{ both prime to } p]. \end{aligned}$$

Now, in both cases, since $p = 4\rho + 1 = 8n + 1$ in case of $\mathfrak{F}_\epsilon$, or $= 4n + 1$ in case of $\mathfrak{F}_\omega$,

$$\left(\frac{n_1}{p} \right) = +1, \dagger \quad \text{and} \quad n^n \equiv +1, \ddagger \pmod{p},$$

and $\epsilon^{n+1} = +1$ or -1 , according as $n = 4\nu - 1$ or $4\nu + 1$; therefore

$$\begin{aligned} \mathfrak{F}_\epsilon &\equiv \frac{1}{(n_1)^{2n}} \left\{ \left(\frac{x}{p} \right) + \left(\frac{y}{p} \right) \right\}, \pmod{p}, \\ \mathfrak{F}_\omega &\equiv \frac{1}{n_1^n} \left\{ \left(\frac{x}{p} \right) + \epsilon^{n+1} \left(\frac{y}{p} \right) \right\}, \pmod{p}. \end{aligned}$$

* Part of this proof was communicated to the author by Mr. Bickmore.

† This follows from the law of quadratic reciprocity.

‡ See *Educational Times*, Vol. LXV., p. 78, Question 13058, by C. E. Bickmore, proving that $(\rho - 1)$ is divisible by $p = 4\rho + 1$.

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Hence, using the abbreviations Q.R. = quadratic residue of p ,
Q.N.R. = quadratic non-residue of p ,

$$\mathfrak{F} \equiv 0, \pmod{p}, \text{ when one of } x, y \text{ is Q.R. and the other Q.N.R. of } p, \quad (15a)$$

$$\mathfrak{F}_\omega \equiv 0, \pmod{p}, \text{ if } n = 4\nu - 1, \text{ when one of } x, y \text{ is Q.R.,} \\ \text{and the other Q.N.R. of } p \left\{ \right. \\ \text{also, if } n = 4\nu + 1, \text{ when } x, y \text{ are both Q.R.,} \\ \text{or both Q.N.R. of } p \left. \right\}. \quad (15b)$$

Hence it follows that—

“When $\mathfrak{P}_\epsilon$ or $\mathfrak{P}_\omega$ is divisible by $p = (4\rho + 1)$, either the Aurifeuillian (N), or some of the other submultiples of $\mathfrak{P}_\epsilon$, $\mathfrak{P}_\omega$, is divisible by p .” (15)

Also, writing N' for the Ant-Aurifeuillian (Art. 4) conjugate to N , and $\mathfrak{F}'$ for the multiple thereof conjugate to N_ω ; then, by similar reasoning to above,

$$\mathfrak{F}'_\omega = \{ (n_1 x^2)^n - \iota^{n+1} (n_2 y^2)^n \}, \text{ where } n = n_1 n_2 \\ \equiv \frac{1}{n} \left\{ \left(\frac{x}{p} \right) - \iota^{n+1} \left(\frac{y}{p} \right) \right\}, \pmod{p}, \text{ where } p = 4n + 1 \\ \text{[here both } x, y \text{ prime to } p] \\ \equiv 0, \pmod{p}, \text{ if } n = 4\nu - 1, \text{ when } x, y \text{ are both Q.R.,} \\ \text{or both Q.N.R. of } p, \quad (15a') \\ \text{also, if } n = 4\nu + 1, \text{ when one of } x, y \text{ is Q.R.,} \\ \text{and the other Q.N.R. of } p; \quad (15b')$$

and therefore

“When $\mathfrak{P}'_\omega$ is divisible by $p = 4\rho + 1$, either the Ant-Aurifeuillian N' , or some of the other submultiples of $\mathfrak{P}'_\omega$, is divisible by p .” (15')

7. *Quotient-Aurifeuillians*.—If an Aurifeuillian $\mathfrak{A}$ be divisible by another Aurifeuillian (N) of the same order, then the quotient (Q) will be, in general, decomposable into two factors, say S, T . For, if $\mathfrak{X}, \mathfrak{Y}$ and L, M be the Aurifeuillian factors of $\mathfrak{A}, N$ respectively, then

$$Q = \mathfrak{A} \div N = \mathfrak{X} \cdot \mathfrak{Y} \div L \cdot M = S \cdot T,$$

so that the factors $\mathfrak{X}, \mathfrak{Y}$ contain between them all the factors of L, M .

Also the quadratic forms in which $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Y}; N, L, M$ are expressible are conformal (because $\mathfrak{Y}, N$ are Aurifeuillians of same

order). Hence also, the quadratic forms in which Q, S, T are expressible are *conformal* with them; and Q, S, T can be actually expressed in those quadratic forms by the process* of *conformal division*, (because they are quotients of conformals). The quotient Q thus has properties quite similar to those of Aurifeuillians.

7a. *Quotient-Aurifeuillians* (continued).—A specially important case of Quotient-Aurifeuillians is obtained by writing

$X^\omega, Y^\omega, x^\omega, y^\omega$ for X, Y, x, y respectively,

in the Aurifeuillian functions of Art. 2. Hereby, in the abbreviated notation of (3, i), (4, i),

$$N = \mathfrak{F}(n, X^2, Y^2) \text{ gives } \mathfrak{F} = \dagger \mathfrak{F}(n, X^{2\omega}, Y^{2\omega}), \quad (16a)$$

$$N = \mathfrak{F}(n, X, Y) \text{ gives } \mathfrak{F} = \dagger \mathfrak{F}(n, X^\omega, Y^\omega). \quad (16b)$$

Thus the new $\mathfrak{F}$ is seen to be an Aurifeuillian of the *same order* ($2n$ or n) as the old N , and the new $\mathfrak{E}, \mathfrak{M}$ are found by changing X, Y , &c., into X^ω, Y^ω , &c. Also $\mathfrak{F}$ is plainly divisible by N (since ω is *odd*). And, in this case, it happens in general that

$\mathfrak{E}$ contains L or M ; and $\mathfrak{M}$ contains M or L ,

so that the quotient Q is resolvable into factors (S, T) , such that

$$Q = \mathfrak{F} \div N = S.T; \quad S = \mathfrak{E} \div L, \text{ or } \mathfrak{E} \div M; \quad T = \mathfrak{M} \div M, \text{ or } \mathfrak{M} \div L. \quad (17)$$

Hence also Q, S, T are expressible (by the process of conformal division) in quadratic forms conformal with those of $\mathfrak{F}, \mathfrak{E}, \mathfrak{M}$. And, lastly, the factors S, T are interrelated by the same relation (7), which connects $\mathfrak{E}$ and $\mathfrak{M}$, or L and M . Thus *this* Quotient-Aurifeuillian has all the properties of an Aurifeuillian.

7b. *Reduced Quotient-Aurifeuillians*.—The Quotient-Aurifeuillians (Q), just treated of, are not the *Reduced* A.P.F. of their subject-functions (X^2, Y^2) or (X, Y) with regard to the complete index $n\omega$, except when ω is *prime*; the extension to the case when ω is *composite* is made as follows.

* *I.e.*, division with *preservation of form*; see Art. 15–23 of the author's paper "On the Connexion of Quadratic Forms," above quoted.

† Here the new $\mathfrak{F}$ is not the A.P.F. of the subject-function with regard to the complete index $n\omega$, but only with regard to the index n , (compare Art. 1).

Let $N(\omega)^*$ be what the Aurifeuillian functions of Art. 2 become when $X^\omega, Y^\omega, x^\omega, y^\omega$ are substituted for X, Y, x, y respectively therein,

$$\text{so that } \left. \begin{aligned} N(\omega) &= \mathfrak{F}(n, X^{2\omega}, Y^{2\omega}) \\ N(1) &= \mathfrak{F}(n, X^2, Y^2) \end{aligned} \right\}, \text{ when } 2nXY = \square;$$

$$\text{so that } \left. \begin{aligned} N(\omega) &= \mathfrak{F}(n, X^\omega, Y^\omega) \\ N(1) &= \mathfrak{F}(n, X, Y) \end{aligned} \right\}, \text{ when } nXY = \square.$$

Then, changing ω into $\lambda, \lambda\mu, \lambda\mu\nu$, &c. in succession, (λ, μ, ν , &c., being unequal odd primes), the rules of Art. 1 give

$$\mathfrak{F}N(\lambda) = N(\lambda) \div N(1), \quad (18a)$$

$$\mathfrak{F}N(\lambda\mu) = N(\lambda\mu) N(1) \div N(\lambda) N(\mu), \quad (18b)$$

$$\mathfrak{F}N(\lambda\mu\nu) = \frac{N(\lambda\mu\nu) N(\lambda) N(\mu) N(\nu)}{N(1) N(\mu\nu) N(\nu\lambda) N(\lambda\mu)}, \text{ \&c.}, \quad (18c)$$

and the law of formation is obvious. Now each of these functions $\mathfrak{F}$ is (by Art. 1) the *Reduced* A.P.F. of its subject-function with regard to the complete index $n\omega$: and is composed of the product of the Quotient-Aurifeuillians described in Art. 7a, divided by a product of similar functions, e.g.,

$$\mathfrak{F}N(\lambda\mu\nu) = \left(\frac{N(\lambda\mu\nu)}{N(\mu\nu)} \frac{N(\lambda)}{N(1)} \right) \div \left(\frac{N(\nu\lambda)}{N(\nu)} \frac{N(\lambda\mu)}{N(\mu)} \right). \quad (18c')$$

Hence the result, which may be termed a *Reduced Quotient-Aurifeuillian* (Q), is (for reasons similar to those used in Art. 7, 7a) resolvable into two factors (S, T), and the three functions Q, S, T are expressible as before in quadratic forms conformal with those of $N(\omega)$ and of its factors $\mathfrak{F}, \mathfrak{F}$. And S, T are connected by the relation (7). Thus these Reduced Quotient-Aurifeuillians have the same properties as Aurifeuillians.

Examples.—The above properties are of great use in factorising high numbers of form $(u^t \pm v^t)$ where $t = 2\omega$ or ω , and ω is composite. Thus, if $t = 105 = 3 \cdot 5 \cdot 7$, then the *Reduced* A.P.F. of $N = (u^{105} \pm v^{105})$ is only of order

$$(105 + 3 + 5 + 7) - (15 + 21 + 35 + 1) = 48$$

in the subject-functions u, v . And writing $n\lambda\mu = 105$, $N = (u^n + v^{n+1} v^n)$ will be an Aurifeuillian of order n when $n\omega = \square$. And the *Reduced* A.P.F. of N , i.e., $\mathfrak{F}N(\lambda\mu)$, will be a Quotient-Aurifeuillian, and therefore resolvable into factors S, T as above. The table shows the resolvable number N .

* Here $N(\omega)$ is not the A.P.F. of the subject-function with regard to the complete index $n\omega$, but only with regard to the index n , (compare Art. 1).

$$\begin{array}{l} \text{Order } n = \quad 3, \quad 5, \quad 7, \quad 15, \quad 21, \quad 35, \quad 105 \\ \lambda\mu = \quad 35, \quad 21, \quad 15, \quad 7, \quad 5, \quad 3, \quad 1 \\ N = \left\{ \begin{array}{l} 3^{105} + 1, \quad 5^{105} - 1, \quad 7^{105} + 1, \quad 15^{105} + 1, \quad 21^{105} - 1, \quad 35^{105} + 1, \quad 105^{105} - 1 \\ 5^{105} + 3^{105}, \quad 7^{105} - 3^{105}, \quad 7^{105} + 5^{105}, \quad 35^{105} - 3^{105} \\ 21^{105} - 5^{105} \\ 15^{105} - 7^{105} \end{array} \right. \end{array}$$

7c. *Quotient Ant-Aurifeuillians*. — The quotient of one Ant-Aurifeuillian $\mathfrak{A}$ by another (N) of the same order may be treated of in a manner quite similar to that of Art. 7—7b, and the results will be, *mutatis mutandis*, quite similar: the difference being that the quotient (Q) will be expressible as a *sum of two squares* (instead of as a difference of squares = product of two factors). And Q will possess all the properties of an Ant-Aurifeuillian.

CHAPTER II., BIN-AURIFEUILLIANS.

8. *Bin-Aurifeuillians*.—When, in Results (3; $3a, b$; $5a, b$),

$$n = n_1 = n_2 = 1, \quad X = x^2, \quad Y = 2y^2,$$

the function N becomes the *Bin-Aurifeuillian*

$$N = x^4 + 2^2 \cdot y^4 \quad [x \text{ taken* odd, and prime to } y; \text{ see } (3a, b)], \quad (19)$$

and its quadratic partitions are

$$N^o = (x^2)^2 + (2y^2)^2; \quad N' = (x^2 - 2y^2)^2 + (2xy)^2. \quad (19a)$$

Its Aurifeuillian factors (L, M), and their quadratic partitions, are

$$L = x^2 - 2xy + 2y^2, \quad M = x^2 + 2xy + 2y^2, \quad (20)$$

$$L' = (x - y)^2 + y^2, \quad M' = (x + y)^2 + y^2. \quad (20a)$$

Also, under Art. 6, N is divisible by 5 (except when one of x, y contains 5); and thus *one* of L, M contains 5, viz.,

$$\left. \begin{array}{l} L \text{ is divisible by } 5, \text{ when } x + y = 5k, \text{ or } x + 2y = 5k \\ M \text{ is divisible by } 5, \text{ when } x - y = 5k, \text{ or } x - 2y = 5k \end{array} \right\}. \quad (21)$$

[It is worth noting that 5 is itself a Bin-Aurifeuillian (since $5 = 1^4 + 4 \cdot 1^4$); and its $L = 1, M = 5$].

Again, since every 4th power ends in 1, 5, or 6, and x is taken *odd*,

$$\text{Every Bin-Aurifeuillian } (x^4 + 4y^4) \text{ ends in } 1, 5, \text{ or } 9. \quad (21a)$$

9. *Bin-Aurifeuillian Dimorphs*.—The question now arises whether the Bin-Aurifeuillian partition of a number (N) is *unique*, or whether

* Hereby N will be of form $(4f + 1)$.

it may be *dimorphic*. Let, if possible,

$$N = x^4 + 4y^4 = \xi^4 + 4\eta^4, \quad (\text{where } \xi \neq x, \eta \neq y); \quad (22)$$

$$\text{therefore } N = (x^2 - 2y^2)^2 + (2xy)^2 = (\xi^2 - 2\eta^2)^2 + (2\xi\eta)^2. \quad (22a)$$

A *general* solution of these seems difficult: but a somewhat limited solution may be obtained by equating separately the *parts* (a, b) of two partitions ($a^2 + b^2$) of *different* kinds, say

$$\xi^2 - 2\eta^2 = x^2, \quad \xi\eta = y^2.$$

These involve (ξ, η being supposed *relative primes*)

$$\xi = \tau^2, \quad \eta = \nu^2, \quad x = t, \quad y = \tau\nu,$$

$$\text{whence} \quad \tau^4 - 2\nu^4 = t^2. \quad (23)$$

Any solution of this last (23) gives a solution of (22); now the *general* solution of (23) is known* to be

$$\tau = \tau_0^4 + 2\nu_0^4, \quad \nu = 2\tau_0\nu_0 \cdot t_0, \quad \text{where } t_0 = \sqrt{\tau_0^4 - 2\nu_0^4}. \quad (23a)$$

Hence every solution of $\tau_0^4 - 2\nu_0^4 = t_0^2$ gives a solution of (23); and, since t is of the same form as one of the forms of t_0 , each set of values of τ, ν, t so found from the base-set τ_0, ν_0, t_0 may be used as a new base-set (*i.e.*, in place of τ_0, ν_0, t_0) from which to generate a *new* set of τ, ν, t ; and this process may be continued *ad infinitum*. Hence, every base-solution (τ_0, ν_0) yields an infinite series of *dimorphic* Bin-Aurifeuillians.

Ex.—The elements of the *lowest* solutions are given in the table below: the two solutions $\tau_0 = 1, \nu_0 = 1$; $\tau_0 = 1, \nu_0 = 13$ are †fundamental, each giving rise to an infinite number of solutions; but the numbers increase in magnitude so very rapidly that it is not worth continuing them.

Data.	Auxiliaries.	Aurifeuillian Partitions.		Bin-Aurifeuillian.
$\tau_0; \nu_0; t_0$	$\tau; \nu; t$	$x; y$	$\xi; \eta$	N
1; 1; 1	3; 2; 7	7; 6	9; 4	7585
3; 2; 7	113; 84; 7967	7967; 9492	12769; 7056	$x^4 + 4y^4$
1; 13; 239	57123; 6214; t	$t; \tau\nu$	$\tau^2; \nu^2$	$\tau^8 + 4\nu^8$

* Euler's *Elements of Algebra*, (Hewlett's transl., 1822), Part II, Art. 211.

† One other ($\tau_0 = 1525, \nu_0 = 1343$) is given in Mr. Lebesgue's paper, "Résolution des Équations Biquadratiques," in the *Journal des Math. Pures et Appliquées*, t. xviii, p. 83.

9a. *Factorisation of above.*—The resolution of these numbers into their Aurifeuillian Factors in two distinct ways, $N = L_1 M_1 = L_2 M_2$, greatly helps their ultimate resolution into prime factors; thus, taking the second number above

$$\begin{aligned} N &= 7967^4 + 4 \cdot 9492^4 = (1525^2 + 9492^2)(17469^2 + 9492^2) = L_1 \cdot M_1 \\ &= 12769^4 + 4 \cdot 7056^4 = (5713^2 + 7056^2)(19825^2 + 7056^2) = L_2 \cdot M_2. \end{aligned}$$

Here $L_1 = 92,423,689$, $M_1 = 394,914,745$; $L_2 = 82,425,505$; $M_2 = 442,817,761$. And it will be found that the G.C.M. of L_1, M_2 is 1553, and the G.C.M. of L_2, M_1 is 5.277. Hence it easily results

$$N = (1553 \cdot 59513)(5 \cdot 277 \cdot 389 \cdot 733) = (5 \cdot 277 \cdot 59513)(389 \cdot 733 \cdot 1553).$$

[This large number, > 36400 billion, would be otherwise difficult to factorise.]

10. *Higher Bin-Aurifeuillians.*—The results of imposing one* of the conditions

$$\text{i. } x^2 \mp 2y^2 = z^2; \quad \text{ii. } 2y^2 - x^2 = z^2, \quad (24)$$

on the Bin-Aurifeuillian $N = x^4 + 4y^4 = LM$, are interesting. The results are

$$\left. \begin{aligned} \text{i. } N &= 2x^4 - 2x^2z^2 + z^4 = \mathfrak{L} \text{ of } \mathfrak{X} \\ \text{ii. } N &= 2x^4 + 2x^2z^2 + z^4 = \mathfrak{M} \text{ of } \mathfrak{X} \end{aligned} \right\}, \quad (25)$$

where $\mathfrak{L}, \mathfrak{M}$ are the Aurifeuillian Factors of the Higher Bin-Aurifeuillian

$$\mathfrak{X} = z^4 + 4x^4 = \mathfrak{L}\mathfrak{M}.$$

This gives three (easily recognizable) cases in which $\mathfrak{X}$ is resolvable into three factors, viz.,

$$\text{i. } \mathfrak{X} = LM \cdot \mathfrak{M}; \quad \text{ii. } \mathfrak{X} = \mathfrak{L} \cdot LM.$$

The general solutions of the Diophantines i, ii of (24), in terms of two arbitrary integers (m, n) , take the following four forms†

$$\left. \begin{aligned} \text{ia. } x &= m^2 - 2n^2, & y &= 2mn, & z &= m^2 + 2n^2, & (m \text{ odd}) \\ \text{ib. } x &= m^2 + 2n^2, & y &= 2mn, & z &= m^2 - 2n^2, & (m \text{ odd}) \\ \text{iiia. } x &= m^2 + 2mn - n^2, & y &= m^2 + n^2, & z &= n^2 + 2mn - m^2 \\ \text{iiib. } x &= n^2 + 2mn - m^2, & y &= m^2 + n^2, & z &= m^2 + 2mn - n^2 \end{aligned} \right\}. \quad (26)$$

[m odd and n even in iia, iiib]

* The results obtained by changing z^2 into $2z^2$ will be found reducible to those given by i and ii.

† Noting that Cases a, b are interconvertible by interchange of x, z .

Examples.—The following are the *lowest* examples of each case:—

Case.	Data.	Parts.	$\mathcal{P} = (x^3 + 4x^2)$.	$\mathcal{P} = \mathcal{L}\mathcal{M}$.	$\mathcal{L}$ or $\mathcal{M} = N$ $= LM$.
	$m; n$	$x; y; z$		$\mathcal{L}; \mathcal{M}$	$N \quad L \quad M$
ia	1; 1	-1; 2; 3	6,565	65; 101	$\mathcal{L} = 5; 13$
ib	1; 1	3; 2; -1	26,245	145; 181	$\mathcal{L} = 5; 29$
ia	1; 2	-7; 4; 9	66,105,925	3,425; 19,301	$\mathcal{L} = 25; 137$
ib	1; 2	9; 4; -7	177,955,685	7,585; 23,461	$\mathcal{L} = 41; 185$
iiia	1; 2	1; 5; 7	5,764,805	2,305; 2,501	$\mathcal{M} = 41; 61$
iiib	1; 2	7; 5; 1	23,059,205	4,705; 4,901	$\mathcal{M} = 29; 169$
iiia	3; 2	17; 13; 7	27,908,794,565	141,121; 197,765	$\mathcal{M} = 185; 1069$
iiib	3; 2	7; 13; 17	7,008,816,645	60,001; 116,645	$\mathcal{M} = 205; 569$

11. *Bin-Aurifeuillians; Quartic Forms.*—A Bin-Aurifeuillian being a quartic form, it becomes interesting to inquire whether it can be expressed in *other* quartic forms. In the first place, it is known that a Bin-Aurifeuillian $(x^4 + 4y^4)$ cannot be* a perfect square, and *a fortiori* not a perfect fourth power. The forms considered below are

Form $(X^4 - Y^4)$, Art. 11a, Form $(X^4 + Y^4)$, Art. 11b,

Form $(U^4 + W^4)(V^4 + W^4)$, Art. 11c.

11a. *Form $(X^4 - Y^4)$.*—Let, if possible,

$$N = x^4 + 4y^4 = X^4 - Y^4. \quad (27)$$

A *general* solution of this seems difficult; but a somewhat limited solution may be obtained by expressing $(x^4 + 4y^4)$ as a difference of squares, and assuming this expression to be *identical* with the other form $(X^4 - Y^4)$: thus

$$N = (x^2 + 2y^2)^2 - (2xy)^2 = (X^2)^2 - (Y^2)^2;$$

whence

$$x^2 + 2y^2 = X^2, \quad 2xy = Y^2.$$

These involve (since x, y are supposed *mutually prime*)

$$x = f^2, \quad y = 2g^2, \quad X = h, \quad Y = 2fg,$$

where

$$f^4 + 8g^4 = h^2. \quad (28)$$

Any solution of this last (28) gives a solution of (27): now the *general* solution of (28) may be shown† to be

$$f = f_0^4 - 8g_0^4, \quad g = 2f_0g_0h_0, \quad \text{where } h_0 = \sqrt{f_0^4 + 8g_0^4}. \quad (28a)$$

* See Euler's *Elements of Algebra*, Art. 209-1.

† By a process similar to that used in Euler's *Elements of Algebra* (Howlett's transl.), Part II, Art. 211, for solution of $\tau^4 - 2\nu^4 = t^2$: compare Art. 9 above.

Hence every solution of $f_0^4 + 8g_0^4 = h_0^4$ gives a solution of (28); and, since h is of the same form as h_0 , each set of values of f, g, h so formed from the base-set f_0, g_0, h_0 may be used as a new base-set (i.e., in place of f_0, g_0, h_0) from which to generate a new set of f, g, h ; and this process may be continued *ad infinitum*. Hence every solution f_0, g_0, h_0 yields an infinite series of Bin-Aurifeuillians expressible in the required form $X^4 - Y^4$.

Examples.—The elements of the lowest solutions are given in the table below: the solutions $f_0 = 1, g_0 = 1$; $f_0 = 239, g_0 = 13$ are fundamental, each giving rise to an infinite number of solutions; but the numbers increase so very rapidly that it is not worth while continuing them.

Data.	Auxiliaries.	Aurifn. Parts.	Quartic Parts.	Bin-Aurifn.
$f_0 ; g_0 ; h_0$	$f ; g ; h$	$x ; y$	$X ; Y$	N
$\cdot ; \cdot ; \cdot$	$1 ; 1 ; 3$	$1 ; 2$	$3 ; 2$	65
$1 ; 1 ; 3$	$7 ; 6 ; 113$	$49 ; 72$	$113 ; 84$	113,260,225
$7 ; 6 ; 113$	$7967 ; 9492 ; (f^4 + 8g^4)^{\frac{1}{4}}$	$f^2 ; 2g^2$	$h ; 2fg$	$f^8 + 64g^8$

11a'. *Connexion with Dimorph Aurifeuillians.*—The two Diophantine equations

$$(23), t^2 = \tau^4 \sim 2v^4 ; \quad (28), h^2 = f^4 + 8g^4$$

are known* to be so connected that every solution of either gives one solution of the other. Now the former has been shown to lead to the *dimorph* Bin-Aurifeuillian

$$N_1 = x_1^4 + 4y_1^4 = (\tau^2)^4 + 4(v^2)^4,$$

and the latter has been shown to lead to the Bin-Aurifeuillian

$$N_2 = x_2^4 + 4y_2^4 = X^4 - Y^4 ;$$

so that these two kinds of Bin-Aurifeuillians must be connected. Comparing Art. 9, 11, it is seen that

$$\text{either} \quad \left. \begin{aligned} x_1 = f, \quad y_1 = g ; \quad x_2 = f^2 = x_1^2 ; \quad y_2 = 2g^2 = 2y_1^2 \end{aligned} \right\} \quad (29)$$

$$\text{or} \quad \tau = h = X, \quad v = Y$$

Thus "Every dimorph Bin-Aurifeuillian (N_1) leads to a Bin-Aurifeuillian (N_2) expressible as $(X^4 - Y^4)$, and *vice versa*." (29a)

11b. *Form $(X^4 + Y^4)$.*—Let, if possible,

$$N = x^4 + 4y^4 = (x^2 - 2y^2)^2 + (2xy)^2 = X^4 + Y^4.$$

As $Y^2 \neq 2y^2$, the algebraic solution involves

$$x^2 \sim 2y^2 = X^2, \quad 2xy = Y^2.$$

These involve (since x is supposed *odd*, and prime to y)

$$x = t^2, \quad y = 2u^2, \quad Y = 2tu, \quad t^4 \sim 8u^4 = X^2.$$

But the latter results, $(t^4 \sim 8u^4) = \text{square}$, are known* to be impossible forms: thus a Bin-Aurifeuillian cannot take the form $(X^4 + Y^4)$.

* See Mr. Lebesgue's paper quoted above, (footnote † of Art. 9).

11c. Form $(U^4 + W^4)(V^4 + W^4)$.—Let, if possible,

$$N = x^4 + 4y^4 = (U^4 + W^4)(V^4 + W^4). \quad (30)$$

A somewhat limited solution may be obtained* by equating the Aurifeuillian Factors (L , M) of N separately to the factors $(U^4 + W^4)$, $(V^4 + W^4)$; thus

$$L = (x-y)^2 + y^2 = U^4 + W^4, \quad M = (x+y)^2 + y^2 = V^4 + W^4,$$

giving

$$x-y = U^2, \quad x+y = V^2, \quad y = W^2,$$

whence

$$x = \frac{1}{2}(V^2 + U^2), \quad y = \frac{1}{2}(V^2 - U^2) = W^2. \quad (30a)$$

The general solution of the latter Diophantine in terms of two arbitraries (s , t) is

$$U = s^2 - 2t^2, \quad V = s^2 + 2t^2, \quad W = 2st, \quad (30b)$$

whence

$$x = s^4 + 4t^4, \quad y = (2st)^2. \quad (30c)$$

And, since x is supposed odd and prime to y , therefore s must be odd and prime to t . Every such pair of values of s , t gives a value of N of the required form; thus the computation of such is direct, and the number of such forms is infinite.

Note that x itself is a Bin-Aurifeuillian, and is therefore a multiple of 5 (Art. 8), except when one of s , t contains 5 (in which case y is a multiple of 5): hence N cannot contain 5 in any case.

Examples.—The lowest solutions are given in the table below: the numbers will be seen to rise very rapidly; most of the results (N) are so high that it has not been thought worth while to multiply out the product $L \cdot M = N$.

Data.	Quartic Parts.	Aurifn. Parts.	Factors of N .
s ; t	U ; V ; W	x ; y	L ; M
1 ; 1	1 ; 3 ; 2	5 ; 4	17 ; 97
1 ; 2	7 ; 9 ; 4	65 ; 16	2657 ; 6817
1 ; 3	17 ; 19 ; 6	325 ; 36	84,817 ; 131,617
3 ; 1	7 ; 11 ; 6	85 ; 36	3697 ; 15937
3 ; 2	1 ; 17 ; 12	145 ; 144	20,737 ; 104,257
5 ; 1	23 ; 27 ; 10	629 ; 100	289,841 ; 541,441
5 ; 2	17 ; 33 ; 20	689 ; 400	243,521 ; 1345,921

12. Bin-Aurifeuillian Factors.—Let N_1 , N_2 be two numbers whose product (N) is a Bin-Aurifeuillian. The conditions that N_1 , N_2 should be themselves the Aurifeuillian Factors (L , M) of N are

$$N_1 = L = x^3 - 2xy^2 + 2y^3, \quad N_2 = M = x^3 + 2xy^2 + 2y^3. \quad (31)$$

Hence $x^2 + 2y^2 = \frac{1}{2}(N_2 + N_1)$, $2xy = \frac{1}{2}(N_2 - N_1)$,

whence $x^2 - 2y^2 = \frac{1}{2}\{4N_1N_2 - (N_2 - N_1)^2\}^{\frac{1}{2}}$.

* This solution was given by the author in his Paper "On 2 as a 16-ic Residue," Art. 19, in *Proc. Lond. Math. Soc.*, Vol. XXVII.

Hence *three* conditions must obtain that any two numbers N_1, N_2 may be the Aurifeuillian Factors of a Bin-Aurifeuillian (N), viz.,

$$\{4N_1N_2 - (N_2 - N_1)^2\} = \{2(x^2 - 2y^2)\}^2 = (2Z)^2, \text{ an even square, } (32a)$$

$$(N_1 + N_2) \pm 2Z = (2x)^2, \text{ an even square, } (32b)$$

$$\frac{1}{2} \{(N_1 + N_2) \mp 2Z\} = (2y)^2, \text{ an even square. } (32c)$$

When N_1, N_2 are given numbers (satisfying the above conditions), the values of x, y giving the Aurifeuillian partition of N , and therefore, also, the quadratic partitions of N, L, M (Art. 5), are given directly.

13. Conformal Bin-Aurifeuillian Products.—The question now arises whether the *product* of two or more Bin-Aurifeuillians can be a Bin-Aurifeuillian; or, conversely, whether a Bin-Aurifeuillian can be resolved into the product of two or more Bin-Aurifeuillians.

$$\text{Let } N = x^4 + 4y^4, \quad N_1 = x_1^4 + 4y_1^4, \quad N_2 = x_2^4 + 4y_2^4, \quad (33)$$

$$\text{and, if possible, let } N = N_1N_2, \text{ (where } N_1 < N_2\text{). } (34)$$

The *general* solution of (34) seems difficult; but a somewhat limited solution may be obtained by first equating the Aurifeuillian Factors (L, M) of N separately to N_1, N_2 , thus obtaining *two* equations

$$L = N_1, \quad M = N_2. \quad (35)$$

Let L, M, N_1, N_2 be now expressed in the quadratic form ($a^2 + b^2$); herein the forms of L, M are algebraically *unique* (L', M'), whilst those of N_1, N_2 are algebraically *dimorph* (N_1°, N_1' ; N_2°, N_2'); see Art. 5, 8. Hence the pair of equations (35) lead to the four systems

$$\text{Systems } \left\{ \begin{array}{ll} \text{I. } L' = N_1^\circ, & M' = N_2^\circ; \\ \text{II. } L' = N_1', & M' = N_2'; \\ \text{III. } L' = N_1^\circ, & M' = N_2'; \\ \text{IV. } L' = N_1', & M' = N_2'. \end{array} \right\}. \quad (35a)$$

By now equating the *parts* (a, b) of L' separately to the *parts* (a, b) of the corresponding N_1 (i.e., of N_1° or N_1'); and also the *parts* (a, b) of M' separately to the *parts* (a, b) of the corresponding N_2 (i.e., of N_2° or N_2'), each system (I, II, III, IV) yields a group of *four* equations between the six quantities x, y, x_1, y_1, x_2, y_2 . Also, in each system, four such groups (of four equations each) may be formed; since *either* part (a, b) of L', M' may be equated to *either* part (a, b) of the corresponding $N_1^\circ, N_1', N_2^\circ, N_2'$. Also, since the parts (a, b) are defined only by their squares (a^2, b^2), each of the parts (a, b) may have either a $\pm$ sign; so that every group (as above formed) may be

formed in *four* ways, by the mere change of sign of each a, b . Thus finally 64 sets of *four* equations each can be formed, 16 in each system.

The four equations in each of the 64 sets are linear in x, y , and quadratic in x_1, y_1, x_2, y_2 . Hence, x, y can always be *linearly* eliminated, thus leaving *two* equations quadratic in x_1, y_1, x_2, y_2 . From these two the value of either pair $(x_1, y_1), (x_2, y_2)$ can be solved (by solution of quadratics) in terms of the other pair in a form involving (in the general case) *three* radicals. Thus, *three* conditions must, in general, be satisfied (viz., that the expressions under the three radicals should be perfect squares), in order that this form of solution may be *possible*.

Thus, whenever either of N_1, N_2 is *given*, and also the Aurifeuillian partition (x_1, y_1) or (x_2, y_2) thereof, these criteria can be *directly* applied; and, if satisfied, the other pair (x_2, y_2) , or (x_1, y_1) , and thence also N_2 or N_1 , and, lastly, x, y , and N can be *directly* calculated. So that this problem is *direct*, and the solution is *unique*.

The solution of the problem *ab initio* is, however, much more difficult. In fact, in the 48 cases of Systems II, III, IV, the radicals are such complex functions that a solution seems difficult; but the 16 cases of System I are so simple in form that it is easy to recognize that 12* of the 16 cases involve some impossibility. The remaining *four* cases (here denoted i, ii, iii, iv) of System I turn out to be *completely* solvable, and to be interrelated in an interesting way.

13a. *System I*.—This system is defined (see 35a) by

$$L' = N_1^2, \quad M' = N_2^2,$$

$$\text{or } (x-y)^2 + y^2 = (x_1^2)^2 + (2y_1^2)^2, \quad (x+y)^2 + y^2 = (x_2^2)^2 + (2y_2^2)^2.$$

The equation of the separate parts ($\pm a, \pm b$) gives rise to the four possible Cases i, ii, iii, iv.

$$\text{i. } \left. \begin{aligned} x-y &= x_1^2, & x+y &= x_2^2, & y &= 2y_1^2 = 2y_2^2, & x &= \frac{1}{2}(x_2^2 + x_1^2) \end{aligned} \right\}, \quad (36)$$

with *two* conditions $x_2^2 - x_1^2 = (2y_1)^2 = (2y_2)^2$

$$\text{ii. } \left. \begin{aligned} y-x &= x_1^2, & y+x &= x_2^2, & y &= 2y_1^2 = 2y_2^2, & x &= \frac{1}{2}(x_2^2 - x_1^2) \end{aligned} \right\}, \quad (36)$$

with *two* conditions $x_2^2 + x_1^2 = (2y_1)^2 = (2y_2)^2$

$$\text{iii. } \left. \begin{aligned} x-y &= 2y_1^2, & x+y &= 2y_2^2, & y &= x_1^2 = x_2^2, & x &= y_2^2 + y_1^2 \end{aligned} \right\}, \quad (36)$$

with *two* conditions $y_2^2 - y_1^2 = x_1^2 = x_2^2$

$$\text{iv. } \left. \begin{aligned} y-x &= 2y_1^2, & y+x &= 2y_2^2, & y &= x_1^2 = x_2^2, & x &= y_2^2 - y_1^2 \end{aligned} \right\}. \quad (36)$$

with *two* conditions $y_2^2 + y_1^2 = x_1^2 = x_2^2$

* It seems unnecessary to give the detail.

Thus there are *two* conditions of similar form in each of the four cases,

One of type $y_1 = y_2$ (in i, ii), or $x_1 = x_2$ (in iii, iv);

One of type $x_2^2 \mp x_1^2 = (2y_1)^2$ (in i, ii), or $y_2^2 \mp y_1^2 = x_1^2$ (in iii, iv).

This latter condition is equivalent to $c^2 = a^2 + b^2$.

Now the *general* solution of this Pythagorean equation is expressible in terms of *two* arbitrary integers m, n , viz.,

$$c = m^2 + n^2, \quad a = m^2 \sim n^2 \text{ or } 2mn, \quad b = 2mn \text{ or } m^2 \sim n^2.$$

Here m, n should be so taken that x_1, x_2, x may be all *odd*, and $x_1, y_1; x_2, y_2; x, y$ should be relatively prime; this involves

$$m, n \text{ mutual primes; and either one } \textit{odd}, \text{ one } \textit{even}, \text{ or both } \textit{odd}. \quad (37a)$$

The general solutions of i, ii, iii, iv are

$$\text{i.} \quad \left. \begin{aligned} x_2 &= m^2 + n^2, \quad x_1 = m^2 \sim n^2, \quad y_1 = y_2 = mn \\ x &= m^4 + n^4, \quad y = 2m^2n^2; \quad x_1, x_2, x \text{ all } \textit{odd} \end{aligned} \right\}, \quad (37)$$

$$\text{ii.} \quad \left. \begin{aligned} 2y_1 &= 2y_2 = m^2 + n^2, \quad x_1 \text{ or } x_2 = m^2 \sim n^2, \quad x_2 \text{ or } x_1 = 2mn \\ &[\text{This solution rejected, as giving } x_1, x_2, x \text{ all } \textit{even}] \end{aligned} \right\}, \quad (37)$$

$$\text{iii.} \quad \left. \begin{aligned} y_2 &= m^2 + n^2, \quad x_1 = x_2 = m^2 \sim n^2, \quad y_1 = 2mn \\ x &= m^4 + 6m^2n^2 + n^4, \quad y = (m^2 \sim n^2)^2; \quad x_1, x_2, x \text{ all } \textit{odd} \end{aligned} \right\}, \quad (37)$$

$$\text{iv.} \quad \left. \begin{aligned} x_1 &= x_2 = m^2 + n^2, \quad y_1 \text{ or } y_2 = m^2 \sim n^2, \quad y_2 \text{ or } y_1 = 2mn \\ x &= m^4 - 6m^2n^2 + n^4, \quad y = (m^2 + n^2)^2; \quad x_1, x_2, x \text{ all } \textit{odd} \end{aligned} \right\}. \quad (37)$$

The solutions when m, n are *both odd* lead eventually only to the *same* results as above, so need not be here detailed; in fact, writing for distinctness m', n' in this case, the two solutions of i, ii, iii, iv are interconvertible by the relations

$$m \text{ or } n = \frac{1}{2}(m' + n'), \quad n \text{ or } m = \frac{1}{2}(m' \sim n'); \quad m' = m + n, \quad n' = m \sim n.$$

Thus there are only *three* distinct general solutions (Nos. i, iii, iv) practically available. These solutions are *general* and *complete*, as they give the means of *direct* computation of Bin-Aurifeuillians N_1, N_2, N , such that $N_1 \cdot N_2 = N$ for any arbitrary values of m, n (one *odd*, one *even*, and relatively prime).

Examples.—The following table shows the elements and the results of a few of the *lowest* of such numbers $N_1 \cdot N_2 = N$ in each of the solvable cases (i, iii, iv). As N is hereby resolved algebraically into *four* factors, it becomes possible to factorise such numbers (N) *completely*, when $< (9 \text{ million})^4 = 6561 \times 10^{24}$, as the present factor-tables extend to 9 million.

Case.	Data.	Aurifeuillian Parts.		Aurifeuillians.	Aurif. Parts of N . [$N = N_1 N_2$]
		Of N_1 .	Of N_2 .		
	$m ; n$	$x_1 ; y_1$	$x_2 ; y_2$	$N_1 ; N_2$	$x ; y$
i	1 ; 2	3 ; 2	5 ; 2	145 ; 689	17 ; 8
	1 ; 4	15 ; 4	17 ; 4	51,649 ; 84,545	257 ; 32
	1 ; 6	35 ; 6	37 ; 6	1,505,809 ; 1,879,345	1297 ; 72
	3 ; 2	5 ; 6	13 ; 6	5,809 ; 33,745	97 ; 72
	3 ; 4	7 ; 12	25 ; 12	85,345 ; 473,569	337 ; 288
	5 ; 2	21 ; 10	29 ; 10	234,481 ; 747,281	641 ; 200
iii	1 ; 2	3 ; 4	3 ; 5	1105 ; 2581	41 ; 9
	1 ; 4	15 ; 8	15 ; 17	67,009 ; 384,709	353 ; 225
	3 ; 2	5 ; 12	5 ; 13	83,569 ; 114,869	313 ; 25
	3 ; 4	7 ; 24	7 ; 25	1,329,505 ; 1,564,901	1201 ; 49
	5 ; 2	21 ; 20	21 ; 29	834,481 ; 3,023,605	1241 ; 441
iv	1 ; 2	5 ; 4	5 ; 3	1649 ; 949	7 ; 25
	1 ; 4	17 ; 8	17 ; 15	99,905 ; 286,021	161 ; 289
	3 ; 2	13 ; 12	13 ; 5	111,505 ; 31,061	119 ; 169
	3 ; 4	25 ; 24	25 ; 7	1,717,729 ; 400,229	527 ; 625
	5 ; 2	29 ; 20	29 ; 21	1,347,281 ; 1,485,205	41 ; 841

14. *Conformal Product of three Bin-Aurifeuillians*.—Let N_1, N_2, N_3 ; N_{12}, N_{123} be Bin-Aurifeuillians, defined by

$$\left. \begin{aligned} N_1 &= x_1^4 + 4y_1^4, & N_2 &= x_2^4 + 4y_2^4, & N_3 &= x_3^4 + 4y_3^4 \\ N_{12} &= x_{12}^4 + 4y_{12}^4, & N_{123} &= x_{123}^4 + 4y_{123}^4 \end{aligned} \right\}. \quad (38)$$

Also let N_1, N_2 be such that

$$N_1 \cdot N_2 = N_{12}. \quad (38a)$$

Also, if possible, let N_{12}, N_3 be such that

$$N_{12} \cdot N_3 = N_{123}. \quad (39)$$

Then N_1, N_2, N_{12} are here related precisely as the N_1, N_2, N of Art. 13, 13a; the N, x, y of that Article being here replaced by N_{12}, x_{12}, y_{12} , the suffixes (12) being introduced to show distinctly that N_{12}, x_{12}, y_{12} arise from N_1, N_2 .

Then, multiplying together $(x-y), (x+y)$ of the results (36) of Art. (13a), and changing x, y thereof into the x_{12}, y_{12} of the new notation,

$$\left. \begin{aligned} \text{Case i. } x_{12}^2 - y_{12}^2 &= (x_1 x_2)^2; & \text{Case ii. } y_{12}^2 - x_{12}^2 &= (x_1 x_2)^2 \\ \text{Case iii. } x_{12}^2 - y_{12}^2 &= (2y_1 y_2)^2; & \text{Case iv. } y_{12}^2 - x_{12}^2 &= (2y_1 y_2)^2 \end{aligned} \right\}. \quad (40)$$

These four results may be included under $x_{12}^2 \sim y_{12}^2 = z^2$ (a square).

Next, taking $y_3 = x, \quad x_3 = x_{12},$

it will be found that *both* the conditions that $N_{123} = N_{12} \cdot N_3$, i.e., that N_{123}, N_{12}, N_3 should be related similarly to the N, N_1, N_2 of Art. 13a, are satisfied at once in Cases iii, iv of Art. 13a (as may be seen by writing x_{12}, y_{12}, x_3, y_3 for x_1, y_1, x_2, y_2 in the conditions there formed); and the x, y of those results (36, iii, iv) become the x_{123}, y_{123} here required. Thus

i. After forming N_1, N_2, N_{12} by Case i, Art. 13a, then

$$x_{12}^2 - y_{12}^2 = (x_1 x_2)^2; \quad \text{take } y_3 = x_1 x_2, \quad x_3 = x_{12}. \quad (41)$$

Then x_{12}, y_{12}, x_3, y_3 (substituted for x_1, y_1, x_2, y_2 in Case iv, Art. 13a) satisfy the two conditions of Case iv, Art. 13a, so that the x, y of the latter become the required

$$x_{123} = y_3^2 - y_{12}^2, \quad y_{123} = x_{12}^2 = x_3^2. \quad (41a)$$

ii. This case is rejected as leading to x_1, x_2, x_{12} , all *even*; see (37, ii).

iii. After forming N_1, N_2, N_{12} by Case iii, Art. 13a, then

$$x_{12}^2 - y_{12}^2 = (2y_1 y_2)^2; \quad \text{take } y_3 = 2y_1 y_2, \quad x_3 = x_{12}. \quad (42)$$

Then, x_{12}, y_{12}, x_3, y_3 satisfy the two conditions of Case iv, as before; and similarly

$$x_{123} = y_3^2 - y_{12}^2, \quad y_{123} = x_{12}^2 = x_3^2. \quad (42a)$$

iv. After forming N_1, N_2, N_{12} by Case iv, Art. 13a, then

$$y_{12}^2 - x_{12}^2 = (2y_1 y_2)^2; \quad \text{take } y_3 = 2y_1 y_2, \quad x_3 = x_{12}. \quad (43)$$

Then x_{12}, y_{12}, x_3, y_3 (substituted for x_1, y_1, x_2, y_2 in Case iii, Art. 13a) satisfy the two conditions of Case iii, Art. 13a; and the x, y of the latter become the required

$$x_{123} = y_{12}^2 + y_3^2, \quad y_{123} = x_{12}^2 = x_3^2. \quad (43a)$$

The general values of $x_{12}, y_{12}; x_3, y_3; x_{123}, y_{123}$ in terms of the arbitrariness m, n of Art. 13a are now easily written down; thus the present x_{12}, y_{12} are simply the x, y of Results (37, i, iii, iv), and the rest $x_3, y_3, x_{123}, y_{123}$ can be obtained from above.

Examples.—See end of next Article.

15. Conformal Continued Product of Bin-Aurifeuillians.—The law of formation of a Bin-Aurifeuillian ($N_{123\dots r}$) which shall be the product of any number (r) of Bin-Aurifeuillians ($N_1 N_2 N_3 \dots N_r$) is now easily seen. Starting with any values of m, n (mutual primes, one *odd*, one

even), and using any of the three known Cases (37, i, iii, iv) to form the first product $N_{12} = N_1 \cdot N_2$; then (by Art. 14) Cases i, iii lead to Case iv, whilst Case iv leads to Case iii; hence N_3 , and N_{123} may be computed as in Art. 14. The formation of the successive new factors N_4 , N_5 , &c., and new products N_{1234} , N_{12345} , &c., falls alternately under Cases iii, iv (Art. 14). It will be seen that the process may be continued *ad infinitum*, and that the result at each step is *unique*, i.e., any pair of values of m, n yields a *definite* continued product.

Examples.—Three examples are given below, starting from the *lowest* values of m, n ($m = 1, n = 2$) in each of Cases i, ii, iii, and carried to the fourth order (N_{1234}), showing also the intermediate steps, (thus giving *eight* algebraic factors) :

$$N_{12} = N_1 \cdot N_2; \quad N_{123} = N_{12} \cdot N_3 = N_1 N_2 N_3; \quad N_{1234} = N_{123} \cdot N_4 = N_1 N_2 N_3 N_4. \quad (44)$$

The numbers rise so rapidly that it is not worth while pushing to higher orders. The larger numbers (N_{1234}) here factorised would be otherwise very difficult to resolve completely; (Ex. 3 runs to 26 figures).

Ex. 1.—Take $m = 1, n = 2$.

Then, (Art. 13a, Case i),

$$N_1 = 3^4 + 4 \cdot 2^4 = 145 = 5 \cdot 29; \quad N_2 = 5^4 + 4 \cdot 2^4 = 689 = 13 \cdot 53;$$

$$\text{therefore} \quad N_{12} = N_1 N_2 = 17^4 + 4 \cdot 8^4 = 99,905 = (5 \cdot 29)(13 \cdot 53).$$

Hence, (Art. 14, Case iv),

$$N_3 = 17^4 + 4 \cdot 15^4 = 286,021 = 229 \cdot 1249;$$

$$\text{therefore} \quad N_{123} = N_{12} \cdot N_3 = 161^4 + 4 \cdot 289^4.$$

Hence, (Art. 14, Case iii),

$$N_4 = 161^4 + 4 \cdot 240^4 = 63,841 \cdot 218,401$$

$$N_{1234} = N_{123} \cdot N_4 = 141,121^4 + 4 \cdot 25,921^4$$

$$= (5 \cdot 29)(13 \cdot 53)(229 \cdot 1249)(63,841 \cdot 218,401).$$

Ex. 2.—Take $m = 1, n = 2$.

Then, (Art. 13a, Case iii),

$$N_1 = 3^4 + 4 \cdot 4^4 = 1105 = (17)(5 \cdot 13); \quad N_2 = 3^4 + 4 \cdot 5^4 = 2581 = 29 \cdot 89;$$

$$\text{therefore} \quad N_{12} = N_1 \cdot N_2 = 41^4 + 4 \cdot 9^4 = 2,852,005 = \{17 \cdot (5 \cdot 13)\} \cdot (29 \cdot 89).$$

Hence, (Art. 14, Case iv),

$$N_3 = 41^4 + 4 \cdot 40^4 = 13,065,761 = 1601 \cdot 8161;$$

$$\text{therefore} \quad N_{123} = N_{12} \cdot N_3 = 1519^4 + 4 \cdot 1681^4 = \{17 \cdot (5 \cdot 13)\} (29 \cdot 89)(1601 \cdot 8161).$$

Hence, (Art. 14, Case iii), $N_4 = 1519^4 + 4 \cdot 134,480^4$;

$$\begin{aligned} \text{therefore} \quad N_{1234} &= N_{123} \cdot N_4 = 2,827,361^4 + 4 \cdot 2,307,361^4 \\ &= \{17 \cdot (5 \cdot 13)\} (29 \cdot 89)(1601 \cdot 8161) \cdot N_4. \end{aligned}$$

Ex. 3. Take $m = 1, n = 2$.

Then, (Art. 13a, Case iv),

$$N_1 = 5^4 + 4 \cdot 3^4 = 949 = 13 \cdot 73; \quad N_2 = 5^4 + 4 \cdot 4^4 = 1649 = 17 \cdot 97;$$

$$\text{therefore} \quad N_{12} = N_1 \cdot N_2 = 7^4 + 4 \cdot 25^4 = 1,564,901 = (13 \cdot 73)(17 \cdot 97).$$

* Here $N_4 = L_4 \cdot M_4 = (132,961^2 + 134,480^2)(135,999^2 + 134,480^2)$; but the further resolution of L_4, M_4 is not obvious.

Hence, (Art. 14, Case iii),

$$N_3 = 7^4 + 4 \cdot 24^4 = 1,329,505 = (5 \cdot 173) \cdot (29 \cdot 53);$$

therefore

$$N_{123} = N_{12} \cdot N_3 = 1201^4 + 4 \cdot 49^4.$$

Hence, (Art. 14, Case iv),

$$N_4 = 1201^4 + 4 \cdot 1200^4 = (337 \cdot 4273) \cdot (7,204,801);$$

therefore

$$\begin{aligned} N_{1234} &= N_{123} \cdot N_4 = 1,437,599^4 + 4 \cdot 1,442,401^4 \\ &= (13 \cdot 73) (17 \cdot 97) \{ (5 \cdot 173) (29 \cdot 53) \} \cdot \{ (337 \cdot 4273) \cdot (7,204,801) \}. \end{aligned}$$

16. *Quotient Bin-Aurifeuillians*.—Writing 2^ω , x^ω , y^ω for 2, x , y in the expression (19); (compare Art. 7),

$$N = x^4 + 2^2 y^4 \text{ becomes } \mathfrak{N} = x^{4\omega} + 2^{2\omega} y^{4\omega} = (x^\omega)^4 + 2^2 \cdot (2^{\omega-1} y^\omega)^4. \quad (45)$$

Hence $\mathfrak{N}$ is a Bin-Aurifeuillian, and it obviously contains N , so that $Q = \mathfrak{N} \div N$ is an algebraic integer, being in fact the A.P.F. of $\mathfrak{N}$ when ω is a *prime*.

One of its $(a^2 + b^2)$ quadratic forms is given by *conformal division* (of $\mathfrak{N}$ by N) in the form

$$Q = \frac{\mathfrak{N}}{N} = \left(\frac{x^{2\omega+2} \pm 2^{\omega+1} y^{2\omega+2}}{x^4 + 2^2 y^4} \right)^2 + (2x^2 y^2)^2 \cdot \left(\frac{x^{2\omega-2} \mp 2^{\omega-1} y^{2\omega-2}}{x^4 + 2^2 y^4} \right)^2, \quad (46)$$

using the upper signs when $\omega = 4\mu + 1$, and lower signs when $\omega = 4\mu + 3$.

The other $(a^2 + b^2)$ quadratic form is also similarly expressible, but the criterion of sign is not so easily expressible in a general form, *i.e.*, except for particular values of ω .

The Aurifeuillian Factors $\mathfrak{E}$, $\mathfrak{H}$ of $\mathfrak{N}$ are

$$\mathfrak{E} = x^{2\omega} - 2^{\frac{1}{2}(\omega+1)} x^\omega y^\omega + 2^\omega y^{2\omega}, \quad \mathfrak{H} = x^{2\omega} + 2^{\frac{1}{2}(\omega+1)} x^\omega y^\omega + 2^\omega y^{2\omega} \quad (47)$$

$$= (x^\omega - 2^{\frac{1}{2}(\omega-1)} y^\omega)^2 + (2^{\frac{1}{2}(\omega-1)} y^\omega)^2, \quad = (x^\omega + 2^{\frac{1}{2}(\omega-1)} y^\omega)^2 + (2^{\frac{1}{2}(\omega-1)} y^\omega)^2. \quad (48)$$

The factors S , T of Q are the *integer* quotients of $\mathfrak{E}$ by L or M , and of $\mathfrak{H}$ by M or L . To discover which quotients are integral, it suffices to write $x = 1$, $y = 1$ in $\mathfrak{E}$, $\mathfrak{H}$, L , M ; whereon L , M reduce to 1, 5; showing* that

$$\begin{aligned} S &= \mathfrak{E} \div L, \quad T = \mathfrak{H} \div M, \quad \text{when } \omega = 8i \pm 1 = \text{prime}, \\ &\text{i.e., when } \left(\frac{2}{\omega} \right) = +1, \quad (49a) \end{aligned}$$

$$\begin{aligned} S &= \mathfrak{E} \div M, \quad T = \mathfrak{H} \div L, \quad \text{when } \omega = 8i \pm 3 = \text{prime}, \\ &\text{i.e., when } \left(\frac{2}{\omega} \right) = -1. \quad (49b) \end{aligned}$$

* Adapted from a similar rule for Aurifeuillian Factors, given in Art. 30 of Mr. Bickmore's first paper, above quoted.

Expressing $\mathfrak{E}, \mathfrak{M}, L, M$ in the $(a^2 + b^2)$ quadratic form, S and T can now also be expressed in that form by *conformal division*, in a form similar to (48); but the criterion of sign cannot be so easily expressed in general, *i.e.*, except for particular values of ω .

All the quantities $\mathfrak{N}, N; \mathfrak{E}, \mathfrak{M}, L, M; Q, S, T$ are also expressible in at least two other quadratic forms depending on the nature of ω , when ω is a *given small* number (Art. 5).

When ω is not a prime, the Reduced A.P.F. of $\mathfrak{M}$ may be formed by the formulæ of Art. 7b, and is a *Reduced Quotient Bin-Aurifeuillian* having the *same* properties as a Bin-Aurifeuillian.

As the quadratic partitions of Q, S, T are completely reducible only when ω is a *given small* number, it is better to pass on at once to the simple cases when $\omega = 3, 5, 7$, &c., as these alone are of *practical* interest. These may be styled *Cubic, Quintic, &c., Bin-Aurifeuillians*.

16a. Cubic Bin-Aurifeuillians.—Writing $\omega = 3$ in the formulæ of Art. 16,

$$Q = \frac{\mathfrak{N}}{N} = \frac{x^{12} + 2^6 y^{12}}{x^4 + 2^2 y^4} = x^8 - 2^2 x^4 y^4 + 2^4 y^8. \quad (50)$$

Here $\mathfrak{N}, \mathfrak{L}, \mathfrak{M}$, and therefore also Q, S, T , are capable of the three quadratic partitions $(a^2 + b^2)$, $(A^2 + 3B^2)$, $(C^2 - 3D^2)$; those of S, T are *monomorph*,* whilst those of Q are* *dimorph*, (since $Q = ST$). The results are

$$Q = (x^4 - 4y^4)^2 + (2x^2 y^2)^2 = (x^4 - 2x^2 y^2 + 4y^4)^2 + (2xy)^2 (x^2 - 2y^2)^2 \quad (50a)$$

$$= (x^4 - 2y^4)^2 + 3(2y^4)^2 = (x^4 - 6x^2 y^2 + 4y^4)^2 + 3(2xy)^2 (x^2 - 2y^2)^2 \quad (50b)$$

$$= (x^4 + 4y^4)^2 - 3(2xy)^2 = (x^4 + 6x^2 y^2 + 4y^4)^2 - 3(2xy)^2 (x^2 + 2y^2)^2. \quad (50c)$$

$$S = \frac{\mathfrak{L}}{M} = \frac{x^6 - 4x^3 y^3 + 8y^6}{x^2 + 2xy + 2y^2}; \quad T = \frac{\mathfrak{M}}{L} = \frac{x^6 + 4x^3 y^3 + 8y^6}{x^2 - 2xy + 2y^2} \quad (51)$$

$$= (x^4 + 2x^2 y^2 + 4y^4) - 2xy(x^2 + 2y^2); \quad = (x^4 + 2x^2 y^2 + 4y^4) + 2xy(x^2 + 2y^2) \quad (51')$$

$$= x^2(x - y)^2 + y^2(x - 2y)^2; \quad = x^2(x + y)^2 + y^2(x + 2y)^2 \quad (51a)$$

$$= (x^2 - xy - y^2)^2 + 3y^2(x - y)^2; \quad = (x^2 + xy - y^2)^2 + 3y^2(x + y)^2 \quad (51b)$$

$$= (x^2 - xy + 2y^2)^2 - 3(xy)^2; \quad = (x^2 + xy + 2y^2)^2 - 3(xy)^2. \quad (51c)$$

16b. Quintic Bin-Aurifeuillians.—Writing $\omega = 5$ in the formulæ of Art. 16,

$$Q = \frac{\mathfrak{N}}{N} = \frac{x^{20} + 2^{10} y^{20}}{x^4 + 2^2 y^4} = x^{16} - 2^2 x^{12} y^4 + 2^4 x^8 y^8 - 2^6 x^4 y^{12} + 2^8 y^{16}. \quad (52)$$

Here $\mathfrak{N}, \mathfrak{L}, \mathfrak{M}$, and therefore also Q, S, T , are capable of the *three* quadratic partitions $(a^2 + b^2)$, $(u^2 - 5v^2)$, $(U^2 + 5V^2)$; those of S, T are *monomorph*,* whilst those of Q are *dimorph*,* (since $Q = ST$). The results are

* Any $(C^2 - 3D^2)$, $(u^2 - 5v^2)$ or $(U^2 - 7V^2)$ partition can of course be transformed into the *automorphs* $(C_k^2 - 3D_k^2)$, $(u_k^2 - 5v_k^2)$, or $(U_k^2 - 7V_k^2)$, by conformal multiplication by any integer power (k) of the unit-form $(2^2 - 3 \cdot 1^2)$, $(9^2 - 5 \cdot 4^2)$, or $(8^2 - 7 \cdot 3^2)$ respectively (see the author's paper quoted in footnote § to Art. 5).

$$Q = (x^8 - 2^2x^4y^4 + 2^4y^8)^2 + (2xy)^2 (x^4 - 2^2y^4)^2 \quad (52a)$$

$$= (x^8 - 2x^6y^2 - 2^2x^4y^4 - 2^3x^2y^6 + 2^4y^8)^2 + (2xy)^2 (x^8 - 2^3y^6)^2 \quad (52b)$$

$$= (x^8 - 2x^4y^4 + 2^4y^8)^2 - 5 (2x^4y^4)^2 \quad (52c)$$

$$= (x^8 + 10x^6y^2 + 28x^4y^4 + 40x^2y^6 + 16y^8)^2 - 5 (2xy)^2 (x^6 + 4x^4y^2 + 8x^2y^4 + 8y^6)^2 \quad (52d)$$

$$= (x^8 - 12x^4y^4 + 16y^8)^2 + 5 (2x^2y^2)^2 (x^4 - 4y^4)^2 \quad (52e)$$

$$= (x^8 - 10x^6y^2 + 28x^4y^4 - 40x^2y^6 + 16y^8)^2 + 5 (2xy)^2 (x^8 - 4x^4y^2 + 8x^2y^4 - 8y^6)^2, \quad (52f)$$

$$S = \mathfrak{H} \div M = (x^{10} - 2^3x^5y^5 + 2^5y^{10}) \div (x^2 + 2xy + 2y^2) \quad (53)$$

$$= x^8 - 2x^7y + 2x^6y^2 - 4x^4y^4 + 8x^2y^6 - 16xy^7 + 16y^8 \quad (53a)$$

$$= (x^4 - x^3y + 2xy^3 - 4y^4)^2 + (xy)^2 (x^2 - 2xy + 2y^2)^2 \quad (53b)$$

$$= (x^4 - x^3y + 3x^2y^2 - 2xy^3 + 4y^4)^2 - 5 (xy)^2 (x^2 - xy + 2y^2)^2 \quad (53c)$$

$$= (x^4 - x^3y - 2x^2y^2 - 2xy^3 + 4y^4)^2 + 5 (xy)^2 (x^2 - 2y^2)^2, \quad (53d)$$

$$T = \mathfrak{H} \div L = (x^{10} + 2^3x^5y^5 + 2^5y^{10}) \div (x^2 - 2xy + 2y^2) \quad (53')$$

$$= x^8 + 2x^7y + 2x^6y^2 - 4x^4y^4 + 8x^2y^6 + 16xy^7 + 16y^8 \quad (53a')$$

$$= (x^4 + x^3y - 2xy^3 - 4y^4)^2 + (xy)^2 (x^2 + 2xy + 2y^2)^2 \quad (53b')$$

$$= (x^4 + x^3y + 3x^2y^2 + 2xy^3 + 4y^4)^2 - 5 (xy)^2 (x^2 + xy + 2y^2)^2 \quad (53c')$$

$$= (x^4 + x^3y - 2x^2y^2 + 2xy^3 + 4y^4)^2 + 5 (xy)^2 (x^2 - 2y^2)^2. \quad (53d')$$

16c. *Septimic Bin-Aurifeuillians*.—Writing $\omega = 7$ in the formulæ of Art. 16,

$$Q = \frac{\mathfrak{H}}{N} = \frac{x^{28} + 2^{14}y^{14}}{x^4 + 2^2y^4} = x^{24} - 2^2x^{20}y^4 + 2^4x^{16}y^8 - 2^6x^{12}y^{12} + 2^8x^8y^{16} - 2^{10}x^4y^{20} + 2^{12}y^{24}. \quad (54)$$

Here $\mathfrak{H}$, $\mathfrak{L}$, $\mathfrak{H}$, and therefore also Q , S , T , are capable of the *three* quadratic partitions $(a^2 + b^2)$, $(w^2 + 7z^2)$, $(W^2 - 7Z^2)$; those of Q being *dimorph**, whilst those of S , T are *monomorph** (since $Q = ST$). The results are

$$Q = \{ (x^4 - 2^2y^4)(x^8 + 2^4y^8) \}^2 + (2x^2y^2)^2 (x^8 - 2^2x^4y^4 + 2^4y^8)^2 \quad (54a)$$

$$= \{ x^{12} + 2^6y^{12} - 2x^4y^4 (x^4 + 2^2y^4) \}^2 + 7 (2x^4y^4)^2 (x^4 - 2^2y^4)^2 \quad (54b)$$

$$= (x^4 + 2^2y^4)^6 - 7 (2x^2y^2)^2 (x^8 + 2^2x^4y^4 + 2^4y^8)^2, \quad (54c)$$

$$S = \mathfrak{L} \div L = (x^{14} - 2^4x^7y^7 + 2^7y^{14}) \div (x^2 - 2xy + 2y^2), \quad (55)$$

$$T = \mathfrak{H} \div M = (x^{14} + 2^4x^7y^7 + 2^7y^{14}) \div (x^2 + 2xy + 2y^2) \quad (55')$$

$$= x^{12} - 2x^{11}y + 2x^{10}y^2 - 4x^8y^4 + 8x^7y^5 - 8x^6y^6 + 16x^5y^7 - 16x^4y^8 + 32x^2y^{10} - 64xy^{11} + 64y^{12} \quad (55a')$$

$$= x^2 (x^6 - x^4y + 2x^2y^3 - 4xy^4 + 4y^6)^2 + y^2 (x^6 - 2x^4y + 2x^2y^3 - 4xy^4 + 8y^6)^2 \quad (55b')$$

$$\dagger = (x^6 - x^5y - 3x^4y^2 + 4x^3y^3 - 6x^2y^4 + 4xy^5 + 8y^6)^2 + 7x^2y^2 (x^4 - x^3y + 2xy^3 - 4y^4)^2 \quad (55c')$$

$$\dagger = (x^6 - x^5y + 4x^4y^2 - 10x^3y^3 + 8x^2y^4 - 4xy^5 + 8y^6)^2 - 7x^2y^2 (x^4 - 2x^3y + 2x^2y^2 - 4xy^3 + 4y^4)^2 \quad (55d')$$

$$\quad (55d')$$

The value of S and the quadratic partitions of S may be obtained by writing $-y$ for y in those of T ; the other (dimorph) partitions of Q may be obtained by con-formal multiplication of the isomorphs of S , T .

17. *Fermat's Numbers*. (Base 2).—These are numbers of form

$$N_n = 2^n + 1.$$

Writing $x = 1$, $y = 2^\nu$, (where ν may be any integer), the Bin-Aurifeuillian $N = x^4 + 4y^4$ becomes

$$N = 1 + 4 \cdot 2^{4\nu} = 1 + 2^{4\nu+2} = N_{4\nu+2}. \quad (56)$$

* See the footnote * to Art. 16a, 16b.

† These two partitions were communicated by Mr. Bickmore.

Hence all Fermat's numbers of this form are resolvable into two factors L, M , of which one is always a multiple of 5, (Art. 8),

$$L = (2^{2^{\nu+1}} - 2^{\nu+1} + 1), \quad M = (2^{2^{\nu+1}} + 2^{\nu+1} + 1), \quad (56a)$$

by the formulæ of Art. 8; and can be completely resolved into their *prime* factors up to $n = 46$ (since at this limit the larger factor $M < 9^*$ million). When $2\nu + 1 = \omega = 3, 5, 7, \&c.$, they can be completely resolved into their *prime* factors by the formulæ of Arts. 16-16c up to $n = 66$ (since at this limit the larger factor S or T is still $< 9^*$ million. With the aid of "special divisors"† the resolution into *prime* factors can be carried far beyond this limit.

Examples.—The following Table‡ shows the complete factorisation of the Reduced A.P.F. of Fermat's Numbers $N_n = 2^n + 1$, where $n = 4\nu + 2$, complete as far as $n = 102$, and—at intervals—as far as $n = 210$. The factorisation of the Aurifeuillian factors L, M or S, T is shown separately, and prime factors § only are shown; the semicolon (;) indicates that the factorisation is complete.

[*Note*, that the highest number ($2^{210} + 1$), here completely factorised, contains 64 figures.]

FACTORS OF THE REDUCED A.P.F. OF $(2^n + 1)$, [when $n = 4\nu + 2$].

n	L or S	M or T	n	L or S	M or T	n	L or S	M or T
2	1 ;	5 ;	26	1613 ;	53.157 ;	50	101.8101 ;	5.268,501 ;
6	1 ;	13 ;	30	61 ;	1321 ;	54	246,241 ;	279,073 ;
10	5 ;	41 ;	34	137.953 ;	26,317 ;	58	107,367,629 ;	536,903,681 ;
14	113 ;	29 ;	38	229.457 ;	525,313 ;	62	5581.384,773 ;	8681.49,477 ;
18	37 ;	109 ;	42	14,449 ;	1429 ;	66	312,709 ;	4,327,489 ;
22	397 ;	2113 ;	46	277.30,269 ;	1013.1657 ;	70	47,392,381 ;	7,416,361 ;

* The present limit of the large Factor-Tables.

† Reuschle's *Neue zahlentheoretische Tabellen*, Stuttgart, 1856, contains (pp. 42-53) a table of "special prime divisors" of $(2^n - 1)$, up to $p < 6000$. The author has a MS. extension (incomplete) to a much higher limit.

‡ Adapted from the Table at pp. 9, 10 of Lucas's *Memoir* (of 1879) above quoted. The Table is there (p. 7) ascribed to M. Le Lasseur (the results as far as $n = 62$ had been given by M. Landry in his *Décomposition des nombres* $(2^n \pm 1)$ en leurs facteurs premiers, Paris, 1869). It has been necessary to recompute it entirely, as Lucas does not show the factors of L, M or S, T separately, and shows only the *special* divisors for each value of n , (i.e., divisors not belonging to any lower values of n); thus Lucas's Table omits the factor 5 when $n = 10$ and 50, and the factor 13 when $n = 78$. Only one mistake was discovered in the part of Lucas's Table hereby verified, viz., the omission of the factor 12,112,549 when $n = 82$. The large factor of M when $n = 146$ was found in Lucas's *Théorèmes d'Arithmétique*, Turin, 1878, p. 11; it is not in the 1879 Memoir.

§ The determination of most of the large primes (of twelve to sixteen figures) is ascribed to M. Le Lasseur in Ed. Lucas's *Théorèmes d'Arithmétique*, Turin, 1878, p. 11. The author's work has verified the large primes up to 36 millions.

FACTORS OF THE REDUCED A.P.F. OF $(2^n + 1)$, [when $n = 4\nu + 2$]*—continued.*

n	L or S	M or T
74	149 . 184,481,113 ;	593 . 231,769,777 ;
78	13 . 313 . 1249 ;	3121 . 21,841 ;
82	181,549 . 12,112,549 ;	10169 . 43,249,589 ;
86	1,759,217,765,581 ;	173 . 101,653 . 500,177 ;
90	181 . 54,001 ;	29,249,661 ;
94	140,737,471,578,113 ;	3761 . 7,484,047,069 ;
98	4,981,857,697,937 ;	197 . 19,707,683,773 ;
102	409 . 3061 . 13669 ;	1,326,700,741 ;
110	415,878,438,361 ;	3,630,105,520,141 ;
114	131,101 . 160,969 ;	275,415,303,169 ;
118	1181 . 3541 . 157,649 . 174,877 ;	
122	733 . 1709 . 368,140,581,013 ;	
126	118,750,098,349 ;	40,388,473,189 ;
130	108,140,989,558,681 ;	521 . 51,481 . 34,110,701 ;
138	70,334,392,823,809 ;	5,415,624,023,749 ;
146		293 . 9929 . 649,301,712,182,209 ;
150	63,901 . 13,334,701 ;	1201 . 1,182,468,601 ;
174	349 . 29,581 . 27,920,807,689 ;	
210	1,041,815,865,690,181 ;	421 . 146,919,792,181 ;

The following small Table gives “special divisors” (p) of such of Fermat’s Numbers of form $(2^{4\nu+2} + 1)$ as are omitted from the previous table, within Lucas’s limit $n = 210$. The discovery of a few more such “special divisors” would enable the factorisation of some of these numbers to be completed.

$n \dots$	106 ; 134 ; 142 ; 154 ; 158 ; 162 ; 166 ; 170 ; 178 ; 182 ;
$p \dots$	. ; 269 ; 569 ; 8317 ; 317 ; . ; 997 ; 1021 . 4421 ; 1069 ; 1093 . 4733 ;
$n \dots$	186 ; 190 ; 194 ; 198 ; 202 ; 206
$p \dots$	373 ; 761 . 54,721 ; 389 . 3881 . 4657 . 5821 ; . ; 809 ; . ;

18. Fermat’s Numbers. (Higher Bases).—Writing in Formulæ (3,3a)

$$X = 1, Y = 2y^2, \text{ and } 2n = 2\omega = 2(2\nu + 1),$$

ν being any integer, then

$$\mathfrak{Y} = 1 + Y^{2n} = 1 + (2y^2)^{4\nu+2} = 1^4 + 2^2 \cdot \eta^4, \text{ where } \eta = 2^\nu \cdot y^{2\nu+1}. \quad (57)$$

Here $\mathfrak{Y}$ is a *Bin-Aurifeuillian*, so that every such number, and its A.P.F., viz.,

$$\mathfrak{Y} \div N = (1 + Y^{2n}) \div (1 + Y^2), \text{ where } Y = 2y^2,$$

and also its “Reduced A.P.F.” (when further reducible), admit of resolution into the two factors L, M , or S, T , of Arts. 8, 16.

The existing Factor-Tables (extending to 9 million) enable the complete factorisation (*i.e.*, into prime factors) of such numbers to be effected up to the following limits:—

$$\begin{aligned} 2n &= 2 \quad ; \quad 6 \quad ; \quad 10 \quad ; \quad 14 \quad ; \quad 18 \quad ; \quad 22 \quad ; \quad 26 \text{ to } 66 \\ Y = 2y^2 &= 2 \cdot 2121^2 \quad ; \quad 2 \cdot 38^2 \quad ; \quad 2 \cdot 5^2 \quad ; \quad 2 \cdot 2^2 \quad ; \quad 2 \cdot 2^2 \quad ; \quad 2 \cdot 2^2 \quad ; \quad 2 \cdot 1^2 \\ &= 8,997,282 \quad ; \quad 2888 \quad ; \quad 50 \quad ; \quad 18 \quad ; \quad 8 \quad ; \quad 8 \quad ; \quad 2 \end{aligned}$$

and, by the aid of "special divisors," *† it can be carried far beyond these limits.

Examples.—The following Table ‡ shows the complete factorisation (*i.e.* into prime factors) of the following ($Y = 2y^2$ throughout) :

$$\begin{aligned} N &= 1 + Y^2 \text{ to } Y = 2 \cdot 40^2 = 3200, § \\ N &= (1 + Y^6) + (1 + Y^2) \text{ to } Y = 2 \cdot 39^2 = 3042, \\ N &= (1 + Y_{2\omega}) + (1 + Y^2) \text{ when } 2\omega = 10, 14, 22, 26 \text{ to various high limits,} \\ N &= (1 + Y^{18}) + (1 + Y^6). \end{aligned}$$

The factorisation of the Aurifeuillian factors L , M is shown separately, and prime¶ factors only are shown; the semicolon (;) in the Tables of $2\omega = 10, 14, 18, 22, 26$, indicates that the factorisation is complete.

* Reuschlo's *Tafeln complexer Primzahlen*, Berlin, 1875, give (p. 467 *et seq.*) the least values of Y for which $(1 + Y^{2\omega})$ is divisible by any given prime $p < 1000$.

† The author has prepared MS. Tables (which it is hoped to publish hereafter) giving the least value of Y for which $N = (1 + Y^n)$ is divisible by any given prime as follows:—

$$n = 5, 10, \text{ and } 15, p < 10,000; \quad n = 7, p < 4,000; \quad n = 11, p < 3,700;$$

by which all prime divisors within these limits can be detected at once for those powers (ω) and for the doubles thereof (*i.e.* 2ω). Most of the large factors (> 1000) of the larger numbers (*i.e.*, when L or $M > 9$ million) here factorised were detected by these MS. tables.

‡ Specially computed by the author for this Paper.

§ The highest number of this kind ($N = 1 + Y^2$) factorisable by the existing Factor-Tables is

$$1 + 8,997,282^2 = 1^4 + 2^2 \cdot 2121^4 = (8,993,041) (25 \cdot 13 \cdot 27697),$$

and, with the aid of "special divisors," this could be pushed much further. Euler, in *Commentationes Arithmeticae* (Petropol., 1849), p. 369 *et seq.*, gives a table of the small prime factors of $N = 1 + Y^2$, as far as $Y = 1500$ only: this shows the great advance in factorisation since Euler's time. Note that these numbers ($N = 1 + Y^2$) form a *Bin-Aurifeuillian Chain*, (Art. 20), *i.e.* the M of each number = the L of the next higher number.

¶ The high primes (> 9 million) occurring when $Y = 32, 128, 512$ are given on authority of Lucas's Table (quoted in footnote † to Art. 17). Those occurring among the factors of $(1 + Y^{10})$ were detected, when < 100 million, by the author's MS. Tables above mentioned; and, when > 100 million, by the method developed in Mr. F. W. Lawrence's paper on "Determination of certain Primes" in Vol. xxviii. of these *Proceedings* (with some improvements communicated to the author in MS. by Mr. Bickmore): this method is most effective in factorising and in detecting the prime character of numbers up to about 12 figures, when given as factors of a known number ($a^n \mp 1$) where a is small and n large.

FACTORISATION OF $N = (1 + Y^{2w}) + (1 + Y^2)$, [when $Y = 2y^2$].

y	$Y = 2y^2$	$(1 + Y^2)$		$(1 + Y^6) + (1 + Y^2)$		$(1 + Y^{10}) + (1 + Y^2)$	
		L	M	L	M	L	M
1	2	1	5	1	13	5;	41;
2	8	5	13	37	109	41.61;	5.1321;
3	18	13	25	229	457	5.15.101;	145.501;
4	32	25	41	13.61	1321	101.8101;	5.268,501;
5	50	41	61	13.157	3061	5.122,541;	7.622,561;
6	72	61	5.17	13.337	6133	5.4.552,601;	31,720,981;
7	98	5.17	113	8317	13.853	61.1.311,341;	5.701.30,341;
8	128	113	5.29	14,449	13.1429	5.47,392,381;	41.7.416,361;
9	162	5.29	181	23,473	13.37.61	421.1.464,241;	5.41.3,753,661;
10	200	181	13.17	97.373	44,221	1447,960,181;	41.43,120,981;
11	242	13.17	5.53	193.277	64153	5.4561.137341;	881.4.262,941;
12	288	5.53	313	13.5869	37.2437		5.181.281.29,401;
13	338	313	5.73	105,769	123,397	5.	61.881.262,261;
14	392	5.73	421	143,053	13.12697		5.
15	450	421	13.37	189,421	216,481	941.40,768,781;	61.701.1,025,021;
16	512	13.37	5.109	246,241	279,073	5.181.1321.54,001;	41.61.29,247,661;
17	578	5.109	613	13.24229	37.61.157	181.581,433,521;	5.
18	648	613	5.137	13.30553	443,917	5.61.761.718,621;	41.
19	722	5.137	761	13.109.349	549,481	881.15401.19001;	5.821.69,772,181;
20	800	761	29.29	37.16453	13.73.709	41.	
21	882	29.29	25.37	741,721	13.62761	5.	41.61.253,765,901;
22	968	25.37	1013	895,357	13.241.313		5.61.8821.341,521;
23	1058	1013	5.13.17	61.17569	1,169,137		
24	1152	5.13.17	1201	1,272,913	829.1669		
25	1250	1201	1301	13.37.3121	1,626,301		
26	1352	1301	5.281	97.18133	1069.1777		
27	1458	5.281	17.89	109.18793	13.169,693		
28	1568	17.89	125.13	37.97.661	337.7561		
29	1682	125.13	1741	73.37.441	2,928,421		
30	1800	1741	1861	13.193.1249	3,349,861		
31	1922	1861	5.397	13.73.3769	1597.2389		
32	2048	5.397	2113	13.312,709	4,327,489		
33	2178	2113	5.449	1009.4561	13.13.28933		
34	2312	5.449	2381	73.97.733	13.423,457		
35	2450	2381	2521	313.18,637	13.37.12,841		
36	2592	2521	5.13.41	6,534,361	6,907,753		
37	2738	5.13.41	29.97	7,296,769	7,702,069		
38	2888	29.97	5.593	13.241.2593	61.229.613		
39	3042	5.593	3121	37.243,769	73.130,057		
40	3200	3121	3281	9,987,121	13.807,637		

[Note, that the highest number $(722^{10} + 1)$, here completely factorised, contains 29 figures.]

FACTORISATION OF $N = (1 + Y^{20}) + (1 + Y^2)$, [when $Y = 2y^2$]*—continued.*

y	$Y = 2y^2$	$(1 + Y^{14}) + (1 + Y^2)$		$(1 + Y^{18}) + (1 + Y^6)$	
		L	M	L	M
1	2	113 ;	29 ;	37 ;	109 ;
2	8	29.14449 ;	113.1429 ;	246241 ;	279073
3	18	29.1623823 ;	1373.17837 ;		37.37.25309 ;
4	32	29.47392381 ;	113.7416361 ;	109.181.54001 ;	37.29247661 ;
5	50	29.		37.73.5761801 ;	
6	72	29.1597.3550541 ;		37.181.	
7	98		29.953	37.73	
8	128	4981857697937 ;	197.19707683773 ;	37.118750098349 ;	109.40388473189 ;

y	$Y = 2y^2$	$(1 + Y^{22}) + (1 + Y^2)$	
		L	M
1	2	397 ;	2113 ;
2	8	2113.312,709 ;	397.4,327,489 ;
3	18		89.
4	32	2113.415,878,438,361 ;	397.3,630,105,520,141 ;
5	50	881.	

y	$Y = 2y^2$	$(1 + Y^{26}) + (1 + Y^2)$	
		L	M
1	2	1613 ;	53.157 ;
2	8	13.53.157.313.1249 ;	1613.3121.21,841 ;
3	18	53.	
4	32	53.157.108,140,989,558,681 ;	521.1613.51,481.34,110,701 ;
5	50		53.

19. *Common Aurifeuillian Factors of Bin-Aurifeuillians.*—It will now be shown that every number F of form $(a^2 + b^2)$ is an Aurifeuillian Factor of four different Bin-Aurifeuillians, say N_1, N_2, N_3, N_4 . It suffices to take

$$\left. \begin{aligned} N_1 &= (b \sim a)^4 + 4a^4, & N_2 &= (b + a)^4 + 4a^4 \\ N_3 &= (a \sim b)^4 + 4b^4, & N_4 &= (a + b)^4 + 4b^4 \end{aligned} \right\}, \quad (58)$$

and, resolving N_1, N_2, N_3, N_4 into their Aurifeuillian Factors (L, M),

$$\left. \begin{aligned} N_1 &= \{(b - 2a)^2 + a^2\} (b^2 + a^2), & N_2 &= (b^2 + a^2) \{(b + 2a)^2 + a^2\} \\ N_3 &= \{(a - 2b)^2 + b^2\} (a^2 + b^2), & N_4 &= (a^2 + b^2) \{(a + 2b)^2 + b^2\} \end{aligned} \right\}, \quad (59)$$

the common factor $(a^2 + b^2)$ is seen to be an Aurifeuillian Factor of *each* of them. Hence many simple relations may be derived between such a set of four Bin-Aurifeuillians, *e.g.*,

$$\left. \begin{aligned} N_1 + N_4 &= N_2 + N_3 = 6(a^2 + b^2)^2 = 6F^2 \\ N_2 - N_1 &= N_4 - N_3 = 8ab(a^2 + b^2) = 8abF \\ N_1 - N_3 &= N_2 - N_4 = 4(a^4 - b^4) \end{aligned} \right\}. \quad (60)$$

In order that the four Bin-Aurifeuillians should fulfil the limitations of Art. 8, *i.e.*, that each x should be *odd* and prime to the corresponding y , it suffices that a, b should be relative primes, one *odd*, one *even*.

The above theorem is true whether F be prime, or composite; but if F be a composite number of form $(a^2 + b^2)$, it may be considerably extended. For every factor of F must necessarily be of form $(a^2 + \beta^2)$; and F will be expressible in several ways as a sum of squares $(a^2 + b^2)$; and the above theorem will be true for *each* of these ways: so that, if r be the number of different ways in which F is expressible as a sum of squares, then F will be a common Aurifeuillian Factor of $4r$ different Bin-Aurifeuillians; and, if $F = a^2 + b^2$ be any one of these partitions, the Results (58) give the values of the four Bin-Aurifeuillians corresponding.

Again, since every Bin-Aurifeuillian (N) is the product of two factors (L, M), each of which is of form $(a^2 + b^2)$, it follows from the above that every Bin-Aurifeuillian (N) is a common Aurifeuillian Factor of *eight* different Bin-Aurifeuillians. Thus, if

$$N = x^4 + 4y^4$$

be the base Bin-Aurifeuillian, then, taking its *two-fold* quadratic partition $(a^2 + b^2), (a'^2 + b'^2)$ from (19a), and substituting these for a, b , in the four formulæ (58), the *eight* Bin-Aurifeuillians (say $N_1, N_2, N_3, \dots, N_8$) which contain N as common Aurifeuillian Factor (L or M) are obtained, *viz.*,

$$\left. \begin{aligned} N_1 &= (2y^2 - x^2)^4 + 4(x^2)^4, & N_2 &= (2y^2 + x^2)^4 + 4(x^2)^4 \\ N_3 &= (x^2 - 2y^2)^4 + 4(2y^2)^4, & N_4 &= (x^2 + 2y^2)^4 + 4(2y^2)^4 \\ N_5 &= (2xy - x^2 + 2y^2)^4 + 4(x^2 - 2y^2)^4 \\ N_6 &= (2xy + x^2 - 2y^2)^4 + 4(x^2 - 2y^2)^4 \\ N_7 &= (x^2 - 2y^2 - 2xy)^4 + 4(2xy)^4 \\ N_8 &= (x^2 - 2y^2 + 2xy)^4 + 4(2xy)^4 \end{aligned} \right\}, \quad (61)$$

and writing, as usual, $N_r = L_r M_r$, the *common* Aurifeuillian Factor (N) is $M_1 = L_2 = M_3 = L_4 = M_5 = L_6 = M_7 = L_8 = N$, (61a)

and the other Aurifeuillian Factors are, (see Art. 8),

$$\left. \begin{aligned} L_1 &= x^4 + 4(y^2 \sim x^2)^2, & M_2 &= x^4 + 4(x^2 + y^2)^2 \\ L_3 &= (x^2 \sim 4y^2)^2 + 4y^4, & M_4 &= (x^2 + 4y^2)^2 + 4y^4 \\ L_5 &= (x^2 - 2y^2)^2 + 4(xy - x^2 + 2y^2)^2, & M_6 &= (x^2 - 2y^2)^2 + (xy + x^2 - 2y^2)^2 \\ L_7 &= (x^2 - 2y^2 - 4xy)^2 + 4(xy)^2, & M_8 &= (x^2 - 2y^2 + 4xy)^2 + (2xy)^2 \end{aligned} \right\} \quad (61b)$$

COR.—The above results (61, 61b) furnish an easy solution of the problem of Art. 18: for the four Bin-Aurifeuillians N_1, N_2, N_3, N_4 each contain a Bin-Aurifeuillian (N) as *one* of their Aurifeuillian Factors, and the *other* Aurifeuillian Factor (*i.e.*, L_1, M_2, L_3, M_4) of any *one* of them becomes itself a Bin-Aurifeuillian by determining x, y so that

$$\text{one of } (y^2 \sim x^2), (y^2 + x^2), (x^2 \sim 4y^2), (x^2 + 4y^2) = z^2 \text{ (a square)}; \quad (61c)$$

and these are the conditions of some one or other of the Cases i, ii, iii, iv of Art. 13a, (the x, y, N of this Article corresponding to the x_1, y_1, N_1 or x_2, y_2, N_2 of Art. 13a, with other interchanges of notation, as necessary).

Examples.—The following Table shows the *quartets* of Bin-Aurifeuillians (N_1, N_2, N_3, N_4) which contain the common Aurifeuillian Factors $F = 5, 13, 17, 25, 37$; and also the *octet* of Bin-Aurifeuillians which contain the common Bin-Aurifeuillian Factor $F = 65$.

Factor.	Aurifeuillian Parts.				Bin-Aurifeuillians.			
	$x_1; y_1$	$x_2; y_2$	$x_3; y_3$	$x_4; y_4$	N_1	N_2	N_3	N_4
$a; b; F$								
1; 2; 5	1; 1	3; 1	1; 2	3; 2	5	85	65	145
3; 2; 13	1; 3	5; 3	1; 2	5; 2	325	949	65	689
1; 4; 17	3; 1	5; 1	3; 4	5; 4	85	629	1105	1649
3; 4; 25	1; 3	7; 3	1; 4	7; 4	325	2725	1025	3425
1; 6; 37	5; 1	7; 1	5; 6	7; 6	629	2405	5809	7585
1; 8; 65	7; 1	9; 1	7; 8	9; 8	2405	6565	18785	22945
7; 4; 65	3; 7	11; 7	3; 4	11; 4	9685	24245	1105	15665

20. *Bin-Aurifeuillian Chains.*—Let a series of Bin-Aurifeuillians $N_1, N_2, N_3, \dots, N_r$ be taken, so that

$$N_r = L_r M_r = x_r^4 + 4y_r^4, \text{ (for each value of } r), \quad (62)$$

and let the parts (x_r, y_r) of the successive functions be taken, so that
Case i.

$$y_1 = y_2 = y_3 = \dots y_r,$$

$$x_2 = x_1 + 2y_1, \quad x_3 = x_1 + 4y_1, \quad x_4 = x_1 + 6y_1, \text{ \&c. } \dots x_r = x_{r-1} + 2y_1.$$

Case ii.

$$x_1 = x_2 = x_3 = \dots x_r,$$

$$y_3 = x_1 + y_1, y_4 = 2x_1 + y_1, y_5 = 3x_1 + y_1, \&c. \dots y_r = x_1 + y_{r-1}.$$

Then, in Case i,

$$M_{r-1} = (x_{r-1} + y_{r-1})^2 + y_{r-1}^2 = (x_{r-1} + y_1)^2 + y_1^2,$$

$$L_r = (x_r - y_r)^2 + y_r^2 = (x_{r-1} + y_1)^2 + y_1^2.$$

And, in Case ii,

$$M_{r-1} = (x_{r-1} + y_{r-1})^2 + y_{r-1}^2 = y_r^2 + y_{r-1}^2,$$

$$L_r = (x_r - y_r)^2 + y_r^2 = y_{r-1}^2 + y_r^2.$$

Then, in each Case, $M_{r-1} = L_r$, (63)

or, in other words, every two successive members N_{r-1} , N_r have a common Aurifeuillian Factor. Such a series may be called a *Bin-Aurifeuillian Chain*, and has many interesting properties: thus

$$N_r : N_{r-1} = L_r M_r : L_{r-1} M_{r-1} = M_r : L_{r-1}. \quad (64)$$

$$\text{Similarly, } N_{r-2} : N_{r-3} = M_{r-2} : L_{r-3},$$

Hence in every successive quartet (N_{r-3} , N_{r-2} , N_{r-1} , N_r)

$$N_r N_{r-2} : N_{r-1} N_{r-3} = M_r : L_{r-3}. \quad (64a)$$

Similarly, giving to r every integer value from 1 to $2r$,

$$\frac{N_{2r} N_{2r-3} \dots N_6 N_4 N_2}{N_{2r-1} N_{2r-2} \dots N_5 N_3 N_1} = \frac{M_{2r}}{L_1}. \quad (64b)$$

Ex.—See the Table of Factorisation of $(1 + Y^2)$, where $Y = 2y^2$, (Art. 18); these numbers $N = (1^4 + 4y^4)$ will be seen to form a *Bin-Aurifeuillian Chain*.

CHAPTER III., TRIN-AURIFEUILLIANS.

21. *Trin-Aurifeuillians*.—When, in Results (4; $4a, b$; $6a, b$), $n_1 = 1$, $n = n_2 = 3$, $X = x^3$, $Y = 3y^3$, the function N becomes the *Trin-Aurifeuillian*,

$$N = (x^6 + 3^3 y^6) \div (x^3 + 3y^3) = x^4 - 3x^2 y^3 + 9y^4 \left. \vphantom{N = (x^6 + 3^3 y^6) \div (x^3 + 3y^3)} \right\}. \quad (65)$$

x being prime* to $3y$, (see Art. 2)

It has four quadratic partitions of form $(A^2 + 3B^2)$, one (N^0) being

* Hereby N will be of form $(6f + 1)$.

general, and three (N' , N'' , N''') alternative, see Art. 5, viz.,

$$N^{\circ} = (x^3 - 3y^3)^3 + 3(xy)^3, \text{ in all cases,} \quad (66^{\circ})$$

$$N' = (x^3 - \frac{3}{2}y^3)^3 + 3(\frac{3}{2}y^3)^3, \text{ when } y \text{ is even,} \quad (66')$$

$$N'' = (\frac{1}{2}x^3 - 3y^3)^3 + 3(\frac{1}{2}x^3)^3, \text{ when } x \text{ is even,} \quad (66'')$$

$$N''' = \left(\frac{x^3 + 3y^3}{2}\right)^3 + 3\left(\frac{x^3 - 3y^3}{2}\right)^3, \text{ when } x, y \text{ both odd.} \quad (66''')$$

Its two Aurifeuillian Factors (L , M) are

$$L = x^3 - 3xy + 3y^3, \quad M = x^3 + 3xy + 3y^3, \quad (67)$$

and are each expressible in three (alternative) ways

$$L' = (x - \frac{3}{2}y)^3 + 3(\frac{1}{2}y)^3; \quad M' = (x + \frac{3}{2}y)^3 + 3(\frac{1}{2}y)^3; \quad (67')$$

$$L'' = (\frac{1}{2}x)^3 + 3(\frac{1}{2}x - y)^3; \quad M'' = (\frac{1}{2}x)^3 + 3(\frac{1}{2}x + y)^3; \quad (67'')$$

$$L''' = \left(\frac{x - 3y}{2}\right)^3 + 3\left(\frac{x - y}{2}\right)^3; \quad M''' = \left(\frac{x + 3y}{2}\right)^3 + 3\left(\frac{x + y}{2}\right)^3. \quad (67''')$$

Here L' , M' , N' apply when y is even; L'' , M'' , N'' when x is even;

L''' , M''' , N''' apply when both x , y are odd.

Also, by conformal multiplication,

$L'M'$ yields both N' , N° ; $L''M''$ yields both N'' , N° ;

$$L'''M''' \text{ yields both } N''', N^{\circ}. \quad (68)$$

Thus, when the Aurifeuillian Factors (L , M) of any particular Trin-Aurifeuillian (N) are both prime, they have only one quadratic partition each, of form $(A^2 + 3B^2)$; and the Trin-Aurifeuillian (N) has in that case only two such partitions (one of which is always N°). If, however, either or both of L , M be composite, all their factors must necessarily be of form $(A^2 + 3B^2)$; and L or M (or both L , M), and therefore also N , will then be polymorphs of form $(A^2 + 3B^2)$: but these partitions will, as a rule, be only arithmetically (not algebraically) possible; and are therefore not of so much interest as those given. In conformity with the remark at end of Art. 5, the three analogous forms (N' , N'' , N'''), or (L' , L'' , L'''), or (M' , M'' , M'''), although arithmetically distinct, differ from each other only by the unit-factor $\frac{1}{4}(1^2 + 3.1^2)$, so that any two of a trio can be derived from the third by conformal multiplication by that unit-form: e.g.,

$$\frac{1}{4}(1^2 + 3.1^2) N' \text{ yields both } N'', N''', \&c., \&c.$$

22. Trin-Ant-Aurifeuillians.—When, in Results (4; 4a, b; 6a, b; 8a, b), $n_1 = 1$, $n = n_2 = 3$, $X = x^3$, $Y = -3y^3$, the function N becomes the *Trin-Ant-Aurifeuillian*.*

$$\left. \begin{aligned} N &= (x^5 - 3^8 \cdot y^5) \div (x^2 - 3y^3) = x^4 + 3x^2y^3 + 9y^4 \\ x &\text{ being prime}\dagger \text{ to } 3y, \text{ (see Art. 2)} \end{aligned} \right\}. \quad (69)$$

It has the following quadratic partitions: one of form $(a^2 + b^2)$, one (N^0) of form‡ $(C^2 - 3D^2)$, and three (N' , N'' , N''') alternative of form $A^2 + 3B^2$.

$$N = (x^2 - 3y^3)^2 + (3xy)^2, \quad (70)$$

$$N^0 = (x^2 + 3y^3)^2 - 3(xy)^2, \quad (70^0)$$

$$N' = (x^2 + \frac{3}{2}y^3)^2 + 3(\frac{3}{2}y^3)^2, \text{ when } y \text{ is even,} \quad (70')$$

$$N'' = (\frac{1}{2}x^2 + 3y^3)^2 + 3(\frac{1}{2}x^2)^2, \text{ when } x \text{ is even,} \quad (70'')$$

$$N''' = \left(\frac{x^2 - 3y^3}{2}\right)^2 + 3\left(\frac{x^2 + 3y^3}{2}\right)^2, \text{ when } x, y \text{ both odd,} \quad (70''')$$

The above forms correspond to the forms (LM), N^0 , N' , N'' , N''' of the Trin-Aurifeuillian (Art. 21) respectively; and the conjugate forms are interconvertible by changing $\pm y^3$ into $\mp y^3$.

23. Fermat's Divisors.—Under Art. 6a, b the conditions of divisibility by 7 and 13 are

Every Trin-Aurifeuillian is divisible by 7 when (and only when) neither of x , y , $(x^2 + 3y^3)$ contains 7; also $N \equiv 3x^4 \pmod{7}$, when $x^2 + 3y^3 \equiv 0 \pmod{7}$, provided neither of x , y contain 7. (71a)

Also $\left. \begin{aligned} L &\text{ contains 7 when } (x+y) \text{ or } (2x-y) = 7k \\ M &\text{ contains 7 when } (x-y) \text{ or } (2x+y) = 7k \end{aligned} \right\}. \quad (71b)$

Every Trin-Aurifeuillian is divisible by 13 when xy is a quadratic non-residue of 13, and at the same time neither of x , y , $(x^2 + 3y^3)$ contains 13. (71c)

Every Trin-Ant-Aurifeuillian is divisible by 13 when xy is a quadratic residue of 13, and at the same time neither of x , y , $(x^2 - 3y^3)$ contains 13. (71d)

[It is worth note that both 7 and 13 are Trin-Aurifeuillians; for $7 = (1^6 + 3 \cdot 1^6) \div (1^2 + 3 \cdot 1^2)$, and $13 = (2^6 + 3 \cdot 1^6) \div (2^2 + 3 \cdot 1^2)$; whilst 13 is also a Trin-Ant-Aurifeuillian, for $13 = (1^6 - 3 \cdot 1^6) \div (1^2 - 3 \cdot 1^2)$.]

Again, since the square and the sixth power of every number end in

* This modification of the Trin-Aurifeuillian is due to Mr. Bickmore (see Art. 4).

† Hereby N will be of form $(12f+1)$.

‡ See footnote * to Art. 16a.

the *same digit*, which is always one of 0, 1, 4, 5, 6, 9, it follows that

All Trin-Aurifeuillians and all Trin-Ant-Aurifeuillians end in one of the digits 1, 3, 7, 9. (71*e*)

When one of them (N) ends in 3 or 7, the conjugate (N') ends in 7 or 3, so that $N + N' \equiv 0, (\text{mod } 10)$; also, when one of them ends in 1 or 9, then either $N - N'$ or $N + N' \equiv 0, (\text{mod } 10)$. (71*f*)

Again, from $66^\circ - 66'''$, and $70^\circ - 70'''$, it follows that—

Every prime factor (F) of a Trin-Aurifeuillian or Trin-Ant-Aurifeuillian is of form $(8f + 1)$, $(12f + 1)$ respectively. (71*g*)

[*Note*, that properties (71*af*) refer only to the simple functions $N = (x^3 \pm 3^{3/6}) + (x^2 \pm 3y^2)$, and not to the Quotient-Functions $(\mathfrak{N} + N)$ treated of in Art. 26, 26*a*, 27.]

24. Trin-Aurifeuillian Dimorphs.—The question now arises whether the Trin-Aurifeuillian partition of a number (N) is *unique*, or whether it may be *dimorphic*.

$$\text{Let } N_1 = x_1^4 - 3x_1^2y_1^2 + 9y_1^4, \quad N_2 = x_2^4 - 3x_2^2y_2^2 + 9y_2^4 \quad (72)$$

be, if possible, two *different* Trin-Aurifeuillian forms of the *same* number N ; so that

$$N = N_1 = N_2. \quad (73)$$

The *general* solution of this seems difficult, but a limited solution may be obtained by identifying two of the *different* quadratic partitions of type $(A^2 + 3B^2)$ of which N_1, N_2 are capable, Art. 21; say

$$N_1 = N_1^\circ; \quad N_2 = N_2' \text{ or } N_2'' \text{ or } N_2''' \text{ (alternatively).} \quad (73a)$$

Thus *three* trial systems arise—

$$\text{i. } N_1^\circ = N_2'; \quad \text{ii. } N_1^\circ = N_2''; \quad \text{iii. } N_1^\circ = N_2'''. \quad (73b)$$

Identifying the *like* quadratic parts (A_1, A_2) , (B_1, B_2) separately, each system gives two equations—

$$\text{i. } A_1^\circ = A_2', \quad B_1^\circ = B_2'; \quad \text{ii. } A_1^\circ = A_2'', \quad B_1 = B_2''; \quad \text{iii. } A_1^\circ = B_2''', \quad B_1^\circ = B_2'''. \quad (73c)$$

*Referring to Art. 21, the *first** system gives

$$x_1^2 - 3y_1^2 = x_2^2 - \frac{3}{2}y_2^2, \quad x_1y_1 = \frac{3}{2}y_2^2, \quad (74)$$

$$\text{whence} \quad (x_1 + \frac{1}{2}y_1)^2 - 13 \left(\frac{y_1}{2} \right)^2 = x_2^2. \quad (74a)$$

These, together with the working conditions, involve

$$y_1, y_2 \text{ both even; } x_1, x_2 \text{ both odd, and not multiples of 3.} \quad (74b)$$

* Systems ii, iii appear to involve impossibilities.

The Diophantine (74a) has two *general* solutions in terms of two arbitrary quantities m, n , (to be taken relative primes):—

$$\left. \begin{array}{l} \text{ia.} \quad x_1 + \frac{1}{2}y_1 = m^2 + 13n^2, \quad \frac{1}{2}y_1 = 2mn, \quad x_2 = m^2 - 13n^2 \\ \text{whence} \quad x_1 = m^2 - 2mn + 13n^2 = (m-n)^2 + 3(2n)^2 \end{array} \right\}. \quad (75)$$

where m, n are one *odd*, one *even*

$$\left. \begin{array}{l} \text{ib.} \quad x_1 + \frac{1}{2}y_1 = \frac{1}{2}(m^2 + 13n^2), \quad \frac{1}{2}y_1 = mn, \quad x_2 = m^2 - 13n^2 \\ \text{whence} \quad x_1 = \frac{1}{2}(m^2 - 2mn + 13n^2) = \frac{1}{2}\{(m-n)^2 + 3(2n)^2\} \end{array} \right\}. \quad (75)$$

m, n both *odd*; giving x_1 *even*; solution *rejected*

Again, the condition $x_1 y_1 = \frac{3}{2}y_2^2$, together with (74b), involves

$$x_1 = \xi^2, \quad y_1 = 6\eta^2, \quad y_2 = 2\xi\eta. \quad (75a)$$

The value of y_1 ($= 4mn$, or $2mn$) shows that either m or n must be a multiple of 3, and, finally, that m, n must have *one* of the forms

$$\text{ia.} \quad \left\{ \begin{array}{l} m = 3\mu^2, \mu^2, 2\mu^2, 6\mu^2 \\ n = 2\nu^2, 6\nu^2, 3\nu^2, \nu^2 \end{array} \right\}. \quad (75b)$$

$$\text{These give} \quad \text{ia.} \quad y_1 = 4mn = 6(2\mu\nu)^2; \quad (75c)$$

thus satisfying the condition $y_1 = 6\eta^2$; so that it only remains to determine m, n so as to satisfy the condition $x_1 = \xi^2$, where x_1 is given by (75, ia.); but it does not seem easy to do this in a general manner.

*Examples.**—Taking $\mu = 1, \nu = 1; m = 3\mu^2 = 3, n = 2\nu^2 = 2$ in ia.

This gives $x_1 = 3^2 - 2 \cdot 3 \cdot 2 + 13 \cdot 2^2 = 49$, which satisfies $x_1 = \xi^2$.

Here $x_1 = 49, y_1 = 24; x_2 = 43, y_2 = 28$.

Hence $\left. \begin{array}{l} N_1 = L_1 M_1 = 601.7657 = 601(13.19.31) \\ N_2 = L_2 M_2 = 589.7813 = (19.31)(13.601) \end{array} \right\} \text{ and } \Lambda_1 = \Lambda_2.$

25. *Tri - Trin - Ant - Aurifeuillians.* — To investigate whether a number N can be at once a Trin-Aurifeuillian (N_1) and a Trin-Ant-Aurifeuillian N_2 . Let, if possible,

$$\left. \begin{array}{l} N = N_1 = N_2, \\ \text{where} \quad N_1 = x_1^2 - 3x_1^2 y_1^2 + 9y_1^4, \quad N_2 = x_2^2 + 3x_2^2 y_2^2 + 9y_2^4 \end{array} \right\}. \quad (76)$$

A general solution seems difficult, but a *limited* solution may be obtained by expressing N_1, N_2 in the various quadratic forms N_1, N_1'', N_1''' ; N_2', N_2'', N_2''' of type $N = A^2 + 3B^2$ of Art. 21, 22, and then

* From the general properties of Diophantines, it seems probable that any solution should yield an infinite set of solutions, (compare Art. 9, 36); but the author has not been able to determine this case.

equating the parts A_1, B_1 of any one of the forms N'_1, N''_1, N'''_1 separately to the *corresponding* parts A_2, B_2 of any one of the forms N'_2, N''_2, N'''_2 . Thus would arise *nine* systems of equations, each of type $A_1 = A_2, B_1 = B_2$; or, taking into account the double sign ($\pm$) of $A_1, A_2; B_1, B_2$, there would be 36 systems of equations, (two in each). Most of these systems appear to be *impossible*: in fact, the only possible systems appear to be those of the *two* groups (I, II) below.

$$\text{GROUP I. i. } N'_1 = N'_2; \text{ ii. } N''_1 = N''_2; \text{ iii. } N'''_1 = N'_2, \quad (77)$$

which yield, disregarding the ($\pm$) sign of A, B ,

$$\text{i. } A'_1 = A'_2, B'_1 = B'_2; \text{ ii. } A''_1 = A''_2, B''_1 = B''_2; \text{ \&c. } (77a)$$

Taking the values of A, B from Art. 21, 22, and introducing the $\pm$ signs, there is found to be *only one* possible case in each system i, ii, iii, and these three possible systems will be found to yield the *same results*, reducible in each case to the *two* conditions

$$x_1^2 - x_2^2 = 3y_1^2 = 3y_2^2. \quad (77b)$$

The further discussion of this case is reserved to Art. 26.

$$\text{GROUP II. iv. } N'_1 = N'_2; \text{ v. } N''_1 = N'_2; \text{ vi. } N'''_1 = N'_2, \quad (78)$$

which yield, disregarding the ($\pm$) sign of A, B ,

$$\text{iv. } A'_1 = A'_2, B'_1 = B'_2; \text{ \&c. \&c.}$$

Reducing the A, B , and introducing the $\pm$ sign, as in Group I, there is found *only one* possible case in each system iv, v, vi, and these three possible cases lead to the same results, reducible in each case to the two conditions

$$y_1^2 - y_2^2 = \frac{1}{3}x_1^2 = \frac{1}{3}x_2^2;$$

but, as this involves x_1, x_2 being each $= 3\xi$, a case expressly excluded (65, 69), this solution is *rejected*.

26. Various Trin-Aurifeuillians.—The results of imposing *one* of the conditions

$$\text{i. } x^2 - 3y^2 = z^2; \text{ ii. } x^2 + 3y^2 = z^2, \quad (79)$$

on the Trin-Aurifeuillian N , and Trin-Ant-Aurifeuillian N^* , where

$$N = x^4 - 3x^2y^2 + 9y^4, \quad N^* = x^4 + 3x^2y^2 + 9y^4, \quad (80)$$

(already partly considered in Art. 25), are worth further study.

* Note that z is here used (instead of x) in the formula for N^* : this is done to distinguish it here from the x used in the formula for N .

They give

$$\text{i. } N = N' = x^4 - x^2 z^2 + z^4 = \frac{x^6 + z^6}{x^2 + z^2} = N_{vi} \text{ (suppose),} \quad (81)$$

$$\text{ii. } N = x^4 - 3x^2 z^2 + 3z^4 = \mathfrak{F} \text{ of } \mathfrak{Y}, \text{ where } \mathfrak{Y} = z^8 - 3x^4 z^4 + 9x^8, \quad (82a)$$

$$N' = x^4 - 3x^2 z^2 + 3z^4 = \mathfrak{F} \text{ of } \mathfrak{Y}, \text{ where } \mathfrak{Y} = x^8 - 3x^4 z^4 + 9z^8. \quad (82b)$$

Thus, Condition ii gives the two higher Trin-Aurifeuillians $\mathfrak{Y}$ whose smaller Aurifeuillian Factor $\mathfrak{F}$ is itself of one of the forms N or N' .

The results of Condition i are of great interest, viz., that the forms N, N', N_{vi} are *interconvertible*,* thus also providing *factorisable* numbers of forms N' , and N_{vi} ,† both being expressible in form $N = LM$.

There are *two* general solutions of each of the Diophantine conditions (i, ii) of (79) in terms of two arbitrary quantities m, n (here $m \neq 3\mu$, and m, n should be relatively prime).

$$\text{i. } x = m^2 + 3n^2, \quad y = 2mn, \quad z = m^2 - 3n^2; \quad m, n \text{ one odd, one even,} \quad (83)$$

$$\text{giving} \quad L = (m^2 - 3mn)^2 + (3mn - 3n^2)^2 = a_r^2 + b_r^2,$$

$$M = (m^2 + 3mn)^2 + (3mn + 3n^2)^2 = a_m^2 + b_m^2.$$

$$\text{Thus} \quad a_i \sim b_i = a_m \sim b_m = m^2 - 3n^2 = z.$$

$$\text{ia. } x = \frac{1}{2}(m^2 + 3n^2), \quad y = mn, \quad z = \frac{1}{2}(m^2 - 3n^2); \quad m, n \text{ both odd.} \quad (83a)$$

These give L, M each $\frac{1}{4}$ of the L, M of Case i.

$$\text{ii. } x = m^2 - 3n^2, \quad y = 2mn, \quad z = m^2 + 3n^2; \quad m, n \text{ one odd, one even.} \quad (84)$$

$$\text{iaa. } x = \frac{1}{2}(m^2 - 3n^2), \quad y = mn, \quad z = \frac{1}{2}(m^2 + 3n^2); \quad m, n \text{ both odd.} \quad (84a)$$

An interesting sub-case under Condition i arises when

$$x^2 - 3y^2 = z^2 \text{ (constant).} \quad (85)$$

This gives $N = N' = N_{vi}$, and also gives the *Trin-Aurifeuillian Chains* further developed in Art. 34a, q.v.

Sub-Case (Fermat's Numbers). — The sub-case of the last when = 1 is worth separate notice, as giving a factorisable case† of Fermat's Numbers $\mathfrak{Y} = (x^6 + 1)$. Here

$$N = N' = N_{vi}, \text{ where } N_{vi} = (x^6 + 1) \div (x^2 + 1), \quad (86)$$

$$N = x^4 - 3x^2 y^2 + 9y^4, \quad N' = 1 + 3y^2 + 9y^4. \quad (86a)$$

Calling the *successive* numbers $N_1, N_2, N_3, \dots, N_r$, and applying the subscript r to the *parts* x, y , of N_r , the parts x, y of successive numbers N are connected by the relations

$$x_1^2 - 3y_1^2 = x_2^2 - 3y_2^2 = \dots = x_r^2 - 3y_r^2 = 1, \quad (87)$$

* The convertibility of N, N' under Condition i was shown in Art. 25; the x_2 of that Article is the z of this Article.

† This factorisable form of N_{vi} is due to Mr. Bickmore, and was communicated by him to the Oxford Mathematical Society in February, 1898.

whence $x_r = 2x_{r-1} + 3y_{r-1}$, $y_r = x_{r-1} + 2y_{r-1}$, ($x_1 = 2$, $y_1 = 1$), (87a)
and the m , n of the general formulæ are connected by the relations

$$\left. \begin{aligned} m_{2r} &= n_{2r} + n_{2r-1}, & m_{2r+1} &= n_{2r+1} + 2n_{2r} \\ n_{2r} &= \frac{1}{2}(m_{2r-1} + n_{2r-1}), & n_{2r+1} &= m_{2r} + n_{2r} \end{aligned} \right\}, \quad (87b)$$

$$\left. \begin{aligned} \text{These give } L_r &= (m_r^2 \sim 3m_r n_r)^2 + (3m_r n_r \sim 3n_r^2)^2 = a_{lr}^2 + b_{lr}^2 \\ M_r &= (m_r^2 + 3m_r n_r)^2 + (3m_r n_r + 3n_r^2)^2 = a_{mr}^2 + b_{mr}^2 \end{aligned} \right\}, \quad (87c)$$

Thus $a_{lr} \sim b_{lr} = a_{mr} \sim b_{mr} = m_r^2 \sim 3n_r^2 = z = 1$, (for all values of r)

so that L_r , M_r each consist of a sum of two consecutive squares

These numbers N_r , (being only a sub-case of the last, given by $z = 1$), form a *Trin-Aurifeuillian Chain*, i.e., a series such that $M_{r-1} = L_r$ for all values of r (Art. 34a).

Examples.—The upper part of the following table shows the factorisation of the numbers $N_{r1} = (x^6 + 1) + (x^2 + 1)$, falling under both forms N , N' , into their *prime* factors, pushed to a very high limit; the values of L , M show the *Trin-Aurifeuillian Chain* ($M_r = L_{r+1}$).

Below these are given a few more examples, the *lowest* of the numbers $N_{r1} = (x^6 + z^6) + (x^2 + z^2)$ —when $z \neq 1$ —also falling under both forms N , N' .

The lower part of the Table contains a few (the *lowest*) examples of Higher Trin-Aurifeuillians $\mathfrak{N}$ whose smaller Trin-Aurifeuillian Factor $\mathfrak{L}$ is itself a Trin-Aurifeuillian, so that $\mathfrak{N} = (L \cdot M) \cdot \mathfrak{M}$.

Case.	r	Data.	Parts.	$N = L \cdot M$		$N_{r1} = N = N'$
		$m ; n$	$x ; y ; z$	L	M	$(x^6 + z^6) + (x^2 + z^2)$
Sub-Case of i ($z = 1$).	1	1 ; 1	2 ; 1 ; 1	1	13	13
	2	2 ; 1	7 ; 4 ; 1	13	181	2,353
	3	5 ; 3	26 ; 15 ; 1	181	2521	456,301
	4	7 ; 4	97 ; 56 ; 1	2521	13.37.73	88,519,873
	5	19 ; 11	362 ; 209 ; 1	13.37.73	489,061	13,341,285,133
	6	26 ; 15	1351 ; 780 ; 1	489,061	6,811,741	
	7	71 ; 41	5042 ; 2911 ; 1	6,811,741	13.61.181.661	
	8	97 ; 56	18817 ; 10864 ; 1	13.61.181.661 ;		
Case i		1 ; 2	13 ; 4 ; 11	61	373	22,753
		5 ; 1	14 ; 5 ; 11	61	13.37	29,341
		1 ; 3	14 ; 3 ; 13	97	349	33,853
		4 ; 1	19 ; 8 ; 13	97	1009	97,873
Case ii		Data.	Parts.	$N = L \cdot M = \mathfrak{L}$		$\mathfrak{N} = \mathfrak{L} \cdot \mathfrak{M}$ $= L \cdot M \cdot \mathfrak{M}$
		$m ; n$	$x ; y ; z$	$L ; M$	$\mathfrak{M}$	
		1 ; 1	1 ; 1 ; 2	1 ; 7	31	217
		2 ; 1	1 ; 4 ; 7	37 ; 61	2551	5,757,607
		1 ; 2	11 ; 4 ; 13	37 ; 301	133,831	1,490,475,847
		5 ; 1	11 ; 5 ; 14	31 ; 19.19	153,487	
		1 ; 3	13 ; 3 ; 14	79 ; 313	43.5197	

27. Trin-Aurifeuillian Factors.—Let N_1, N_2 be two numbers whose product $\mathfrak{N}$ is a Trin-Aurifeuillian. The conditions that N_1, N_2 should be the Aurifeuillian Factors (L, M) of N are

$$N_1 = L = x^2 - 3xy + 3y^2, \quad N_2 = M = x^2 + 3xy + 3y^2. \quad (88)$$

Hence $x^2 + 3y^2 = \frac{1}{2} (N_2 + N_1), \quad 3xy = \frac{1}{2} (N_2 - N_1),$

whence $x^2 - 3y^2 = \frac{1}{6} \{36N_1N_2 - 3(N_2 - N_1)^2\}^{\frac{1}{2}}.$

Hence *three* conditions must obtain that N_1, N_2 may be the Aurifeuillian factors of N , viz.,

$$\{36N_1N_2 - 3(N_2 - N_1)^2\} = \{6(x^2 - 3y^2)^2\} = (6Z)^2, \text{ a square,} \quad (88a)$$

$$(N_1 + N_2) \pm 2Z = (2v)^2, \text{ an even square,} \quad (88b)$$

$$\frac{1}{6} \{(N_1 + N_2) \mp 2Z\} = (2y)^2, \text{ an even square.} \quad (88c)$$

When N_1, N_2 are given numbers, (satisfying the above conditions), the values of x, y , the Aurifeuillian parts of N , and therewith also the quadratic partitions ($A^2 + 3B^2$) of L, M, N , are hereby given directly.

28. Conformal Trin-Aurifeuillian Products.—The question now arises whether the *product* of two or more Trin-Aurifeuillians can be a Trin-Aurifeuillian; and, conversely, whether a Trin-Aurifeuillian can be resolved into the product of two or more Trin-Aurifeuillians.

Let

$$N = x^4 - 3x^2y^2 + 9y^4, \quad N_1 = x_1^4 - 3x_1^2y_1^2 + 9y_1^4, \quad N_2 = x_2^4 - 3x_2^2y_2^2 + 9y_2^4, \quad (89)$$

and, if possible, let $N = N_1 N_2$, (where $N_1 < N_2$). (90)

The *general* solution of (90) seems difficult, but a somewhat limited solution may be obtained by first equating the Aurifeuillian Factors (L, M) of N separately to N_1, N_2 , thus obtaining the *two* equations

$$L = N_1, \quad M = N_2. \quad (90a)$$

Let $L, M; N_1, N_2$ be now expressed in the quadratic form ($A^2 + 3B^2$); herein the forms of L, M are each *trimorph*, whilst N_1, N_2 are each *tetramorph*, (Art. 21). Observing that a corresponding pair (L, M), (L', M'), (L'', M''), (L''', M''') must necessarily enter together into (90), and, therefore, into any one form of Equation (90a), whilst the forms of N_1, N_2 are *independent*, the pair of equations (90a) may be expressed in 48 ways, which are summarized below.

$$L' = \text{any of } N_1^{\circ}, N_1', N_1'', N_1'''; \quad M' = \text{any of } N_2^{\circ}, N_2', N_2'', N_2''', (16 \text{ pairs}).$$

$$L'' = \text{any of } N_1^{\circ}, N_1', N_1'', N_1'''; \quad M'' = \text{any of } N_2^{\circ}, N_2', N_2'', N_2''', (16 \text{ pairs}).$$

$$L''' = \text{any of } N_1^{\circ}, N_1', N_1'', N_1'''; \quad M''' = \text{any of } N_2^{\circ}, N_2', N_2'', N_2''', (16 \text{ pairs}).$$

By now equating the parts (A, B) of any corresponding pair (L', M') , (L'', M'') , (L''', M''') separately to the parts (A, B) of each of the four forms of N_1, N_2 respectively, each of the 48 systems above yields *one* group of *four* equations between the six quantities x, y, x_1, y_1, x_2, y_2 . Further, as each of the parts A, B may have either a $\pm$ sign, each of the above 48 groups may be formed in *four* ways.

The four equations in each of the 192 sets are *linear* in x, y , and *quadratic* in x_1, y_1, x_2, y_2 . Hence x, y can always be *linearly* eliminated, thus leaving *two* equations *quadratic* in x_1, y_1, x_2, y_2 . From these the value of either pair $(x_1, y_1), (x_2, y_2)$ can be solved (by quadratics) in terms of the other pair in a form involving (in the general case) *three* radicals. Thus *three* conditions must in general be satisfied (viz., that the expressions under the three radicals should be perfect squares) in order that this form of solution may be *possible*.

Thus, whenever either N_1 or N_2 is *given*, and also the Aurifeuillian parts (x_1, y_1) or (x_2, y_2) thereof, these criteria can be *directly* applied; and, if satisfied, the other pair (x_2, y_2) or (x_1, y_1) and thence also N_2 or N_1 , and, lastly, x, y , and N , can be *directly* calculated, so that this problem is *direct*, and the solution is *unique*.

The solution of the problem *ab initio* is, however, much more difficult.

In a good many of the 192 sets of equations above specified it is found, on attempting the solution, that some impossibility* is involved. The following are the only cases which have been found completely solvable:—

$$\left. \begin{array}{l} \text{i. } L' = N'_1, \quad M' = N'_2, \quad (y, y_1, y_2 \text{ all even}) \\ \text{ii. } L' = N''_1, \quad M' = N''_2, \quad (y, x_1, x_2 \text{ all even}) \\ \text{iii. } L''' = N'_1, \quad M''' = N'_2, \quad (y_1 \text{ even, the rest odd}) \\ \text{iv. } L''' = N''_1, \quad M''' = N''_2, \quad (y_2 \text{ even, the rest odd}) \\ \text{v. } L''' = N'_1, \quad M''' = N'_2, \quad (x_1 \text{ even, the rest odd}) \\ \text{vi. } L''' = N''_1, \quad M''' = N''_2, \quad (x_2 \text{ even, the rest odd}) \end{array} \right\}. \quad (91)$$

28a. The detailed solutions are—

System i. $(y, y_1, y_2 \text{ all even})$.—The equations are

$$\begin{aligned} (x - \tfrac{3}{2}y)^2 + 3 \left(\tfrac{y}{2} \right)^2 &= L' = N'_1 = (x_1^2 - \tfrac{3}{2}y_1^2)^2 + 3 \left(\tfrac{3}{2}y_1^2 \right)^2, \\ (x + \tfrac{3}{2}y)^2 + 3 \left(\tfrac{y}{2} \right)^2 &= M' = N'_2 = (x_2^2 - \tfrac{3}{2}y_2^2)^2 + 3 \left(\tfrac{3}{2}y_2^2 \right)^2. \end{aligned}$$

* It seems unnecessary to give the detail.

One of the possible cases is given by equating the parts $(\pm A, \pm B)$ as follows :—

$$x - \frac{3}{2}y = x_1^2 - \frac{3}{2}y_1^2, \quad x + \frac{3}{2}y = x_2^2 - \frac{3}{2}y_2^2, \quad y = 3y_1^2 = 3y_2^2,$$

giving
$$x = \frac{1}{3}(2x_1^2 + x_2^2) \quad \left. \vphantom{x = \frac{1}{3}(2x_1^2 + x_2^2)} \right\} \quad (92, i)$$

with the *two* conditions $x_2^2 - x_1^2 = (3y_1)^2 = (3y_2)^2 \quad \left. \vphantom{x_2^2 - x_1^2 = (3y_1)^2 = (3y_2)^2} \right\}$.

By varying the signs of the parts $(\pm A), (\pm B)$ equated, another possible case is found; but the results differ from the above only in the interchange of x_1, x_2 , and change of sign of x .

System ii. (y, x_1, x_2 all even).—The equations are

$$(x - \frac{3}{2}y)^2 + 3 \left(\frac{y}{2} \right)^2 = L' = N_1'' = \left(\frac{x_1^2}{2} - 3y_1^2 \right)^2 + 3 \left(\frac{x_2^2}{2} \right)^2,$$

$$(x + \frac{3}{2}y)^2 + 3 \left(\frac{y}{2} \right)^2 = M' = N_2'' = \left(\frac{x_2^2}{2} - 3y_2^2 \right)^2 + 3 \left(\frac{x_1^2}{2} \right)^2.$$

One of the possible cases is given by equating the parts $(\pm A), (\pm B)$ as follows :—

$$x - \frac{3}{2}y = \frac{1}{2}x_1^2 - 3y_1^2, \quad x + \frac{3}{2}y = \frac{1}{2}x_2^2 - 3y_2^2, \quad y = x_1^2 = x_2^2,$$

giving
$$x = -(y_1^2 + 2y_2^2) \quad \left. \vphantom{x = -(y_1^2 + 2y_2^2)} \right\} \quad (92, ii)$$

with the *two* conditions $y_2^2 - y_1^2 = x_1^2 = x_2^2 \quad \left. \vphantom{y_2^2 - y_1^2 = x_1^2 = x_2^2} \right\}$.

By varying the signs of the parts $(\pm A), (\pm B)$ equated, another possible case is found; but the results differ from the above only in the interchange of y_1, y_2 , and change of sign of x .

System iii. (y_1 even, the rest odd).—The equations are

$$\left(\frac{x-3y}{2} \right)^2 + 3 \left(\frac{x-y}{2} \right)^2 = L''' = N_1' = (x_1^2 - \frac{3}{2}y_1^2)^2 + 3 \left(\frac{3}{2}y_1^2 \right)^2,$$

$$\left(\frac{x+3y}{2} \right)^2 + 3 \left(\frac{x+y}{2} \right)^2 = M''' = N_2''' = \left(\frac{x_2^2 + 3y_2^2}{2} \right)^2 + 3 \left(\frac{x_2^2 - 3y_2^2}{2} \right)^2.$$

One of the possible cases is given by equating the parts $(\pm A, \pm B)$ as follows :—

$$-(x-3y) = 2x_1^2 - 3y_1^2, \quad x-y = 3y_1^2,$$

$$x+3y = x_2^2 + 3y_2^2, \quad -(x+y) = x_2^2 - 3y_2^2,$$

giving
$$x = y_2^2 + 2y_1^2, \quad y = x_1^2 = x_2^2 \quad \left. \vphantom{x = y_2^2 + 2y_1^2, \quad y = x_1^2 = x_2^2} \right\} \quad (92, iii)$$

with *two* conditions $y_2^2 - y_1^2 = x_1^2 = x_2^2 \quad \left. \vphantom{y_2^2 - y_1^2 = x_1^2 = x_2^2} \right\}$.

System iv.—The results differ from *System iii* only in the interchange of (x_1, y_1) with (x_2, y_2) , and change of sign of x .

System v. (x_1 even, the rest odd).—The equations are

$$\left(\frac{x-3y}{2}\right)^2 + 3\left(\frac{x-y}{2}\right)^2 = L''' = N_1'' = \left(\frac{1}{2}x_1^2 - 3y_1^2\right) + 3\left(\frac{1}{2}x_1^2\right)^2,$$

$$\left(\frac{x+3y}{2}\right)^2 + 3\left(\frac{x+y}{2}\right)^2 = M''' = N_2'' = \left(\frac{x_2^2 + 3y_2^2}{2}\right)^2 + 3\left(\frac{x_2^2 - 3y_2^2}{2}\right)^2.$$

One of the possible cases is given by equating the parts $(\pm A, \pm B)$ as follows:—

$$x - 3y = x_1^2 - 6y_1^2, \quad x - y = x_1^2,$$

$$x + 3y = x_2^2 + 3y_2^2, \quad x + y = x_2^2 - 3y_2^2,$$

$$\left. \begin{array}{l} \text{giving} \quad x = \frac{1}{3}(2x_1^2 + x_2^2), \quad y = 3y_1^2 = 3y_2^2 \\ \text{with two conditions} \quad x_2^2 - x_1^2 = (3y_1)^2 = (3y_2)^2 \end{array} \right\} \quad (92, v)$$

System vi.—The results differ from *System v* only in the interchange of (x_1, y_1) with (x_2, y_2) , and change of sign of y .

The two conditions in all these systems are very similar, viz.,

One of type $y_1 = y_2$ (in i, v, vi); or $x_1 = x_2$ (in ii, iii, iv).

One of type $x_2^2 \sim x_1^2 = (3y_1)^2$, (in i, v, vi); or $y_2^2 \sim y_1^2 = x_1^2$ (in ii, iii, iv).

The latter condition is equivalent to $c^2 = a^2 + b^2$. There are two general solutions of this Pythagorean equation in terms of two arbitrary quantities (m, n) (cf. Art. 13a); but one of the general solutions in each system is found (on actual trial) to violate some of the initial (arithmetical)* conditions as to *oddness* or *evenness* of some of the quantities x_1, y_1, x_2, y_2 , leaving effectively *only one* general solution of each system; and also involving effectively

$$m, n \text{ mutual primes, one odd, one even (in each system).} \quad (93)$$

The general solutions of i, ii, iii, v are

$$\left. \begin{array}{l} \text{i. } x_2 = m^2 + n^2, \quad x_1 = m^2 \sim n^2, \quad y_1 = y_2 = \frac{2}{3}mn \\ \quad x = m^4 - \frac{2}{3}m^2n^2 + n^4 = (y + x_1^2), \quad y = \frac{4}{3}m^2n^2 \\ [m, n \text{ mutual primes, one odd, one even; and one a multiple of 3;} \\ \quad \text{giving } x, x_1, x_2 \text{ all odd; } y, y_1, y_2 \text{ all even}] \end{array} \right\} \quad (94, i)$$

* If these *arithmetical* restrictions be removed, the general solutions here omitted of Systems i, ii, iii, v will be found to merge into the general solutions given of Systems v, iii, ii, i respectively.

$$\begin{aligned}
 &\text{ii. } y_1 = m^2 + n^2, y_2 = m^2 \sim n^2, x_1 = x_2 = 2mn \\
 &\quad x = -(3m^4 - 2m^2n^2 + 3n^4) = -(y + 3y_2^2), y = (2mn)^2 \left. \vphantom{\begin{matrix} y_1 = m^2 + n^2, \\ y_2 = m^2 \sim n^2, \\ x_1 = x_2 = 2mn \end{matrix}} \right\} \quad (94, \text{ii}) \\
 &\quad [m, n \text{ mutual primes, one odd, one even; giving } x, y_1, y_2 \text{ odd;} \\
 &\quad \quad \quad y, x_1, x_2 \text{ even}] \\
 \\
 &\text{iii. } y_2 = m^2 + n^2, y_1 = 2mn, x_1 = x_2 = m^2 \sim n^2 \\
 &\quad x = m^4 + 10m^2n^2 + n^4 = (y + 3y_1^2), y = (m^2 \sim n^2)^2 \left. \vphantom{\begin{matrix} y_2 = m^2 + n^2, \\ y_1 = 2mn, \\ x_1 = x_2 = m^2 \sim n^2 \end{matrix}} \right\} \quad (94, \text{iii}) \\
 &\quad [m, n \text{ mutual primes, one odd, one even, and one a multiple* of 3;} \\
 &\quad \quad \quad \text{giving } x, y, x_1, x_2, y_2 \text{ odd; } y_1 \text{ even}] \\
 \\
 &\text{v. } x_2 = m^2 + n^2, x_1 = 2mn, y_1 = y_2 = \frac{1}{3} (m^2 \sim n^2) \\
 &\quad x = \frac{1}{3} (m^4 + 10m^2n^2 + n^4) = y + x_1^2, y = \frac{1}{3} (m^2 \sim n^2)^2 \left. \vphantom{\begin{matrix} x_2 = m^2 + n^2, \\ x_1 = 2mn, \\ y_1 = y_2 = \frac{1}{3} (m^2 \sim n^2) \end{matrix}} \right\} \quad (94, \text{v}) \\
 &\quad [m, n \text{ mutual primes, one odd, one even; such that one of } (m \mp n) \\
 &\quad \quad \quad \text{is a multiple of 3, giving } x, y, y_1, x_2, y_2 \text{ odd; } x_1 \text{ even}]
 \end{aligned}$$

Examples.—The following Table gives the elements and results of the *lowest* of such numbers $N_1 N_2 = N$ in each of the solvable Cases i, ii, iii, v. The numbers increase so very rapidly that it is not worth continuing. These high numbers (N) would be difficult to factorise otherwise.

Case.	Data.	1st Aurifeuillian.		2nd Aurifeuillian.		Product-Aurifeuillian.	
	$m; n$	$x_1; y_1$	N_1	$x_2; y_2$	N_2	$x; y$	N
i	3; 2	5; 4	1729	13; 4	22,753	73; 48	39,339,937
	3; 4	7; 8	29857	25; 8	307,489	241; 192	9,180,699,073
ii	1; 2	4; 5	4681	4; 3	553	— 43; 16	2,588,593
	3; 2	12; 13	204,777	12; 5	15,561	— 219; 144	3,186,534,897
	1; 4	8; 17	700,297	8; 15	416,521	— 739; 64	
iii	3; 2	5; 12	176,449	5; 13	244,999	457; 25	43,229,828,551
iv	1; 2	4; 1	217	5; 1	559	19; 3	121,303
	1; 4	8; 5	4921	17; 5	67,471	139; 75	332,024,791

29. *Conformal Continued Trin-Aurifeuillian Products.*—The conditions for these are so much more complex than those for the similar problem in Bin-Aurifeuillians that it has not been found possible to extend the above solutions to the product of three Trin-Aurifeuillians.

* Otherwise y, N_2 , and N will be multiples of 9.

30. *Quotient Trin-Aurifeuillians*.—Writing 3^ω , x^ω , y^ω for 3 , x , y (where $\omega \neq 3\kappa$) in the expression (65), it becomes

$$\mathfrak{Y} = \frac{x^{6\omega} + 3^{3\omega}y^{6\omega}}{x^{2\omega} + 3^\omega y^{2\omega}} = x^{4\omega} - 3^\omega x^{2\omega}y^{2\omega} + 3^{2\omega}y^{4\omega} \quad (95)$$

$$= X^4 - 3X^2Y^2 + 3^2Y^4, \text{ where } X = x^\omega, Y = 3^{4(\omega-1)}y^\omega. \quad (95a)$$

Hence $\mathfrak{Y}$ is a Trin-Aurifeuillian; and, writing

$$Q = \frac{\mathfrak{Y}}{N} = \frac{x^{6\omega} + 3^{3\omega}y^{6\omega}}{x^{2\omega} + 3^\omega y^{2\omega}} \cdot \frac{x^2 + 3y^2}{x^6 + 3^3y^6}, \quad (96)$$

it is seen that $\mathfrak{Y}$ contains N , and that Q is the A.P.F. of $(x^{6\omega} + 3^{3\omega}y^{6\omega})$ when ω is prime.

Hence Q is a Quotient Trin-Aurifeuillian, resolvable into factors S , T ; and Q , S , T are expressible in quadratic forms conformal with those of $\mathfrak{Y}$, $\mathfrak{X}$, $\mathfrak{H}$, as in Art. 7-7b.

As the chief interest of these numbers (Q) is for small values of ω , ($= 5, 7$, &c.), it is better to pass on at once to these simple cases, (as general formulæ would be very lengthy).

30a. *Quintic Trin-Aurifeuillian*.—Writing $\omega = 5$ in the formula (95),

$$\mathfrak{Y} = \frac{x^{30} + 3^{15}y^{30}}{x^{10} + 3^5y^{10}} = x^{20} - 3^5x^{10}y^{10} + 3^{10}y^{20}. \quad (97)$$

Here $Q = \mathfrak{Y} \div N$ can have each of the partitions $(A^2 + 3B^2)$, $(u^2 - 5v^2)$, $(V^2 + 15W^2)$, denoted by Q_3 , Q_6 , Q_{15} , (accented as required, Art. 5, 21),

$$Q = \frac{\mathfrak{Y}}{N} = \frac{x^{30} + 3^{15}y^{30}}{x^{10} + 3^5y^{10}} \cdot \frac{x^2 + 3y^2}{x^6 + 3^3y^6} = \frac{x^{20} - 3^5x^{10}y^{10} + 3^{10}y^{20}}{x^4 - 3x^2y^2 + 3^2y^4}, \quad (98)$$

$$= x^{16} + 3x^{14}y^2 - 3^3x^{10}y^6 - 3^4x^8y^8 - 3^5x^6y^{10} + 3^7x^2y^{14} + 3^8y^{16}, \quad (98a)$$

$$Q_3^2 = (x^8 - 3^2x^4y^4 + 3^4y^8)^2 + 3(xy)^2(x^6 + 3x^4y^2 - 3^2x^2y^4 - 3^3y^6)^2, \quad (98b)$$

$$Q_6^2 = (x^8 + 8 \cdot 3x^6y^2 + 13 \cdot 3^2x^4y^4 + 8 \cdot 3^3x^2y^6 + 3^4y^8)^2 - 5(3xy)^2(x^2 + 3y^2)^6, \quad (98c)$$

$$Q_{15}^2 = (x^8 - 2 \cdot 3x^6y^2 + 3 \cdot 3^2x^4y^4 - 2 \cdot 3^3x^2y^6 + 3^4y^8)^2 + 15(xy)^2(x^6 - 3x^4y^2 + 3^2x^2y^4 - 3^3y^6)^2, \quad (98d)$$

$$S = \mathfrak{L} + M = (x^{10} - 3^3x^5y^5 + 3^5y^{10}) \div (x^2 + 3xy + 3y^2) \quad (99)$$

$$= x^8 - 3x^7y + 6x^6y^2 - 3^2x^5y^3 + 3^2x^4y^4 - 3^3x^3y^5 + 2 \cdot 3^3x^2y^6 - 3^4xy^7 + 3^4y^8. \quad (99a)$$

Here S has three alternative forms $(A^2 + 3B^2)$ denoted by S'_3 , S''_3 , S'''_3 ; and two alternative forms of $(u^2 - 5v^2)$ * denoted by S'_5 , S''_5 ; and one form (S_{15}) of $(V^2 + 15W^2)$.

$$S'_3 = (x^4 - \frac{3}{2}x^3y + \frac{3}{2}x^2y^2 - \frac{9}{2}y^4)^2 + 3 \left(\frac{y}{2} \right)^2 (x^3 - 3x^2y + 6xy^2 - 9y^3)^2, \quad (y \text{ even}), \quad (99b)$$

$$S''_3 = \left\{ \frac{1}{2}x(x^3 - 3xy^2 + 9y^3) - 9y^4 \right\}^2 + 3 \left(\frac{x}{2} \right)^2 (x^3 - 2x^2y + 3xy^2 - 3y^3)^2, \quad (x \text{ even}), \quad (99b)$$

$$S'''_3 = \frac{1}{4}(x^4 - 3x^3y + 6x^2y^2 - 9xy^3 + 9y^4)^2 + 3 \cdot \frac{1}{4}(x^4 - x^3y + 3xy^3 - 9y^4)^2, \quad (x, y \text{ both odd}), \quad (99b)$$

$$S'_5 = (x^4 - \frac{3}{2}x^3y + \frac{15}{2}x^2y^2 - \frac{9}{2}xy^3 + 9y^4)^2 - 5 \left(\frac{3xy}{2} \right)^2 (x^2 - xy + 3y^2)^2, \quad (x \text{ or } y \text{ even}), \quad (99c)$$

$$S''_5 = (\frac{3}{2}x^4 - 6x^3y + 15x^2y^2 - 18xy^3 + \frac{9}{2}y^4)^2 - 5 \left(\frac{1}{2}x^4 - 3x^3y + 6x^2y^2 - 9xy^3 + \frac{9}{2}y^4 \right)^2, \quad (x, y \text{ both odd}), \quad (99c)$$

$$S_{15} = \left\{ x^4 - \frac{3}{2}xy(x^2 + 3y^2) + 9y^4 \right\}^2 + 15 \cdot \frac{1}{4}(xy)^2(x^2 - 3y^2)^2. \quad (99d)$$

The values of $T = \mathfrak{H} \div L$, and of its quadratic partitions T'_3 , T''_3 , T'''_3 ; T'_5 , T''_5 ; T_{15} may be derived from those of S by writing $-y$ for y , (Art. 3).

* See footnote * to Art. 16b.

The quadratic partitions of Q , viz., Q_3^2 , Q_3' , Q_3'' , Q_3''' ; Q_6^2 , Q_6' , Q_6'' ; Q_{15}^2 , Q_{15}' may be obtained by conformal multiplication of the similarly accented quadratic forms of S , T ; each such product yielding two forms of Q , viz., Q^2 and one of Q' , Q'' , Q''' .

31. Fermat's Numbers. (Base 3).—These are numbers of form $\mathfrak{F}_n = 3^n + 1$. Writing $x = 1$, $y = 3^\nu$ (ν any integer), the Trin-Aurifeuillian N (Art. 21) becomes

$$N = (1 + 3^3 \cdot 3^{6\nu}) \div (1 + 3 \cdot 3^{2\nu}) = (1 + 3^{6\nu+3}) \div (1 + 3^{2\nu+1}). \quad (100)$$

Hence the A.P.F. of all Fermat's Numbers $\mathfrak{F}_n = 3^n + 1$, where $n = 6\nu + 3$, is resolvable into the two factors L , M of Art. 21, viz.,

$$L = 3^{2\nu+1} - 3^{\nu+1} + 1, \quad M = 3^{2\nu+1} + 3^{\nu+1} + 1, \quad (100a)$$

and these (L , M) can be at once *completely* resolved into their *prime* factors as far as $n = 39$.* Also the A.P.F. of $\mathfrak{F}_{45} = (3^{45} + 1)$ is *completely* resolvable by the formulæ of Art. 30a. Beyond this limit the factorisation into *prime* factors requires the aid of "special† divisors." When $\frac{1}{3}n = 2\nu + 1$ is *composite*, the *Reduced A.P.F.* of $\mathfrak{F}_n$ is similarly resolvable into two factors (S , T) by Art. 7b.

Examples.—The following Table shows the *complete* resolution‡ of the *Reduced A.P.F.* of these numbers into their *prime* factors, all but complete§ up to $n = 105$: the factorisation of the Aurifeuillian Factors L , M or S , T being shown separately: prime factors only are shown; the semicolon (;) denotes that the factorisation is *complete*.

[Note that the highest number $(3^{105} + 1)$, here *completely* factorised, contains 51 figures.]

* At this limit the *larger* Aurifeuillian Factor M is > 9 million, the present limit of the large Factor-Tables.

† Reuschle's *Neue zahlentheoretische Tabellen*, above quoted, gives (pp. 42-46) a table of "special prime divisors" (p) of $(3^n - 1)$ up to $p < 1000$.

‡ The whole of this Table has been worked out by the author for this paper, though much of it had been previously obtained by others. The initials B , C , L (in the columns of L or S , M or T) indicate the *original* computer (so far as known to the author) who first succeeded in *completely* factorising the Aurifeuillian Factors into their *prime* factors, viz.,

B denotes Mr. C. B. Bickmore;

L denotes M. Le Lasseur;

C denotes Lt.-Col. A. Cunningham, (the present author).

(1) Mr. Bickmore gives the results when $n = 3, 9, 15, 21, 27 : 87, 105$ in his first paper quoted, pp. 8, 43; (except that, when $n = 105$, S is not factorised). The remaining results marked B in the Table were communicated to the author by Mr. Bickmore in MS.: besides this, Mr. Bickmore has considerably aided in, and to some extent verified, the author's results marked C .

(2) The results when $n = 87$ are ascribed to M. Le Lasseur in Ed. Lucas's *Théorèmes d'Arithmétique*, Turin, 1878, p. 12.

(3) Most of the factorisations and detection of high primes done by the author (C) were effected by the method developed in Mr. Lawrence's Paper (see footnote § to Art. 18).

§ Complete except that the factorisation of the smaller Aurifeuillian Factor L , when $n = 31$, is undetermined.

FACTORS OF THE REDUCED A.P.F. OF $(3^n + 1)$. [$n = 6\nu + 3$].

n	L or S	M or T
3	1 ;	7 ;
9	19 ;	37 ;
15	31 ;	271 ;
21	7 · 43 ;	2269 ;
27	19,441 ;	19927 ;
33	176,419 ;	25411 ;
39	79 · 2887 ;	157 · 1041 ;
45	387,631 ;	755,551 ;
51	613 · 30,091 ;	129,159,847 ;
57	3079 · 53,923 ;	1,162,320,517 ;
63	127 · 883 · 2521 ;	550,554,229 ;
69	139 · 5107 · 132619 ;	9109 · 1,476,463 ;
75	26,251 · 119,101 ;	2551 · 1,530,601 ;
81	163 · 208,657 · 224,209 ;	1297 · 5,879,415,781 ;
87	261,697 · 37,464,463 ;	349 · 142,159 · 1,383,301 ;
93		373 · 541,447 · 3,058,399 ;
99	199 · 4357 · 337,448,233 ;	397 · 378,450,588,583 ;
105	24151 · 3,369,031 ;	211 · 1051 · 3,454,081 ;
111	223 ·	
117	937 ·	1171 ·

Note.—To complete the factorisation of these numbers

$\mathfrak{P}_n = (3^n + 1)$, the factorisation of the algebraic factors of form $F = 3^\omega + 1$, where $\omega \nmid 31$, is required. This factorisation, when $\omega < 19$, is given in Mr. Bickmore's first paper, above quoted, p. 43. The remaining results, still required, are*

$\frac{1}{4}(3^{10} + 1) = 2851 \cdot 101,917$;
 $\frac{1}{4}(3^{23} + 1) = 23,635,794,707$;
 $\frac{1}{4}(3^{31} + 1) = 6883 \cdot 22,434,744,889$.

32. *Fermat's Numbers.* (Higher Bases).—Writing in Formulæ (65) $X = 1$, $Y = 3y^2$, $n = 3\omega = 6\nu + 3$, (ν being any integer), then

$$\mathfrak{Y} = \frac{1 + Y^{3\omega}}{1 + Y^\omega} = \frac{1 + (3y^2)^{6\nu+3}}{1 + (3y^2)^{2\nu+1}} = \frac{1 + 3^3 \cdot \eta^6}{1 + 3 \cdot \eta^2}, \quad \text{where } \eta = 3^\nu \cdot y^{2\nu+1}. \quad (101)$$

Here $\mathfrak{Y}$ is a *Trin-Aurifeuillian*, so that every such number, and also its A.P.F., viz.,

$\mathfrak{Y} \div N = \{(1 + Y^{3\omega})(1 + Y)\} \div \{(1 + Y^\omega)(1 + Y^3)\}$, where $Y = 3y^2$; and also its "Reduced A.P.F.," (when further reducible), admit of resolution into the two factors L , M , or S , T of Art. 7a, b, 30.

The existing Factor-Tables (extending to 9 million) enable the complete factorisation (i.e., into prime factors) of such numbers to be effected up to the following limits:—

$$\begin{aligned} 3\omega &= 3, \quad 9, \quad 15, \quad 21, \quad 27 \text{ to } 45 \\ Y = 3y^2 &= 3 \cdot 1731^2, \quad 3 \cdot 8^2, \quad 3 \cdot 4^2, \quad 3 \cdot 2^2, \quad 3 \cdot 1^2 \\ &= 8,989,083, \quad 192, \quad 48, \quad 12, \quad 3 \end{aligned}$$

and, by the aid of "special divisors," ‡§ it can be carried far beyond these limits.

Examples.—The following table shows the resolution|| into prime factors of

* The first result is due to Mr. Bickmore; the other two are due to the author.

† Reference note (†) at foot of p. 430.

‡ Reuschle's *Tafeln komplexer Primzahlen* give (p. 193 et seq.) the least values of Y for which $(1 + Y^{3\omega})$ is divisible by any given prime $p < 1000$.

§ See footnote † to Art. 18 about the author's MS. Tables of these.

|| The whole of this Table has been worked out by the author for this paper. The initials u, c indicate the original computer, as in footnote † to Art. 31, q.v.

$N = \text{A.P.F. of } (1 + Y^n)$, where $Y = 3y^2$ and $n = 3\omega$ to various high limits, for the index $n = 3\omega = 3\uparrow, 9, 15, 21, 33$.

The factorisation of the Aurifeuillian Factors L, M or S, T is shown separately, and *prime* factors only are shown: the semicolon (;) where $n = 9, 15, 21, 27, 33$ indicates that the factorisation is *complete*.

FACTORISATION OF THE A.P.F. OF $(1 + Y^{3\omega})$, [where $Y = 3y^2$].

y	$Y = 3y^2$	$(1 + Y^3) \div (1 + Y)$		$(1 + Y^9) \div (1 + Y^3)$		$\{(1 + Y^{15})(1 + Y)\} \div \{(1 + Y^5)(1 + Y^3)\}$	
		L	M	L	M	$L \text{ or } S$	$M \text{ or } T$
1	3	1	7	19;	37;	31;	271;
2	12	7	19	1657;	1801;	31.421;	35,671;
3	27	19	37	19,241;	19,927;	387,631;	755,551;
4	48	37	61	110,017;	19.5851;	31.134.731;	6,887,341;
5	75	61	7.13	127.3313;	423.001;	331.78,781;	31.151.8311;
6	108	127	127	181.6949;	19.66403;	115,693,471;	161,466,931;
7	147	127	13.13	19.167,023;	3.179,611;	151.241.11,161;	12,511.43,201;
8	192	13.13	7.31	7,073,281;	19.372,763;	1,202,388,841;	211.7,317,091;
9	243	7.13	271	37.387,631;	19.755,551;	26,251.119,101;	2551.1,530,601;
10	300	271	331	19.487.2917;	37.37.109.181;	151.4231.11,491;	31.541.534,661;
11	363	331	397	19.37.68,023;	73.655,399;	31.512,130,481;	241.79,011,151;
12	432	397	7.67	37.2,178,541;	19.4,244,059;	31.61.32,386,261;	61.151.181,211,5281;
13	507	7.67	547	19.6,858,109;	73.109.16,381;	61.5821.313,711;	151.9721.87,541;
14	588	547	631	109.1,865,341;	109.1,865,341;	61.751.4,242,421;	31.
15	675	631	7.103	73.103.25,849;	937.483,481;	31.61.172,938,811;	
16	768	7.103	19.43	37.		2221.12,841.18,691;	
17	867	19.43	919	127.577.8893;		541.1561,577,671;	
18	972	919	13.79				

† Reference note (†) at foot of p. 430.

[Note that the highest number $(1 + Y^{15})$, here *completely* factorised, contains 45 figures: the factorisation of $(1 + Y^5)$ is required to render these really complete; it is hoped to publish these hereafter.]

FACTORISATION OF THE A.P.F. OF $(1 + Y^{30})$, [where $Y = 3y^2$].—Continued.

y	$Y = 3y^2$	$\{(1 + Y^{21})(1 + Y)\} \div \{(1 + Y^7)(1 + Y^3)\}$	
		L or S	M or T
1	3	7 · 43 ;	2269 ;
2	12	1,885,339 ; B	7 · 43 · 17,011 ; B
3	27	127 · 883 · 2521 ; B	550,554,229 ; C
4	48	43 ·	
5	75	883 ·	
6	108		
7	147	43 ·	757 ·
8	192	421 ·	7 · 43 · 211 · 1051 · 3,454,081 ; D
9	243	2269 · 24,151 · 3,369,031 ; C	463 ·
10	300		

y	$Y = 3y^2$	$(1 + Y^{27}) + (1 + Y^9)$	
		L	M
1	3	19,441 ; B	19,927 ; B
2	12	109 · 47,336,293 ; B	271 · 487 · 39,037 ; B
3	27	163 · 208,657 · 224,209 ; C	1297 · 5,879,415,781 ; C

y	$Y = 3y^2$	$\{(1 + Y^{33})(1 + Y)\} \div \{(1 + Y^{11})(1 + Y^3)\}$	
		L or S	M or T
1	3	176,419 ; B	25,411 ; B
2	12	67 · 199 · 1453 · 5479 ; B	
3	27	199 · 4357 · 337,448,233 ; C	397 · 378,450,588,583 ; C

† Refer to †, pp. 428, 429. The highest number of this kind, $N = (1 + Y^3) \div (1 + Y)$, factorisable by the existing Factor Tables is

$$(1 + 8,989,083^3) \div (1 + 8,989,083) = (7 \cdot 73 \cdot 17581) \cdot (19 \cdot 473,383).$$

It will be noticed that these numbers form a *Trin-Aurifeuillian Chain* (Art. 34), i.e., the M of each number = the L of the next higher number.

33. Common Aurifeuillian Factor of Trin-Aurifeuillians.—It will now be shown that every number F of form (A^2+3B^2) is an Aurifeuillian Factor of six different Trin-Aurifeuillians.

Take N_r as the type of six different Trin-Aurifeuillians, such that

$$N_r = (x_r^2 + 3y_r^2) \div (x^2 + 3y^2) = L_r M_r, \quad [r = 1, 2, 3, 4, 5, 6]. \quad (102)$$

Then, taking $N_1 < N_2$, $N_3 < N_4$, $N_5 < N_6$, and $L_r < M_r$ in general, F will be a common factor of all the six N s, when

$$F = M_1 = L_2 = M_3 = L_4 = M_5 = L_6. \quad (103)$$

Next expressing the three L s and three M s by pairs in the three quadratic forms (A^2+3B^2) of Art. 21,

$$F = A^2 + 3B^2 = M'_1 = L'_2 = M''_3 = L''_4 = M'''_5 = L'''_6. \quad (103a)$$

Equating each quadratic part (A_r, B_r) of the L s and M s separately to the parts (A, B) of F in turn, these give the relations

$$\left. \begin{aligned} A &= x_1 + \frac{3}{2}y_1 = x_3 \sim \frac{3}{2}y_3 = \frac{1}{2}x_3 = \frac{1}{2}x_4 = \frac{1}{2}(x_5 + 3y_5) = \frac{1}{2}(x_6 \sim 3y_6) \\ B &= \frac{1}{2}y_1 = \frac{1}{2}y_2 = \frac{1}{2}x_3 + y_3 = \frac{1}{2}x_4 \sim y_4 = \frac{1}{2}(x_5 + y_5) = \frac{1}{2}(x_6 \sim y_6) \end{aligned} \right\} \quad (104)$$

whence, finally,

$$\left. \begin{aligned} x_1 &= A - 3B, & y_1 &= 2B; \\ x_2 &= A + 3B, & y_2 &= 2B; \\ x_3 &= 2A, & y_3 &= B \sim A; \\ x_4 &= 2A, & y_4 &= B + A; \\ x_5 &= 3B \sim A, & y_5 &= A \sim B; \\ x_6 &= 3B + A, & y_6 &= A + B; \\ L'_1 &= (A \sim 6B)^2 + 3B^2, & M'_1 &= A^2 + 3B^2 \\ L'_2 &= A^2 + 3B^2, & M'_2 &= (A + 6B)^2 + 3B^2 \\ L'_3 &= A^2 + 3(2A \sim B)^2, & M''_3 &= A^2 + 3B^2 \\ L'_4 &= A^2 + 3B^2, & M''_4 &= A^2 + 3(2A + B)^2 \\ L'_5 &= (3B \sim 2A)^2 + 3(2B \sim A)^2, & M'''_5 &= A^2 + 3B^2 \\ L'_6 &= A^2 + 3B^2, & M'''_6 &= (3B + 2A)^2 + 3(A + 2B)^2 \end{aligned} \right\} \quad (105)$$

Hence many simple relations may be found between such a set of

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six Trin-Aurifeuillians, e.g.,

$$\left. \begin{aligned} N_1 + N_4 &= N_2 + N_3 = N_5 + N_6 = 14(A^2 + 3B^2) = 14F^2 \\ N_2 - N_1 &= N_4 - N_3 = \frac{1}{2}(N_5 - N_6) = 24ABF \end{aligned} \right\}. \quad (106)$$

In order that the six Trin-Aurifeuillians should fulfil the limitations of (65), it suffices that A, B should be one *odd*, one *even*, and A prime to $3B$; these make each x prime to its own $3y$, and N *odd*.

The above theorem is true whether F be prime, or composite; but, if $F = A^2 + 3B^2$ be a *composite*, it may be considerably extended. For every factor of F must in this case be of form $(a^2 + 3b^2)$, and F will then be expressible in several ways in form $(A^2 + 3B^2)$, and the above theorem will be independently true for *each* of these different modes of expression of F .

Again, since every Trin-Aurifeuillian (N) is the product of *two* factors (L, M), each of which is of form $(A^2 + 3B^2)$, it follows from the above that every Trin-Aurifeuillian (N) is an Aurifeuillian factor of 12 different Trin-Aurifeuillians.

Examples.—The following Tables show the 6 Trin-Aurifeuillians which contain the common Aurifeuillian Factor $F=7$, or 13, and the set of 12 Trin-Aurifeuillians which contain the Trin-Aurifeuillian $F=133$.

Data.		Trin-Aurifeuillians.					
F	A, B		N_1	N_2	N_3	N_4	N_5
7	2, 1	$x, y =$	-1, 2	5, 2	-4, 1	4, 3	1, 1
		$L, M =$	19. 7	7. 67	31. 7	7. 79	1. 7
		$N =$	133	469	217	553	7
13	1, 2	$x, y =$	-5, 4	7, 4	2, 1	2, 3	5, 1
		$L, M =$	133. 13	13. 181	1. 13	13. 49	43. 13
		$N =$	1729	2353	13	637	559
133	11, 2	$x, y =$	-5, 4	17, 4	22, 9	22, 13	-5, 9
		$L, M =$	13. 133	133. 541	1321. 133	133. 1849	403. 133
		$N =$	1729	71953	175693	245917	53599
	5, 6	$x, y =$	13, 21	23, 12	10, 1	10, 11	13, 1
		$L, M =$	1069. 133	133. 1789	73. 133	133. 793	211. 133
		$N =$	142177	237937	9709	165469	28063
							23, 11
							133. 1651
							199583

34. Trin-Aurifeuillian Chains.—Let a series of Trin-Aurifeuillians, $N_1, N_2, N_3, \dots, N_r$, be taken, so that

$$N_r = x_r^4 - 3x_r^2y_r^2 + 9y_r^4 = L_r M_r, \text{ (for all values of } r), \quad (107)$$

and let the parts (x_r, y_r) of the successive functions (N_r) be taken, so that

Case i. $y_1 = y_2 = \dots = y_r$, (an even number),

$x_2 = x_1 + 3y_1$, $x_3 = x_2 + 3y_2$, ..., $x_r = x_{r-1} + 3y_{r-1}$, (all odd). (108, i)

Case ii. $x_1 = x_2 = \dots = x_r$, (an even number),

$y_2 = y_1 + x_1$, $y_3 = y_2 + x_2$, ..., $y_r = y_{r-1} + x_{r-1}$, (all odd). (108, ii)

Case iii. $x_1^2 - 3y_1^2 = x_2^2 - 3y_2^2 = \dots = x_r^2 - 3y_r^2 = z^2$, (an odd square),

where x_r, y_r are one odd, one even. (108, iii)

Now, by the formulæ for $L', M'; L'', M''$, Art. 21,

Case i. $M'_{r-1} = (x_{r-1} + \frac{3}{2}y_{r-1})^2 + 3(\frac{1}{2}y_{r-1})^2$, $L'_r = (x_r - \frac{3}{2}y_r)^2 + 3(\frac{1}{2}y_r)^2$,

Case ii. $M''_{r-1} = (\frac{1}{2}x_{r-1})^2 + 3(\frac{1}{2}x_{r-1} + y_{r-1})^2$, $L''_r = (\frac{1}{2}x_r)^2 + 3(\frac{1}{2}x_r + y_r)^2$.

Case iii. See next Article.

Hence, since $L = L' = L''$, and $M = M' = M''$, (the accents merely indicating the quadratic partition),

Cases i, ii, iii. $M_{r-1} = L_r$, (in each case), (109)

or, in other words, every successive pair N_{r-1}, N_r have a common Aurifeuillian Factor. Such a series may be called a *Trin-Aurifeuillian Chain*, and has many interesting properties. Thus

$$N_r : N_{r-1} = L_r \cdot M_r : L_{r-1} \cdot M_{r-1} = M_r : L_{r-1}. \quad (110a)$$

Similarly, $N_{r-2} : N_{r-3} = M_{r-2} : L_{r-3}$.

Hence in every successive quartet $(N_{r-3}, N_{r-2}, N_{r-1}, N_r)$

$$N_r \cdot N_{r-2} : N_{r-1} \cdot N_{r-3} = M_r : L_{r-3}. \quad (110b)$$

Similarly, giving to r every integer value from 1 to $2r$,

$$\frac{N_{2r} \cdot N_{2r-2} \dots N_0 \cdot N_4 \cdot N_2}{N_{2r-1} \cdot N_{2r-3} \dots N_5 \cdot N_3 \cdot N_1} = \frac{M_{2r}}{L_1}. \quad (110c)$$

34a. *Trin-Aurifeuillian Chains.* (Case iii).—This case requires special study. It is defined by

$$x_r^2 - 3y_r^2 = z^2 \quad (\text{an odd square}), \quad [r = 1, 2, 3, \&c.]. \quad (111)$$

Here z must be either an odd prime of form $z = 12\zeta \pm 1$, or a product

of such primes. Now, for any given *prime** value of z of this form > 1 , there are *two minimum solutions*, say (x_1, y_1) , (ξ_1, η_1) , connected by the reciprocal relations

$$\left. \begin{aligned} x_1 &= 2\xi_1 \sim 3\eta_1, & y_1 &= \xi_1 \sim 2\eta_1; & \xi_1 &= 2x_1 \sim 3y_1, & \eta_1 &= x_1 \sim 2y_1 \end{aligned} \right\}. \quad (112a)$$

one of x_1, ξ_1 *odd*, one *even*; y_1, η_1 of opposite affection to x_1, ξ_1

Also each such "base-solution"* gives rise to an infinite series of solutions, the *successive* solutions of either series being connected by the similar relations,

$$x_r = 2x_{r-1} + 3y_{r-1}, \quad y_r = x_{r-1} + 2y_{r-1}; \quad \xi_r = 2\xi_{r-1} + 3\eta_{r-1}, \quad \eta_r = \xi_{r-1} + 2\eta_{r-1}, \quad (112b)$$

$$x_{r-1} = 2x_r - 3y_r, \quad -y_{r-1} = x_r - 2y_r; \quad \xi_{r-1} = 2\xi_r - 3\eta_r, \quad -\eta_{r-1} = \xi_r - 2\eta_r. \quad (112c)$$

The last two relations are similar in form to, but not identical with, Equation (112a); such a pair of series will be called *Conjugate Series*.

$$\text{Now let} \quad N_r = x_r^4 - 3x_r^2 y_r^2 + 9y_r^4 = L_r \cdot M_r,$$

$$\overline{N}_r = \xi_r^4 - 3\xi_r^2 \eta_r^2 + 9\eta_r^4 = \overline{L}_r \cdot \overline{M}_r,$$

be the Trin-Aurifeuillians formed from (x_r, y_r) , (ξ_r, η_r) respectively.

Also, for distinctness, suppose

$$x_1 \text{ odd}; \text{ then } x_1, \eta_1 \text{ are both odd}; \quad y_1, \xi_1 \text{ even};$$

and, by Art. 21,

$$L_1 = (x_1 \sim \frac{3}{2}y_1)^2 + 3(\frac{1}{2}y_1)^2, \quad \overline{L}_1 = (\frac{1}{2}\xi_1)^2 + 3(\frac{1}{2}\xi_1 \sim \eta_1)^2,$$

and, the relations (112a) now give

$$L_1 = \overline{L}_1, \quad (113)$$

so that the lowest Trin-Aurifeuillians $N_1, \overline{N}_1$ of two conjugate series have the *same* smaller Aurifeuillian Factor (i.e., $L_1 = \overline{L}_1$).

Again, the assumption x_1 *odd* involves

$$\begin{aligned} x_1, y_2, x_3, \dots, x_{2r-1}, y_{2r}, x_{2r+1}, & \text{ all odd,} \\ y_1, x_2, y_3, \dots, y_{2r-1}, x_{2r}, y_{2r+1}, & \text{ all even.} \end{aligned}$$

Hence, by Art. 21,

$$\begin{aligned} M'_{2r-1} &= (x_{2r-1} + \frac{3}{2}y_{2r-1})^2 + 3(\frac{1}{2}y_{2r-1})^2; & L''_{2r} &= (\frac{1}{2}x_{2r})^2 + 3(\frac{1}{2}x_{2r} \sim y_{2r})^2, \\ M''_{2r} &= (\frac{1}{2}x_{2r})^2 + 3(\frac{1}{2}x_{2r} + y_{2r})^2; & L_{2r+1} &= (x_{2r+1} \sim \frac{3}{2}y_{2r+1})^2 + 3(\frac{1}{2}y_{2r+1})^2; \end{aligned}$$

* When z is *composite*, these "base-solutions" are more numerous: they are connected in pairs by the reciprocal relation (112a); and each base-solution gives rise to its own *Trin-Aurifeuillian Chain* as when z is *prime*. *Example*.—When $z = 11.13$, there are *four* base solutions; see Table below.

and the relations (112b, c) now give

$$M'_{2r+1} = L''_{2r}, \quad M''_{2r} = L'_{2r+1},$$

which are equivalent to $M_{r-1} = L_r$ (for all values of r). (114)

By similar reasoning, in the conjugate $(\xi_r, \eta_r, \bar{N}_r)$ series, $\bar{M}_{r-1} = \bar{L}_r$. Hence these series are *Trin-Aurifeuillian Chains*, as defined in last Article (Case iii). The property that each number (N_r) of these series is expressible in the *three* forms $N = N' = N_{r1}$ has been already treated of in Art. 25, 26

From the above it is seen that any such Trin-Aurifeuillian N_1 gives rise to *two* chains, in each of which it is the lowest number, viz., one of Case iii and one of Case i or Case ii according as y_1 or x_1 is even. Also any number L_1 of form $(A^2 + 3B^2)$ gives rise to *four* chains in which it is a common Aurifeuillian Factor (L_1) of the lowest number (N_1) of each chain, viz., *one* of Case i, *one* of Case ii, and *two* of Case iii.

Examples.—The table shows the quadratic parts (x_r, y_r) of a few numbers N_1, N_2 , &c. of the Trin-Aurifeuillian Chains arising when $z = 11, 13$, and 143 ($= 11 \cdot 13$). The conjugate chains are in each case bracketed together. Each number N_r can be expressed in the three forms N, N', N_{r1} by the formulæ of Art. 26 from the given x, y, z . The important case when $z = 1$ has been given in Art. 26 in fuller detail.

z	N_1	N_2	N_3	N_4
	$x_1; y_1$	$x_2; y_2$	$x_3; y_3$	$x_4; y_4$
11 {	13; 4	38; 21	139; 80	518; 299
	14; 5	43; 24	158; 91	589; 340
13 {	14; 3	37; 20	134; 77	499; 288
	19; 8	62; 35	229; 132	854; 493
143 {	146; 17	343; 180	1226; 703	&c.
	241; 112	818; 465	3031; 1748	&c.
	151; 28	386; 207	1393; 800	&c.
	218; 95	721; 408	2666; 1537	&c.

35. Bin- and Trin-Aurifeuillians.—These two functions have many properties in common, which are worth study together. A summary is given below. [Here C.F. denotes *Common Factor*.]

ART. *Bin-Aurifeuillians*, N , $\mathfrak{N}$.

9. Dimorphs are possible.
 10. $N = \mathfrak{N}$ or $\mathfrak{N}$ of $\mathfrak{N}$, when
 $x^2 \mp 2y^2$, or $2y^2 - x^2 = z^2$.
 11. $N = X^4 + Y^4$ is impossible. }
 11. $N = X^4 - Y^4$ is possible. }
 13a. $N_1 N_2 = N$, when
 $x_2^2 \mp x_1^2 = (2y_1)^2 = (2y_2)^2$;
 or $y_2^2 \mp y_1^2 = x_1^2 = x_2^2$.
 17. Factorisation of $(2^{4\nu+2} + 1)$.
 18. Factorisation of $(2y^2)^{4\nu+2} + 1$.
 19. $F = a^2 + b^2$ is C.F. of 4 numbers N ,
 $N_1 + N_4 = N_2 + N_3 = 6F^2$.
 Also N is C.F. of 8 numbers $\mathfrak{N}$.
 20. Bin-Aurifeuillian Chain ($M_{r-1} = L_r$),
 when i. $y_r = y_1$,
 and $x_r = x_{r-1} + 2y_1$,
 or ii. $x_r = x_1$,
 and $y_r = x_1 + y_{r-1}$.

ART. *Trin-Aurifeuillians*, N , $\mathfrak{N}$.

24. Dimorphs are possible.
 25. $N = \mathfrak{N}$ of $\mathfrak{N}$, when
 $x^2 - 3y^2 = z^2$.
 26. $N = (x^6 + z^6) + (x^2 + z^2)$, when
 x or $2x = m^2 + 3n^2$.
 28a. $N_1 N_2 = N$, when
 $x_2^2 \sim x_1^2 = y_1^2 = y_2^2$,
 or $y_2^2 \sim y_1^2 = x_1^2 = x_2^2$.
 31. Factorisation of $(3^{6\nu+3} + 1)$.
 32. Factorisation of $(3y^2)^{6\nu+3} + 1$.
 33. $F = A^2 + 3B^2$ is C.F. of 6 numbers N ,
 $N_1 + N_4 = N_2 + N_3 = N_5 + N_6 = 14F^2$.
 Also N is C.F. of 12 numbers $\mathfrak{N}$.
 34. Trin-Aurifeuillian Chain ($M_{r-1} = L_r$),
 when i. $y_r = y_1 = \epsilon$,
 and $x_r = x_{r-1} + 3y_1$,
 or ii. $x_r = x_1 = \epsilon$,
 and $y_r = x_1 + y_{r-1}$.

36. Bi-Trin-Aurifeuillians.—The question now arises as to whether any numbers (N) exist, expressible at once in the Bin-Aurifeuillian (N_2), Trin-Aurifeuillian (N_3), and Trin-Ant-Aurifeuillian (N'_3) forms.

A *general* solution seems very difficult to obtain; but an interesting partial solution is obtainable by Art. 26; from which it is seen that, writing

$$N_{v1} = \frac{x^6 + z^6}{x^2 + z^2}, \quad N_3 = \frac{x^6 + 3^3 y^6}{x^3 + 3y^3}, \quad N'_3 = \frac{z^6 - 3^3 y^6}{z^2 - 3y^2}, \quad (115)$$

$$\text{then the relation} \quad x^3 - 3y^2 = z^2 \quad (116)$$

suffices that the number N_{v1} should be expressible as both N_3 and N'_3 .

$$\text{Next, taking} \quad z = \xi^2, \quad x = 2\xi^2, \quad (116a)$$

the function $N_{v1} = \frac{\xi^{12} + 2^6 \xi^{12}}{\xi^4 + 2^3 \xi^4}$, a cubic Bin-Aurifeuillian, (say N''_2).

The relation (116) thus becomes

$$4\xi^4 - \zeta^4 = 3y^4. \quad (116b)$$

Hence, every solution of this Diophantine gives a number expressible in the *four* ways

$$N_{v1} = N_6 = N_3 = N_2'''. \quad (117)$$

A *general* solution of (116b) seems difficult; but, if ξ, ζ, y be any *given* solution, then a *second* solution is given* by the following formulæ:—

$$\left. \begin{aligned} \xi_1 &= \xi(27y^4 \sim 64\xi^4), & \zeta_1 &= \zeta(27y^4 \sim 4\xi^3) \\ y_1 &= y\{324y^3 \sim 3(4\xi^4 + \zeta^4)\} \end{aligned} \right\}. \quad (118)$$

Similarly a *third* solution (ξ_2, ζ_2, y_2) may be derived; and, repeating the process, an infinity of solutions may be derived from any given solution (ξ, ζ, y).

Examples.—One solution of (116b) is obvious, viz., $\xi = 1, \zeta = 1, y = 1$; giving $z = 1, x = 2$, whence

$$N = 13 = \frac{2^6 + 1^6}{2^2 + 1^2} = \frac{2^6 + 3^3 \cdot 1^6}{2^2 + 3 \cdot 1^2} = \frac{1^6 \sim 3^3 \cdot 1^6}{1^2 \sim 3 \cdot 1^2} = \frac{1^{12} + 2^6 \cdot 1^{12}}{1^4 + 2^2 \cdot 1^4}.$$

Then (118) give a *second* solution, $\xi_1 = 37, \zeta_1 = 23, y_1 = 1551$; giving $z = 529, x = 2738$, whence

$$N = \frac{2738^6 + 529^6}{2738^2 + 529^2} = \frac{2738^6 + 3^3 \cdot 1551^6}{2738^2 + 3 \cdot 1551^2} = \frac{529^6 \sim 3^3 \cdot 1551^6}{529^2 \sim 3 \cdot 1551^2} = \frac{23^{12} + 2^6 \cdot 37^{12}}{23^4 + 2^2 \cdot 37^4}.$$

The next solution (ξ_2, ζ_2, y_2) is in too high numbers to be practically useful.

36a. Factorisation of above numbers.—These numbers, being Aurifeuillians of orders 2 and 3, admit of resolution into Aurifeuillian Factors in *each way*; this greatly helps their ultimate resolution into *prime* factors.

Thus $N = N_2''' = S_2 \cdot T_2$, (Art. 16a); and $N = N_3 = I_3 M_3$, (Art. 21).

Ex.—Taking the above high number $N = N_2''' = N_3$,

$$N_2''' = \frac{23^{12} + 2^6 \cdot 37^{12}}{23^4 + 2^2 \cdot 37^4} = S_2 T_2 = 3,664,453 \cdot 14,785,321.$$

$$N_3 = \frac{2738^6 + 3^3 \cdot 1551^6}{2738^2 + 3 \cdot 1551^2} = I_3 M_3 = 1,973,533 \cdot 27,453,361.$$

* See M. Desboves's "Mémoire sur la résolution en nombres entiers de l'équation $aX^m + bY^m = cZ^m$," Art. 16, *Results* (67): published in *Nouv. Ann. de Mathém.*, 2^e Sér., t. xviii, 1879.

Here T_2, M_3 are both beyond the range of the Factor-Tables; but the resolution of S_2, L_3 supplies the small factors of M_3, T_2 respectively; thus

$$N = N_2''' = (13 \cdot 61 \cdot 4621)(457 \cdot 32,353) = N_3 = (61 \cdot 32,353)(13 \cdot 457 \cdot 4621).$$

This large number (14 figures) could not have been otherwise easily factorised.

N.B.—It is hoped to publish later similar investigations on Aurifeuillians of higher orders, *e.g.*, order 5, 6, &c.

APPENDIX.

37. *High Primes*.—The following high primes* occurring in this paper (see Tables, Art. 18, 31, 32) seem worth separate record. The initials (B, C) indicate the name of the original discoverer, as explained in the footnote ‡ to Art. 31.

Prime	Factor of	Initial	Prime	Factor of	Initial
9,987,121	$3200^6 + 1$	C	337,448,233	$3^{99} + 1$	C
			512,130,481	$363^{15} + 1$	C
31,720,981	$72^{10} + 1$	C	550,554,229	$3^{63} + 1$	C
32,386,261	$507^{15} + 1$	C	581,433,521	$578^{10} + 1$	C
40,768,781	$450^{10} + 1$	C			
43,120,981	$200^{10} + 1$	C	1,162,320,517	$3^{57} + 1$	B, C
47,336,293	$12^{27} + 1$	B	1,202,388,841	$192^{15} + 1$	C
69,772,181	$722^{10} + 1$	C	1,447,960,181	$200^{10} + 1$	C
79,011,151	$363^{15} + 1$	C	1,561,577,671	$972^{15} + 1$	C
			5,879,415,781	$3^{81} + 1$	C
115,693,471	$108^{15} + 1$	C			
129,159,847	$3^{51} + 1$	B, C	22,434,744,889	$3^{31} + 1$	C
161,466,931	$108^{15} + 1$	C	23,535,794,707	$3^{23} + 1$	C
172,938,811	$768^{15} + 1$	C			
253,765,901	$882^{10} + 1$	C	378,450,588,583	$3^{99} + 1$	C

* Believed to be all new. The mode of detection of these primes is explained in the footnotes †, ¶ to Art. 18 and ‡ (3) to Art. 31.