The case proposed for consideration is that in which the asymptotes include an angle of 90° , so that the maximum value of θ is 45° . Substituting this in (32), we find, $v_0^2/g = 1.081$, and thus

$$p = .5405 \cos^2 \theta \pm \sqrt{\{.5405^2 \cos^4 \theta - .0730\}} \dots\dots\dots(33).$$

From this equation we may calculate any number of corresponding values of p and θ , and thus draw the tangents of which the required curves are the envelopes. The annexed table contains a few such pairs of values, sufficient for an indication of the forms of the curves:—

θ	p_1	p_2	$\cos \theta$	$p_1 / \cos \theta$	$p_2 / \cos \theta$
0°	1.0086	.0724	1.0000	1.0086	.0724
9°	.9800	.0744	.9877	.9922	.0753
18°	.8964	.0814	.9511	.9425	.0856
27°	.7625	.0957	.8910	.8557	.1074
36°	.5823	.1253	.8090	.7196	.1549
45°	.2702	.2702	.7071	.3821	.3821

The two last columns give the intercepts on the axis, by means of which and the value of $\cos \theta$ the lines are more readily drawn than from the perpendiculars themselves. The result is shown in Fig. 3.

On a Deduction from the Elliptic-Integral Formula

$y = \sin(A + B + C + \dots)$. By JOHN GRIFFITHS, M.A.

[Read Dec. 13th, 1883.]

The transformation-equation $y = \sin(A + B + C + \dots)$ for elliptic integrals of the first kind, recently communicated by me to the Society, gives also the relations between the corresponding integrals of the second kind.

1. Case of a transformation of odd order n .

Here we have... $\sin A = \frac{x \{2\alpha' + 1 - (1 + \alpha')^2 x^2\}}{1 - \alpha^2 x^2}$,

$$\cos B = \frac{1 - (1 + \beta^2) x^2}{1 - \beta^2 x^2}, \quad \cos C = \frac{1 - (1 + \gamma^2) x^2}{1 - \gamma^2 x^2}, \quad \&c.,$$

where $\alpha, \beta, \gamma, \&c.$ are identical with Jacobi's coefficients arising from the n -division of the complete function K ; viz.,

$$\alpha \equiv k \operatorname{sn} \frac{2sK}{n}, \quad \&c.; \quad \alpha' \equiv \operatorname{dn} \frac{2sK}{n},$$

1833.] the *Elliptic-Integral Formula* $y = \sin (A+B+C+\dots)$. 79

s being an integer from 1 to $\frac{1}{2}(n-1)$. This being so, the equation $y = \sin (A+B+C+\dots)$ leads to the relation

$$\frac{dy}{\sqrt{1-y^2} \cdot 1-\lambda^2 y^2} = M \cdot \frac{dx}{\sqrt{1-x^2} \cdot 1-k^2 x^2},$$

or, say,

$$\int_0^y \frac{dy}{\sqrt{1-\lambda^2 y^2}} = M \int_0^x \frac{dx}{\sqrt{1-k^2 x^2}}; \quad \text{i.e., } v = Mu,$$

$$\text{if} \quad M \operatorname{dn} (v, \lambda) = \operatorname{dn} (u, k) \left\{ 1 + \Sigma \frac{2\alpha'}{1-\alpha^2 \operatorname{sn}^2 (u, k)} \right\}.$$

This latter equation is, in fact, another form of the transformation-equation we have started with.

Now, if $\sin A = \frac{\operatorname{sn} u \{2\alpha' + 1 - (1+\alpha')^2 \operatorname{sn}^2 u\}}{1-\alpha^2 \operatorname{sn}^2 u}$, where $\alpha^2 + \alpha'^2 = 1$,

$$\text{then} \quad \frac{dA}{dk} = \frac{2 \operatorname{sn} u \operatorname{cn} u}{1-\alpha^2 \operatorname{sn}^2 u} \frac{d\alpha'}{dk} + \left\{ \frac{1}{\operatorname{cn} u} + \frac{2\alpha'}{\operatorname{cn} u (1-\alpha^2 \operatorname{sn}^2 u)} \right\} \frac{d}{dk} (\operatorname{sn} u);$$

and, similarly,

$$\frac{dB}{dk} = \frac{2 \operatorname{sn} u \operatorname{cn} u}{1-\beta^2 \operatorname{sn}^2 u} \frac{d\beta'}{dk} + \frac{2\beta'}{\operatorname{cn} u (1-\beta^2 \operatorname{sn}^2 u)} \frac{d}{dk} (\operatorname{sn} u),$$

&c. &c.

$$\text{Hence} \quad \Sigma \frac{dA}{dk} = 2 \operatorname{sn} u \operatorname{cn} u \Sigma \frac{\frac{d\alpha'}{dk}}{1-\alpha^2 \operatorname{sn}^2 u} + M \frac{\operatorname{dn} (v, \lambda)}{\operatorname{cn} u \operatorname{dn} u} \frac{d}{dk} (\operatorname{sn} u),$$

and we arrive at the formula

$$\frac{dv}{d\lambda} = \frac{dk}{d\lambda} \left\{ M \frac{du}{dk} - \frac{2 \operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} (v, \lambda)} \Sigma \frac{\frac{d\alpha'}{dk}}{1-\alpha^2 \operatorname{sn}^2 u} \right\} + u \frac{dM}{d\lambda},$$

$$\text{or, since} \quad M^2 = \frac{nkk'^2}{\lambda \lambda'^2} \cdot \frac{d\lambda}{dk}, \quad kk'^2 \frac{du}{dk} = E - k'^2 F - k^2 \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u},$$

we may write

$$E(v, \lambda) - \lambda'^2 v - \lambda^2 \frac{\operatorname{sn} (v, \lambda) \operatorname{cn} (v, \lambda)}{\operatorname{dn} (v, \lambda)} = \frac{n}{M} \left\{ E(u, k) - k'^2 u - k^2 \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} \right\} \\ - 2 \frac{nkk'^2}{M^2} \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} (v, \lambda)} \Sigma \frac{\frac{d\alpha'}{dk}}{1-\alpha^2 \operatorname{sn}^2 u}$$

where $v = Mu$,

$$+ \lambda \lambda'^2 u \frac{dM}{d\lambda};$$

and

$$M \operatorname{dn} (v, \lambda) = \operatorname{dn} u \left\{ 1 + \Sigma \frac{2\alpha'}{1-\alpha^2 \operatorname{sn}^2 u} \right\}.$$

When $u = K$, $v = n\Lambda$, we have, as a consequence of the above,

$$E(n\Lambda, \lambda) - \lambda^2 \cdot n\Lambda = \frac{n}{M} \{E(K, k) - k^2 K\} + \lambda \lambda^2 K \frac{dM}{d\lambda};$$

$$\text{i.e., } E(\Lambda, \lambda) - \lambda^2 \Lambda = \frac{1}{M} \{E(K, k) - k^2 K\} + \lambda \lambda^2 \frac{K}{n} \frac{dM}{d\lambda}.$$

If $\frac{1}{M}$ be written for M , the relation in question between the complete functions becomes identical with that given by Professor Cayley. (See "Elliptic Functions," p. 223.)

2. Case of a transformation of an even order $2n$.

Here, in the formula $y = \sin(A + B + C + \dots)$, we have to replace the former expression for A by the following,

$$\text{viz., } \sin A = (1 + k') \frac{x \sqrt{1 - x^2}}{1 - k^2 x^2},$$

$$\text{which gives } \frac{dA}{dk} = -\frac{k}{k'} \frac{\text{sn } u \text{ cn } u}{\text{dn}^2 u} + \left\{ 1 + \frac{k'}{\text{dn}^2 u} \right\} \frac{\frac{d}{dk}(\text{sn } u)}{\text{cn } u}.$$

Hence

$$\frac{dv}{d\lambda} = \frac{dk}{d\lambda} \left\{ M \frac{du}{dk} - \frac{\text{sn } u \text{ cn } u}{\text{dn}(v, \lambda)} \left(\frac{dk'}{\text{dn}^2 u} + 2 \frac{\frac{d\beta'}{dk}}{1 - \beta'^2 \text{sn}^2 u} \right) \right\} + u \frac{dM}{d\lambda};$$

$$\text{where } \frac{M \text{dn}(v, \lambda)}{\text{dn } u} = 1 + \frac{k'}{\text{dn}^2 u} + 2 \Sigma \frac{\beta'}{1 - \beta'^2 \text{sn}^2 u},$$

$$M = 1 + k' + 2 \Sigma \beta', \quad v = Mu, \quad \beta \equiv k \text{sn} \frac{sK}{n}, \text{ \&c.,}$$

s being an integer from 1 to $n-1$; and we can deduce a formula similar to that given above for the corresponding elliptic integrals of the second kind, $E(v, \lambda)$ and $E(u, k)$.

As a geometrical result of these formulæ, it follows that an arc of a given ellipse can, in various ways, be expressed in terms of arcs of two other ellipses.

Note by Professor CAYLEY.

Consider, for instance,
the cubic transformation

$$y = \frac{x[1 + 2\alpha' - (1 + \alpha')^2 x^2]}{1 - \alpha^2 x^2}$$

This implies

$$\sqrt{1-y^2} = \frac{\sqrt{1-x^2}[1 - (1 + \alpha')^2 x^2]}{1 - \alpha^2 x^2}$$

viz., $\sqrt{1-y^2}$ = a rational multiple of $\sqrt{1-x^2}$;

and the quadric transformation

$$z = \frac{1 - (1 + \beta^2) x^2}{1 - \beta^2 x^2}.$$

This implies

$$\sqrt{1-z^2} = \frac{\sqrt{1-x^2} \cdot 2\beta' x}{1 - \beta^2 x^2},$$

viz., $\sqrt{1-z^2}$ = a rational multiple of $\sqrt{1-x^2}$.

Hence, assuming

$$u = yz - \sqrt{1-y^2} \sqrt{1-z^2},$$

which is a rational function = $\frac{x(a_0 - a_2 x^2 + a_4 x^4)}{1 - \alpha^2 x^2 \cdot 1 - \beta^2 x^2}$,

we have

$$\sqrt{1-u^2} = y \sqrt{1-x^2} + z \sqrt{1-y^2},$$

which is = $\sqrt{1-x^2}$ into a like rational function.

That is, in defining the a_0, a_2, a_4 , functions of the two arbitrary coefficients α, β , as above, we have in effect so determined them that $\sqrt{1-u^2}$ shall be = $\sqrt{1-x^2}$ into a rational function of x .

We can then further determine a_0, a_2, a_4 in such wise that the change x into $\frac{1}{kx}$ shall change u into $\frac{1}{\lambda u}$; and, this being so, making the change in $\sqrt{1-u^2}$, we obtain $\sqrt{1-\lambda^2 u^2}$ in the form $\sqrt{1-k^2 x^2}$ into a rational function of x ; viz., u is a function of x such that

$$\frac{du}{\sqrt{1-u^2} \cdot 1 - \lambda^2 u^2} = \frac{M du}{\sqrt{1-x^2} \cdot 1 - k^2 x^2}.$$

The theory is thus in effect Jacobi's—with the *novelty* of so combining two lower transformations in such wise that the assumed expression for u as a rational function of x shall give

$$\sqrt{1-u^2} = \sqrt{1-x^2} \text{ into a rational function of } x.$$

It is not necessary that the equations

$$y = \text{rational function of } x \text{ and } z = \text{rational function of } x$$

should be elliptic-function transformations. All that is required is that they should be such as to give $\sqrt{1-y^2}$ and $\sqrt{1-z^2}$ each = $\sqrt{1-x^2}$ into rational function of x .

Thursday, January 10th, 1884.

S. ROBERTS, Esq., F.R.S., Vice-President, in the Chair.

Messrs. D. Brockelbank and Asútosh Mukhopâdhyây, B.A., Calcutta University, were elected Members, and Messrs. Fortey and Heppel were admitted into the Society.

A letter from Prof. Greenhill (the Auditor) was read, in which he stated that he had verified the amount of the moneys standing in the Society's name at the Bank of England.*

The Chairman spoke upon the loss the Society had sustained by the death of Mr. Merrifield, and read the vote of condolence which he, as President, was requested to send to the bereaved family in the name of the Council.

Mr. Buchheim stated "An extension of Pascal's Theorem to Space of Three Dimensions," and communicated a paper "On the Theory of Screws in Elliptic Space"; and Mr. Fortey read a paper "On Contacts and Isolations, a problem in Permutations." Mr. Tucker presented a paper by Prof. H. Lamb, "On the Induction of Electric Currents in Cylindrical and Spherical Conductors," and spoke "On a Group of Circles which are connected with the 'Triplicate-Ratio' Circle."

The following presents were received :—

- "Educational Times" for January, 1884.
- "Cambridge Philosophical Society—Transactions," Vol. xiii., Pt. iii., 1883.
- "Cambridge Philosophical Society—Proceedings," Vol. iv., Pt. vi., 1883.
- "Johns Hopkins University Circulars," Vol. iii., No. 27.
- "Bulletin Astronomique et Météorologique de l'Observatoire Impérial, de Rio de Janeiro," No. 7, July, 1883.
- "Atti della R. Accademia dei Lincei—Transunti," Vol. vii., Fasc. 15 and 16.
- "Jornal de Sciencias Mathematicas e Astronomicas," publicado pelo Dr. F. Gomes Teixeira, Vol. v., Nos. 1 and 2, 8vo; Coimbra.
- "Beiblätter zu den Annalen der Physik und Chemie," B. vii., St. 12.
- "Jahrbuch über die Fortschritte der Mathematik, &c." xiii., 2, Jahrgang 1881.
- "Bulletin des Sciences Mathématiques et Astronomiques," T. vii., Nov. 1883.
- "Anuario del Observatorio Astronómico Nacional de Tacubaya," para el 1884; Mexico, 1883.
- "Nieuw Archief voor Wiskunde," Deel i. to Deel x., St. i., 8vo; Amsterdam, 1875 to 1883.
- "The Mathematical Magazine: a Journal of Elementary Mathematics," edited by Artemas Martin, M.A., Vol. i., No. 8, Oct. 1883; Erie, Pa.
- "American Journal of Mathematics," Vol. vi., No. 2, Dec., 1883; Baltimore.

* See p. 68.