XVI. On the conductivities of certain heterogeneous media for a steady flux having a potential

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On an average it was estimated that the proportion of oxygen would be about 37 per cent. of the whole. The total quantity of diffused gas reckoned at atmospheric pressure was about 300 c.c. per twenty-four hours.

On removal from the pump the gas was introduced into an inverted flask standing over alkali, and with addition of oxygen as required was treated with the electrical discharge from a transformer in connexion with the public supply of alternating current. In this way the nitrogen was gradually oxidized and absorbed. Towards the close of operations the gas was transferred to a smaller vessel, where it was further sparked until no further contraction occurred, and the lines of nitrogen had disappeared from the spectrum. The excess of oxygen was then removed by phosphorus.

It remains only to record the final figures. The residue, free of oxygen and nitrogen, from 3205 c.c. of diffused gas was 39 c.c. The most instructive way of stating the result is perhaps to reckon the argon as a percentage, not of the whole, but of the nitrogen and argon only. Of the 3205 c.c. total, 2020 c.c. would be nitrogen and argon, and of this the 39 c.c. argon would be 1.93 per cent. Since, according to Kellas (Proc. Roy. Soc. vol. lix. p. 67, 1895), 100 c.c. of mixed atmospheric nitrogen and argon contains 1.19 per cent. of argon, we see that in the diffused gas the proportion of argon is about half as great again as in the atmosphere. Argon then passes the indiarubber film more readily than nitrogen, but not in such a degree as to render the diffusion process a useful one for the concentration of argon from the atmosphere.

XVI. *On the Conductivities of certain Heterogeneous Media for a Steady Flux having a Potential.* By CHARLES H. LEES, D.Sc.*

IN the chapter on "Conduction through heterogeneous media," in his 'Electricity and Magnetism,' Maxwell works out the conductivity of a compound medium formed by embedding in a medium of conductivity k_2 a number of small spheres of a medium of conductivity k_1 , for the case in which the total volume of the spheres bears only a small proportion to that of the medium k_2 , and the distances of the spheres apart are so great that they have no mutual effect on one another. He also works out the conductivities for two other cases of compound media: in the first the two media are separated by planes either parallel or perpendicular to

* Communicated by the Physical Society: read November 24, 1899.

the plane equipotential surfaces in the media; in the second the parts of the two media consist of right prisms of rectangular cross-section, one dimension of the cross-section being infinitely great compared to the other, and the length of the prism being again infinitely great compared to this greatest cross-sectional dimension, placed with their edges either parallel or perpendicular to the equipotential surfaces. The difference of order of magnitude of the three dimensions of the elementary parts of the two media in this last case is necessary in order that the disturbance of the flow near the edges of the prism may be neglected in the calculation.

These three cases, although of considerable interest, have only a limited application on account of the restrictions introduced to enable the calculations to be carried out; and I propose here to consider the case of a medium formed of an equal number of infinitely long prisms of square cross-section, of two media having conductivities k_1 and k_2 , arranged as shown in fig. 1, and bounded by two parallel equipotential

Fig. 1.

planes, AB, CD, drawn through the diagonals of the cross-sections of the prisms.

I assume that the flux and potential are continuous at the surfaces of separation of the two media. If, as in the case of the electrical potentials of copper and zinc, a difference of potential may exist between the media on the two sides of the surface of separation without a flux resulting, the potential which is continuous through the surface must be understood to be that to which the flux is due.

From symmetry it is evident in the first place that A'B', C'D', and in the second place that EF, E'F', &c., are equipotential planes. Also that AC, BD, LM, L'M', &c., are lines of flow. Hence the problem is reduced to finding the conductivity of a prism of square cross-section A'B'C'D' (fig. 2), of which A'D' and B'C' are equipotential surfaces, A'B' and D'C' lines of flow, and A'B'C' consists of a medium of conductivity k_2 , A'D'C' of a medium of conductivity k_1 .

Fig. 2.

If v_1 and v_2 are the potentials in the two media respectively, each satisfies the general condition

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0;$$

v_1 becomes equal to a constant along $A'D'$, and has no flux across $D'C'$; v_2 becomes constant along $B'C'$, and has no flux across $A'B'$; v_1 and v_2 become identical and the fluxes due to them equal along $A'C'$.

It might be possible to construct two functions which would satisfy these conditions by adding together particular solutions of the above equation of the form $v = e^{a(x \pm iy)}$, in the way done by Riemann*, but it is easy by a change of coordinates to make the problem depend on the following simple case, the solution of which is known.

Let AC (fig. 3) be the line of separation of two media of

Fig. 3.

conductivities k_1 and k_2 , and let AD be an equipotential and DC a stream-line in the k_1 medium. Let k_1/k_2 have such a value that AB , the stream-line in the second medium passing through A , makes the same angle with AC that AD does. If, then, CB is the equipotential line in the second medium passing through C , the triangle ABC is equal to the triangle ADC .

If z is the vector coordinate of a point in the plane of fig. 3, and z' the vector coordinate of a point in the plane of fig. 2, it is possible by a transformation $z' = f(z)$ to convert the right-angled triangle ADC in the z -plane into the isosceles right-angled triangle $A'D'C'$ in the z' -plane†. Similarly, the same transformation converts the triangle ABC in the z -plane into the triangle $A'B'C'$ in the z' -plane; and contiguous points on opposite sides of AC in the z -plane become contiguous points on opposite sides of $A'C'$ in the z' -plane. Hence the quadrilateral $ABCD$ converts into the quadrilateral $A'B'C'D'$, and the difference of potential and the amount of

* *Partielle Differentialgleichungen*, Abschnitt iv., Bewegung der Wärme

† Forsyth, 'Theory of Functions,' p. 543; Love, *Amer. Journ. of Math.* xi. pp. 164 & 168 (1889).

the flux between any two points of the area ABCD will be identical with the quantities for the corresponding points of the area A'B'C'D'.

If V is the difference of potential between the points A, C of fig. 3 and $AC=a$, the flux bounded by the stream-lines DC and AB

$$= k_1 \frac{a \cos \theta V}{a \sin \theta} = k_1 V \cot \theta, \text{ where } \theta = \angle DAC.$$

Hence the flux bounded by the stream-lines D'C' and A'B' in fig. 2 under the potential-difference V is equal to $k_1 V \cot \theta$.

If k is the apparent conductivity of the square, the flux bounded by the stream-lines D'C' and A'B' is equal to kV ; hence

$$k_1 V \cot \theta = kV, \text{ or } k = k_1 \cot \theta.$$

Now in fig. 3, since the flux is continuous across AC, $k_1 \cot DAC = k_2 \cot BCA$. Also $\angle BAC = \angle DAC$.

$$\therefore k_1 \cot \theta = k_2 \tan \theta;$$

$$\text{i. e. } \cot \theta = \sqrt{\frac{k_2}{k_1}}.$$

Therefore the apparent conductivity k of the square

$$= \sqrt{k_1 k_2}.$$

If now this medium, consisting half of material k_1 , half of material k_2 , be supposed homogeneous, and be combined, in the way indicated in fig. 1, with an equal volume of the k_1 medium, we have k' the conductivity of the new medium,

$$\begin{aligned} &= k^{\frac{1}{2}} k_1^{\frac{1}{2}}, \\ &= k_1^{\frac{3}{4}} k_2^{\frac{1}{4}}. \end{aligned}$$

Similarly, by making other combinations of the new media and the old, we deduce that if a volume p_1 of a medium of conductivity k_1 be combined in this manner with a volume $1-p_1$ of a second medium of conductivity k_2 , the conductivity k of the combination will satisfy the equation

$$k = k_1^{p_1} k_2^{(1-p_1)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\begin{aligned} \log k &= p_1 \log k_1 + (1-p_1) \log k_2, \\ &= \log k_2 + p_1 \log \frac{k_1}{k_2}. \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

That is, for the mixture considered, the logarithms of the conductivities for fluxes parallel to the diagonals in fig. 1 follow the law of means.

Since $\log k = -\log \rho$, where ρ is the resistivity of a medium, we also have

$$\begin{aligned}\log \rho &= p_1 \log \rho_1 + (1-p_1) \log \rho_2 \\ &= \log \rho_2 + p_1 \log \frac{\rho_1}{\rho_2} \dots \dots \dots (3)\end{aligned}$$

as the relation between the resistivity of the mixture and the resistivities and relative volumes of its constituents.

If more than two media are present in volumes p_1, p_2, p_3 , &c. per c.c. of the mixture,

$$k = k_1^{p_1} \cdot k_2^{p_2} \cdot k_3^{p_3} \dots \dots \dots (4)$$

In order to show the principal features of the distribution of the flux and potential in a medium consisting of equal square prisms of two media arranged as indicated in fig. 1, the equipotential lines (or the stream-lines) for the single

Fig. 4.

Fig. 5.

block considered above are shown in fig. 4, and for a strip of the medium in fig. 5. In each case one constituent conducts

six times as well as the other. The curves have been drawn by an approximate method which depends on the fact that near each angular point A, C, the equation which transforms the kite-shaped figure of fig. 3 into the square of fig. 2 takes the form $z' = Az^n$. The accuracy of the diagrams will therefore be least near the centre of each square ABCD, where the position of an equipotential line may be in error by 10 per cent. of its distance from the next one.

Professor Lamb has pointed out to me that by superposing a second flux, the direction of which is at right angles to the first, on the first, a flux with stream and equipotential lines through the centres of the squares parallel to their sides is obtained. The conductivity of the medium for this flux will be readily seen to be given again by the formula

$$k = \sqrt{k_1 k_2}.$$

The equipotential lines (or the stream-lines) for a strip of the medium, when one constituent conducts six times as well as the other, are shown in fig. 6.

Fig. 6.

Since there are four directions in the medium in which the conductivity has the above value, the conductivity of the medium for a flux in any direction has the same value, *i. e.* equation (4) gives the conductivity of a compound medium in the general case where the constituents are present in the form of long prisms with their axes perpendicular to the planes of flow.

XVII. *The Law of Partition of Kinetic Energy.*

By S. H. BURBURY*.

LORD RAYLEIGH'S article† seems to raise the question what is the enunciation of the Maxwell-Boltzmann law of equal partition of energy. It cannot mean that in every case in which a number of particles of unequal masses are in stationary motion each particle has the same mean

* Communicated by the Author.

† *Supra*, p. 98.