(where $M, \lambda; M_1, \lambda_1$, &c. refer to the first, second, &c. transformation) has thus been proved to be true for $n = 3$; and, as a consequence of it, I have shown that, in the case of a prime number, there is an addition-theorem for the various moduli. The latter has also been verified for $n = 3, 5$, and 7 .

As another result of the formula, we may notice $\Sigma M = 0$ or $2n$.

I may also observe that the triplication formula

$$y = \sin(-A + B + C + D)$$

throws some light on the modular equation for the nonic transformation.

This, according to the present theory, may be considered as composed of a cubic and three quadric transformations, and so is, in fact, of the form

$$y = \sin(A + B + C + D).$$

We should thus expect to find that $\lambda = k$ is a root of the modular equation. This is what actually happens, as may be seen from a memoir by Prof. Cayley "On the Transformation of Elliptic Functions," *Phil. Trans.*, t. 164 (1874). The above result is in complete accord with Jacobi's theorem as to the transformations corresponding to a square number.

On the Immediate Application of the Principle of Least Action to Dynamics of a Particle, Catenaries, and other related Problems.

By Professor J. LARMOR.

[Read March 13th, 1884.]

1. It is matter of constant observation that different departments of Mathematical Physics are closely connected together, so that the solution of a question in one branch of the subject admits of being transferred into another branch, and serving as the solution of a corresponding problem there.

To facilitate the general discussion of these relations between different problems, it is desirable to express each in its simplest mathematical terms, so that it shall involve as little reference as possible to the detailed theory of the special subject. Now nearly every such question can be expressed as a problem of maximum or minimum; and this mode of statement probably conveys a clearer and more compact representation of the question, and an easier grasp of its mathematical relations as a whole, than any other. It at once places the question beyond the accident of the particular coordinate system employed. We should therefore expect that it would be a direct and easy method of arriving at dynamical analogies.

The development of this method is the object of the following paper. Many of the examples given are more or less known, but there does not seem to have been any systematic attempt to group them together as illustrations of a single general principle.

2. The complete mathematical statement of every question in Dynamics is involved in either of the forms of the Principle of Least Action, when once the expressions for the potential and kinetic energies of the system are known. The principle admits of easy and elementary proof, and therefore forms a suitable foundation from which to construct dynamical theory. See, for example, Thomson and Tait's *Nat. Phil.*, Vol. i., Art. 327.

The prominent place that this principle directly holds in general dynamical theory is illustrated by the attempts that have been made to place the Second Law of Thermodynamics on a dynamical basis. The investigations of Clausius, Boltzmann, and Szily have shown that the Principle of Least Action in Dynamics is in close relationship with that law. And this agrees with what we might have expected from the above; for the law must have some dynamical interpretation, and the only dynamical relations hitherto discovered that are absolutely general and independent of coordinate systems of representation are the Principle of Energy, of which the First Law of Thermodynamics is a case, and the Principle of Least Action.

Hamilton pointed out the importance and began the development of the general method of reducing the solution of every dynamical question to the consideration of a single function (*Phil. Trans.*, 1834-1835); and it has since been widely extended and carried on by others (*cf.* Cayley's *Report, Brit. Assoc.*, 1857). What we now wish to point out is the ease and simplicity with which the mere statement of the principle, without any consequent analytical work, reveals the existence and nature of many analogies between different problems, whose relationship has been hitherto treated in a more or less isolated manner.

When we wish to compare the motion of one system with the equilibrium of another, we have the variational energy condition of equilibrium at hand, which may, if we please, be itself considered to be a case of the general principle.

3. In what follows we shall develop the relations between different problems relating to the motion of a particle, and the analogies between the motion of a particle and the forms of systems of rays and of catenaries that flow immediately from the minimum principle. The application of the method of inversion and of other methods of transformation flows naturally from it, and we are guided immediately to a large number of generalizations of solvable cases of motion

that have formed the subjects of separate papers by Liouville and others.

It may be observed that the method for brachistochrones, given, from Woodhouse, in the Appendix to Tait and Steele's *Dynamics*, affords also a very simple and elementary geometrical proof of the Principle of Least Action for a particle. For, drawing the system of equipotential surfaces of the field of force, the velocity of the particle between any two consecutive surfaces is invariable, as the energy of the motion is given. To determine that path which makes $\int v ds$ least, we take three consecutive points on the path, and find the position that the intermediate one must have on the equipotential through it, in order to make the variation of the integral zero. We shall be led to the ordinary law of centrifugal force in the orbit; and it follows that the actual orbit is the one possessing the stationary property.

In another paper the same principle will be applied to the discussion of problems connected with general solid and fluid systems, cyclic and acyclic, and of certain close analogies with questions relating to elastic wires and electric currents.*

Application of the Method of Inversion.

4. When a particle moves in a given field of force, the form of its orbit is determined by the condition that $\int v ds$ satisfies the maximum-minimum condition, in which v is given by the equation of energy

$$\frac{1}{2}v^2 + V = E, \text{ a constant,}$$

where V is the potential of the field of force.

Now, if we invert the orbit with respect to any origin, with radius of inversion k , $\int v ds$ is equivalent to $k^3 \int \frac{v}{r^3} ds'$, where dotted letters refer to the inverse orbit; since, by Geometry,

$$ds : ds' = r : r' = k^2 : r^2$$

This new integral therefore satisfies the maximum-minimum condition, when taken along the inverse curve; and therefore this curve is the orbit of a particle in a field of force which makes

$$v' \propto v'^{-2},$$

and therefore

$$\frac{1}{2}v'^2 \propto (E - V) r'^{-4},$$

in which V is now to be expressed in terms of the dotted letters.

The value of V' , the potential of the new field of force, is therefore given by

$$AV' + (E - V) r'^{-4} = \text{constant,}$$

or, what is the same, $r^3 (E' - V') \propto r^3 (E - V)$.

* The original paper has been divided into two, and some other changes have been made, at the suggestion of the referee.

The velocity of projection in the new orbit is determined by the above value of v' .

We notice that, if the original law of force only contain powers of the coordinates higher than the inverse fifth, this velocity-condition will in general be that the particle be projected with the velocity from infinity. (See ex. 2, § 5.)

The ratio of the velocities at corresponding points of the related orbits is clearly inversely proportional to that of the radii vectores.

5. The following are examples :—

(1) With no forces acting, a particle constrained to remain on a fixed surface describes a geodesic with constant velocity; the curve corresponding to this geodesic on the inverse surface satisfies the relation $\delta \int r^{-2} ds = 0$, and is therefore a free path for a particle constrained to remain on the surface, under the action of a force varying as the inverse fifth power of the distance from the centre of inversion, and projected with the velocity from infinity.

On a sphere, the geodesics are great circles. Therefore particles projected on the surface of a sphere under the action of a force varying inversely as the fifth power of the distance from any fixed centre, and with the velocity due to approach from infinity, will describe circles on the sphere, which are its curves of intersection with the system of spheres passing through the centre of force and its inverse point. In particular, if the original sphere becomes a plane, we get Newton's result for free motion in a circle with a centre of force on its circumference.

On a cylinder of any form of cross-section, the geodesics make equal angles with the generators. Therefore on the ring of vanishing aperture, which is the inverse of such a cylinder, the paths of particles projected under the same conditions as before (the centre of force being now at the aperture) will be curves, each of which cuts all the meridians of the surface at the same angle. This result holds for any ring with circular meridians all passing through the same point; in particular, for a common anchor ring of vanishing aperture.

Again, lines of curvature invert into lines of curvature, and umbilici into umbilici; for the directions of the lines of curvature at a point are determined by the nature of the contact with spheres touching at the point, and spheres invert into spheres. Therefore we arrive at the result that, on any inverse of an ellipsoid, and in particular on Fresnel's Surface of Elasticity (in which case the centre of the surface is the centre of force), all particles projected from one umbilicus under the law of the inverse fifth power and with the velocity from infinity, will, when reflected by a line of curvature, pass through the other umbilicus, will then be again reflected through the first, and

so on continually. If the orbit pass through the two umbilici without reflection, it will have the form of a curve winding round and round the surface, and passing through them once in each revolution; if it do not pass through either, it will be such a curve touching the two branches of a line of curvature, one surrounding each umbilicus, once in each revolution. All this, of course, results from the well-known properties of geodesics on an ellipsoid.

Further, the orbits of particles moving on a surface under this law of force with any velocity of projection whatever, become on inversion the orbits described under similar conditions on the inverse surface; while, with the particular velocity discussed above, they become geodesics.

(2) For free motion in a plane the application is easy. Thus, if for the original orbit, the force (repulsive) = μr^n , we have

$$\frac{1}{2}v^2 = \left\{ E + \frac{\mu}{n+1} \left(\frac{k^2}{r'} \right)^{n+1} \right\} r'^{-1},$$

the centre of inversion being the centre of force; and therefore in the inverted orbit the force = $\mu_1 r'^{-5} + \mu_2 r'^{-n-5}$, where the values of μ_1, μ_2 are obvious. If $n > -5$, the velocity will be that from infinity; while, if the velocity in the original orbit was that from the origin when $n > -1$, or to infinity when $n < -1$, we have the further simplification $\mu_1 = 0$.

We may also invert with respect to a point not in the plane of the orbit, and so obtain the fields of force in which curves drawn on a sphere are free orbits with proper velocity of projection. There is no need to supply examples.

6. Transformation by Conjugate Functions.

If a plane area is transformed by writing ξ, η for x, y , each element of length near a point is altered in the ratio λ to 1, where

$$\lambda^2 \equiv \frac{d(\xi\eta)}{d(xy)} \equiv \left(\frac{d\xi}{dx} \right)^2 + \left(\frac{d\xi}{dy} \right)^2 \equiv \left(\frac{d\eta}{dx} \right)^2 + \left(\frac{d\eta}{dy} \right)^2,$$

for in the original case ξ, η were the coordinates, but they are now replaced by x, y .

Hence the relation $\delta \int v ds = 0$,

becomes

$$\delta \int \lambda v d\sigma = 0,$$

where $d\sigma$ is the corresponding element of arc of the transformed curve. The transformed curve is therefore the orbit, with proper velocity, in a field of force given by

$$V' \propto (E - V) \lambda^2.$$

For instance, if the orbit be the straight line

$$x + ay = b,$$

we have

$$V = \text{constant},$$

and therefore the curve $\xi + a\eta = b$

can be described freely in a field $V' \propto \lambda^2$.

On making $\xi = n \log r$, $\eta = n\theta$, we find $\lambda^2 = n^2 r^{-2}$, and therefore the spiral $r = Ae^{-a\theta}$ is the orbit under the law of the inverse cube, with velocity from infinity, as Newton proved.

7. We have here also a ready method of inventing fields of forces, in which the motion can be integrated. For example, motion in a field whose potential is given by

$$V = f(x) + F(y)$$

can obviously be reduced to quadratures. The transformed curves are free orbits in a field whose potential is

$$V' = \lambda^2 \{f(\xi) + F(\eta) - E\},$$

with proper velocity, E being the total energy of the original orbit. But further, being given this law of potential, without *any* velocity-condition, and transforming back again, we arrive at the law

$$V = f(x) + F(y) - E + C\lambda^{-2},$$

where C depends on the velocity; so that, if λ^{-2} is of the form $\phi(x) + \psi(y)$, we can integrate without any restrictions as to initial velocity.

Liouville arrived at these results from a discussion of the equations of motion.

(1) He applies them to the case of motion under two centres of force, by using elliptic coordinates with these centres as foci; thus, if μ, ν represent the semi-axes of the confocals through a point, and $2b$ the distance between the foci, we have

$$r = \mu + \nu, \quad r' = \mu - \nu, \\ ds^2 = (\mu^2 - \nu^2) \{(\mu^2 - b^2)^{-1} d\mu^2 + (b^2 - \nu^2)^{-1} d\nu^2\},$$

so that λ^{-2} is here equal to $\mu^2 - \nu^2$, and we can reduce to quadratures motion under any law

$$V = (\mu^2 - \nu^2)^{-1} \{f(\mu) + F(\nu)\} = (rr')^{-1} \{f(r+r') + F(r-r')\}.*$$

A case of this is easily seen to be

$$V = gr^{-1} + g'r'^{-1} + kR^2,$$

where R is the distance from the centre. (*Vide* Liouville, Vol. 11,

* [Thus $\int v ds = \sqrt{2} \int \sqrt{E - V} . ds = \sqrt{2} \int \sqrt{(E - V)(\mu^2 - \nu^2)} d\sigma$
 $= \sqrt{2} \int \sqrt{E(\mu^2 - \nu^2) - (f\mu + F\nu)} d\sigma,$

where $d\sigma^2 = d\mu'^2 + d\nu'^2$, and this comes under the first form of § 7.]

1846; or Prof. Cayley's "Report on Special Problems of Dynamics," *Brit. Assoc.*, 1862.)

Further, if $\psi(x, y) = c$ be the equation to the orbit under the law $V = f(x, y)$ when the energy of the motion is E , we see that $\psi(\mu', \nu') = c$ will be the equation to the orbit under the law

$$V = (\mu^3 - \nu^3)^{-1} \{f(\mu', \nu') - E\},$$

where $\mu' = \int (\mu^3 - b^3)^{-1} d\mu$, $\nu' = \int (b^3 - \nu^3)^{-1} d\nu$,

the velocity being such that $\frac{1}{2}v^3 + V = 0$.

For instance, in the field,

$$V = A'(\mu^3 - \nu^3)^{-1} = A'(rr')^{-1},$$

particles projected with the velocity from infinity will describe oblique trajectories of the system of confocals.

(2) On taking $\xi = \log r$, $\eta = \theta$, we find $\lambda^3 = r^{-2}$, and we have the solution immediately for the law

$$V = r^{-2} \phi\left(\frac{x}{y}\right) + \psi(r),$$

with any velocity of projection, since $\lambda^{-2} (= x^2 + y^2)$ is of the proper form.

This law has been discussed by Jacobi and Bertrand from the equations of motion.

(3) Adopting the dipolar system of coordinates

$$\xi = \log r - \log r', \quad \eta = \theta - \theta',$$

we find

$$\lambda = b(rr')^{-1},$$

and we have at once the equations in quadratures of the orbits in a field of force given by

$$\begin{aligned} V &= \frac{1}{r^3 r'^3} \left\{ f\left(\frac{r}{r'}\right) + F\left(\frac{r^2 + r'^2 - b^3}{2rr'}\right) \right\} \\ &= \frac{1}{r^3 r'^3} \phi\left(\frac{r}{r'}\right) + \psi(rr'), \text{ say,} \end{aligned}$$

with the velocity of projection which makes $\frac{1}{2}v^3 + V = 0$.

We may notice that, if $V \propto (rr')^{-2}$, particles projected with velocity from infinity will describe paths cutting all circles through the foci at equal angles.

These results might have been arrived at by the method of inversion.

(4) For motion on an ellipsoid, we have

$$ds^2 = \frac{(\mu^3 - \rho^3)(\mu^3 - \nu^3)}{(\mu^3 - h^3)(\mu^3 - k^3)} d\mu^2 + \frac{(\nu^3 - \mu^3)(\nu^3 - \rho^3)}{(\nu^3 - h^3)(\nu^3 - k^3)} d\nu^2,$$

where ρ, μ, ν are the principal semi-axes of the confocals through the

point, and h^2, k^2 the constant differences of the squares of the semi-axes of each surface. Hence $\int v ds$

becomes

$$\int v (\mu^2 - \nu^2)^{\frac{1}{2}} (d\theta^2 + d\phi^2)^{\frac{1}{2}},$$

where

$$\theta = \int \left(\frac{\mu^2 - \rho^2}{\mu^2 - h^2 \cdot \mu^2 - k^2} \right)^{\frac{1}{2}} d\mu,$$

$$\phi = \int \left(\frac{\rho^2 - \nu^2}{\nu^2 - h^2 \cdot \nu^2 - k^2} \right)^{\frac{1}{2}} d\nu.$$

Therefore, if the potential is of the form

$$(\mu^2 - \nu^2)^{-1} \{f(\mu) + F(\nu)\},$$

the equation of the trajectory can always be reduced to quadratures, by §7.* By means of Hamilton's Characteristic equation, Liouville has generalized this result so as to include free motion in space (*Journal*, Vol. 12, 1847).

(5) On a sphere with ordinary polar coordinates,

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2 \theta \cdot d\phi^2.$$

Hence

$$\int v ds$$

becomes

$$\int a v \sin \theta \left\{ \left(\frac{d\theta}{\sin \theta} \right)^2 + d\phi^2 \right\}^{\frac{1}{2}},$$

and similar results may be stated.

This method may clearly be applied very widely; and it possesses the advantage of establishing a complete correspondence between the two systems of orbits, so that, for instance, the stability of either orbit is determined at once when that of the other is known.

8. A curve of quickest descent from one point to another in a given field of force, with given velocity of projection from the first point, is

such that

$$\delta \int \frac{ds}{v} = 0,$$

and is therefore the same as a free orbit in a field which would make the velocity equal to v^{-1} ; this latter implies a definite law of potential and a definite velocity of projection. The converse proposition is of course equally true. Examples of their application have been given at length by Mr. Townsend in the *Quarterly Journal of Mathematics*, Vol. xv.; this short and direct statement has already been given by Sir W. Thomson (*vide* Tait and Steele's *Dynamics*). We notice that the constrained orbit is not brachistochronous beyond the point at which

* [I find that this method has been applied to the discussion of some cases of motion on an ellipsoid by Mr. W. R. W. Roberts in *Math. Soc. Proc.*, 1883, Vol. xiv., pp. 230—5.]

the free orbit becomes unstable; so that simple observation as to whether a consecutive free orbit does or does not intersect the given one between the two fixed points, in Mr. Townsend's examples, determines at once whether the corresponding brachistochrone is real, or only apparent, as corresponding simply to a maximum-minimum.

9. *For rays of light or sound, Fermat's Principle of Least Time,*

$$\delta \int v^{-1} ds = 0, \text{ or } \delta \int \mu ds = 0,$$

follows at once from the notion of waves. The analogy of this to the principle of least action for a particle has long been known, and indeed this case of the principle of action appears to have been suggested to Euler by the interpretation of Fermat's principle according to the corpuscular theory of light. (Serret, *Bulletin de Math.*, II., p. 97.) We can therefore apply to the case of rays all the transformations that have been indicated for the motion of a particle, and all the examples already given can be translated into solutions for the paths of rays in heterogeneous media by simply writing μ for v . The restriction as to velocity of projection that often affected the generality of the transformations is here rather an advantage, as it simplifies the law of the index of refraction. The following are simple examples:—

(1) By inverting the straight rays in a uniform medium, we find that, in a medium for which $\mu \propto r^{-2}$, the rays are circles passing through the origin.

(2) If $\mu \propto \sqrt{\frac{2}{r} - \frac{1}{a}}$, the rays are ellipses with the origin as focus, the major axis of each is $2a$, and the rays can never pass outside an exterior caustic which is an ellipse with the radiant point and the origin as foci. (*Vide* Tait and Steele's *Dynamics*, Ch. iv.)

Further, if in these examples r denote the distance from an axis, the particle can have a constant velocity along the axis in addition to the orbital motion round it; and therefore for a medium distributed in cylindric layers round an axis we have for the rays helical curves,

(1) On circular cylinders,

(2) On elliptic cylinders,

whose forms are completely determined. And we see also that the rays in (2) are bounded externally by a caustic elliptic cylinder, and that in (1) they all pass through a line parallel to the axes; and, as the conjugate foci for small pencils lie on the caustics, they are completely determinable, and each pencil has an infinite number of conjugate foci symmetrically placed.

By inverting this class of cases, we arrive at distributions of refractive index which lead to ring-shaped caustics, with corresponding properties.

10. For a chain of uniform density hanging in a field of force whose potential is V , the figure of equilibrium is determined by the condition

$$\delta \int (V + \lambda) ds = 0,$$

where λ is a constant that has to be adjusted to the given length of the chain, according to the known method of the Calculus of Variations. There is here one condition more than in the case of the motion of a particle. We conclude, however, that the forms assumed by chains of different lengths attached to two fixed points are the trajectories of particles projected from one of the points so as to pass through the other in a field where

$$v = V + \lambda,$$

and the potential is therefore $-\frac{1}{2}(V + \lambda)^2$, λ being variable.

11. When the chain assumes the form of a circle round a centre of force as centre, the determination of its stability is thus at once reduced to Newton's well-known determination of the stability of a circular orbit, by calculation of the apsidal angle. If the apsidal angle be imaginary, the circular form will be stable; if it be real, the circular form will only be stable for arcs shorter than the distance between two consecutive apses. For the condition of stability is that the potential energy be a minimum, and we know, by Jacobi's rule in the Calculus of Variations,* that it can only be a minimum when taken to a limit short of the point in which the curve is met by any consecutive curve of equilibrium of the same length. As was to be expected, the value of the apsidal angle contains λ , and therefore depends on the length of the chain; however, the relation between λ and the radius of the circle is easily determined. In fact, following the analogy with dynamics of a particle, we have, if

$$u = r^{-1},$$

$$\frac{d^2 u}{d\theta^2} + u = \frac{P}{h^2 u^3} = \frac{V + \lambda}{h^2} \frac{dV}{du},$$

since P , the attractive force in the problem of the particle, is the rate of increase of the potential outwards, and is therefore the rate of decrease of $\frac{1}{2}v^2$ outwards from the origin. When the orbit is nearly circular, we have

$$u = a + x,$$

$$V \equiv \phi u = \phi a + x\phi' a,$$

where x is small; and h is a constant for which we may write the

* See Routh's *Rigid Dynamics*, 3rd ed., p. 561.

approximate value $h = vp = (\phi a + \lambda) a^{-1}$.

$$\begin{aligned} \text{Therefore } \frac{d^2x}{d\theta^2} + x + a &= \frac{ha + x\phi'a}{h^3} (\phi'a + x\phi''a) \\ &= \frac{a}{h} \phi'a + x \left\{ \left(\frac{\phi'a}{h} \right)^2 + \frac{a\phi''a}{h} \right\}. \end{aligned}$$

Therefore, if we make $\phi'a = h$,

we have for x the oscillatory value

$$x = A \sin(\mu\theta + a),$$

where

$$\mu^2 = \frac{a\phi''a}{\phi'a},$$

and the apsidal angle is π/μ .

For a force varying as the inverse n^{th} power of the distance acting on the string, we have

$$\phi u = Bu^{n-1},$$

$$\frac{u\phi''u}{\phi'(u)} = n-2,$$

so that, if $n < 2$, the apses will be imaginary; and we conclude that the circular form will be unstable for the law of the inverse square, or any higher inverse power.* Under a lower power, a string originally held in a circular or nearly circular form round the centre of force as centre will maintain that form; under any other power, it will depart from that form unless held fast at points whose distance is less than the distance between two consecutive apses. This appears to be at variance with Mr. Townsend's statement (*Quarterly Journal of Math.*, xiii., p. 238).

12. When the force emanates from a fixed axis, the trajectory of a particle will be a curve, on a cylinder whose cross-section is the plane central orbit, such that the velocity parallel to the axis of the cylinder will be constant. When the cross-section is closed, the trajectory will be a helical curve. So also one of the catenaries for the corresponding law of force will be this helical curve; but any length more than one turn of it would usually be unstable.

The difficulty in establishing a strict analogy between the orbit of a particle and the form of a catenary arises from the fact that between two points an infinite number of catenaries of different lengths can lie, while only a definite number of orbits with given total energy can pass through them. This explains the appearance of the additional constant λ .

The problem of determining what forces must be applied to a chain,

* [Mr. R. R. Webb has given for the period of that vibration of the string in which it is divided into m complete waves, the expression $\frac{2\pi}{m} \left\{ ag \frac{m^2 + n - 2}{m^2 + 1} \right\}^{-\frac{1}{2}}$, the law of force being $g(ar)^{-n}$ (*Cambridge Math. Tripos*, Jan., 1884); which confirms the above statement.]

or how it must be loaded, in order that it may retain a certain given form, is, of course, much less general than the inverse problem of the determination of the system of catenaries that are possible in a given field of force; and accordingly the analogy is easily enough stated for the more special problem, but in that form it applies only to the special curve considered.

The following, however, is a specimen of the kind of results that flow from the more general form of the analogy:—

Suppose rays diverging from a centre in a medium of varying index μ to touch a caustic; the paths of the rays will be the curves assumed by uniform strings, fastened to the fixed centre and wrapped round the caustic, in a field of force whose potential is equal to μ .

The provision that the string has to touch the caustic, here disposes of the extra degree of generality in the catenaries, and limits us to that *one* which passes through the two terminals, and has the same tangent as the ray at one of them.

13. A very interesting *series of natural catenaries* is included in the problem of finding the form assumed by a flexible string carrying an electric current in a given magnetic field. Here the energy condition leads to the solution that for equilibrium the variation of the number of tubes of magnetic force enclosed by the string should be zero, while for stability the number of tubes enclosed must be a maximum.

It follows that all such catenaries are geodesic curves on the surfaces formed by the lines of force which intersect them; for, if not, then, by making them geodesics on wider surfaces, they can be made to include more lines of force.

Let us take the case in which the force is everywhere in a fixed direction. We know that it must then also be uniform. The catenary will be a geodesic on a cylindrical surface, and would therefore become straight when the surface is unrolled. Hence its projection on a cross-section of the surface will also be of constant length, and the cylinder will therefore include the greatest number of tubes of force (*i.e.*, the greatest area) when it is circular. Thus the catenaries will all be circular helices; but it is clear that more than a single turn of the helix will be unstable, unless the string is confined by being enclosed in a non-conducting cylinder, or wrapped round one. Even if this cylinder do not lie along the lines of force, or if it is not circular, the portion of string in contact with it will still lie along a geodesic.

Le Roux has observed the helical form in a platinum wire, rendered flexible by incandescence, and placed in the sensibly uniform magnetic field between two flat poles of an electro-magnet. Professor Stokes has deduced the helical form analytically in connection with the

explanation of the spiral discharges in a rarefied gas, along a magnetic field, observed by Spottiswoode. (*Vide* Chrystal, *Encyc. Brit.*, Art. "Electricity"; Spottiswoode, *Proc. Roy. Soc.*, 1875.) We notice that, to explain the number of convolutions observed in the spirals, we must also take cognizance of the outer and less heated portions of the gas as performing the part of the constraining cylinder mentioned above.

The method which we have just employed shows also that, in a field of force radiating from a centre, the catenary will be a geodesic on a right circular cone; while, to obtain more than a single turn of the spiral, the string must be enclosed in or wrapped round a circular cone. This form of the catenary has been already deduced analytically by Darboux.

A flexible electric current attracted by a very powerful straight current will, if lying in one plane, assume a form such that

$$\cos \psi = 1 - A \log \frac{y}{a},$$

where ψ is the angle its direction makes with that of the straight current, at a point whose distance from it is y .

On the Direct Application of the Principle of Least Action to the Dynamics of Solid and Fluid Systems, and Analogous Elastic Problems. By PROFESSOR J. LARMOR, Queen's College, Galway.

[Read March 13th, 1884.]

1. In the preceding paper it has been observed that the most direct and compendious method of stating the mathematical conditions of a physical problem is to express it as a maximum or minimum relation, and that in dynamics this can always be accomplished by means of the Principle of Least Action. It is natural, then, to employ this principle in the transformation of dynamical problems, and in the investigation of analogies between different departments of the science; and it was so employed in connection with Dynamics of a Particle and related subjects.

We proceed to apply the same method to dynamics of general systems. There are two well-known forms of the minimum principle which we shall use, viz., that if T be the kinetic energy, and V the potential energy of the system, then, between any two fixed configurations,

$$(1) \quad \delta \int (T - V) dt = 0, \text{ when the time of motion is constant,}$$

$$(2) \quad \delta \int T dt = 0, \text{ when the total energy of the motion } E (\equiv T + V) \text{ is constant.}$$