THE FIRST MONTH OF GEOMETRY.

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While the many committees and associations are at work deciding what mathematics shall be taught in the high school, it may be of some value to discuss in the meantime some of our methods of teaching. The following paper is a presentation of one way of introducing geometry to the pupil. In working out this method I have assumed that the difficulties with which the pupil must ordinarily contend (taken in the order in which they occur) are:

1. The technical notation and language consisting of many new words, such as hypothesis, axioms, postulate, perpendicular, vertical (used in a sense in no way relating to the ordinary word), supplement, complement, etc.

2. The logical difficulties, or the pupil's inability to understand why much effort is expended in proving facts that appear obvious to him. Even when the pupil may understand the class discussion, he feels uncertain of himself for surely teachers would not be paid to explain such trivial matters.

3. While struggling with the new words and the apparent unreasonableness of the teacher, the pupil must also learn to present his ideas in a new way, unlike the way he is allowed to present them in an English or even an algebra class, and in a mechanical form with a reason attached to every statement.

The first difficulty is the easiest to overcome as we may proceed slowly enough to let the many polysyllabic words become meaningful. The third difficulty could be overcome by letting the pupil present his ideas in any way he chooses during the first month. This will mean more work for the teacher as more time will be required to read and correct the homework; and any method involving more work for the teacher is, for that very reason, doomed to fail. The second difficulty is considered sufficiently treated by some writers if they present a page of optical illusions showing that lengths of lines must not be judged by appearances. Such illusions are a poor argument for the pupil readily answers that these illusions were made purposely to deceive, and no such purpose exists in his own drawings. Further, any attempt during the first months to make the pupil see the logical setting of the facts seems futile because this appreciation of logic is to be developed by the subject, and hence can not be assumed to exist or be used as a foundation.

The following method is therefore not based on logic but on the idea that nothing shall be introduced until the need for it has been shown.

In the first lessons I present in the standard way a few problems for whose solution a knowledge of geometry is needed such as measuring the distance from A to B on opposite sides of a lake. To visualize the problem to the pupil I draw the figure on top of a table, placing upright pencils at the points, and having pupils sight along the pencils to get them in a line. The next problem, which I use because it emphasizes angles, is that of the farmer who wishes to continue on the further side of a lake, a fence which he has started on one side. This is solved by using a rectangle, not a triangle. The third problem consists in tearing off the corners of a triangle and placing the angles about a point. We see that it is a good guess that the outer edges of the sum lie along a line. "But," I ask, "by what reasoning could you convince a blind man of that fact?" Geometry will furnish the proofs, and will determine the lengths that can not be measured.

The talk thus far has involved triangles, and our first question is "When are triangles equal?" or, better stated, "What information must we have about two triangles in order to decide whether or not they are equal?" The first guess is that they are equal if the three sides of one are equal to the three sides of the other. Evidently correct, for if I take three sticks I can make only one triangle from them. But in making the triangle, I also *simultaneously create* three angles; and how do we know that the three angles which I make will be respectively equal to the three creations of some one else? Hence, before we can discuss triangles further, we must see what we know about angles.

I. *Angles.* In considering angles the ideas to be treated are:

(a) The notation. First we name an angle by stating its vertex and its sides; then we write it as $\angle(BA, AC)$, and finally contract this to $\angle BAC$. If this development of the notation is presented instead of the mere fact, the pupil can see reason for our action.

(b) The size of an angle is independent of the lengths of its sides.

(c) Equality of angles is tested by placing the vertices together, making two sides coincide, and then deciding whether the remaining two sides will or can be made to coincide.

(d) Measurement of angles. We define perpendicular lines

to be lines which meet or cross so that two adjacent angles are equal. This defines a right angle. A unit angle is defined as one ninetieth of a right angle. Here we prove (that is, explain or call attention to) that all right angles are equal, and see the necessity for proving it, namely, in order that the units used by all people may be the same.

(e) The construction for copying an angle, bisecting angles, drawing perpendiculars, postponing the proofs that our constructions are correct.

The words complement and supplement are postponed until we are ready to use them.

II. *Triangles.* We now return to triangles. There were six elements to be considered, three sides and three angles. Evidently if the six of one triangle are equal to the six of the other, the triangles are equal. This we can verify by superposition. I write on the blackboard: Place A on A'; rotate the triangle if necessary until AB lies along A'B'; B' will fall on B because, etc. Then I ask: Did we use all of the six elements? Perhaps the triangles are equal if only five elements of each are equal, or if only four, or three. What is the minimum? We try two. Finally we decide to discuss the theorem:

Two triangles are equal if three elements of one equal three corresponding elements of the other, provided (1) at least one is a line, and (2) the three are properly selected.

The necessity for the first provision can not, of course, be proved rigorously. We can make it look reasonable by drawing similar triangles; but an argument is not useless because it lacks rigor if the pupil recognizes that fact and sees why. An incomplete proof should always be an inducement for further study in the hope of completing the gaps. The second provision refers to two triangles having two sides and an opposite angle equal to these corresponding elements in another triangle. A single illustration is sufficient to show the necessity of the provision. (And at the beginning of the year this usually causes greater astonishment than any optical illusion.) This case nearly always introduces the question: Is it always possible to construct a triangle with any selection of sticks for sides? I also consider here the case wherein two angles and an opposite side are equal to the corresponding elements. Assuming that the sum of the angles of a triangle is 180° , we can see that this choice of elements is satisfactory. If we had a similar relation for the sum of the sides of a triangle, the second provision could

be eliminated entirely. The reader will notice that in using this assumption about the sum of the angles, I am creating additional reasons for the study of the subject.

This discussion will take several days or a week and in order to give written home work during these days, I assign construction problems dealing with angles, bisectors, perpendiculars, etc., and spend half the class period for such matters.

Finally, we are ready to prove the two standard theorems about the two sides and the included angle, and the two angles and the included side. In the statement of these theorems, the text books have abbreviated Euclid but not improved him, for the important idea is not that the two triangles are equal, but that the remaining elements are equal. In various ways I try to impress this fact on the class but the pupil seldom sees the significance of it until he has worked some exercises in which the equality of the triangles is merely a step to some more important equality.

In working the exercises based on these theorems I emphasize that the procedure will be: From some given information about lines and angles, we are to find two triangles which can be proved equal, and then in these two triangles we are to discover that certain other lines and angles are equal. The first exercises should not say, for example: Prove $EF = GH$, or prove $\angle 1 = \angle 2$, but should read: If we know that $AB = AC$, $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, what other lines and angles can we discover to be equal? The pupil when writing the exercise labels one part "Given Information" and finishes with "Discoveries."

III. *The Given Information.* The making of discoveries would be easy if our information always referred directly to the six elements of the triangle. But in most cases the information is camouflaged, or concealed in a forest of apparently

irrelevant ideas. Since it is always so in real life, why should we expect it otherwise in geometry? An illustration will show how this arises. If we are told that $AC=BD$, we can not use this information because neither AC nor BD are sides of the two triangles. The subtraction of BC from each line gives us the desired information $AB=CD$.

Hence, the necessity having arisen, it is at this stage of the work that we begin the study of five axioms which allow us to add, subtract, multiply, divide and equate lines or angles. In the practice on the use of the axioms, the pupil's state of mind should be that of trying to make new equations from old ones, using his five tools. One of the parlor games of long ago was that of seeing who could make the most words out of the letters from some long given word. The pupil should see that he is playing the same kind of a game. Illustrations from algebra, exercises on areas, volumes, weights, etc., should be used as well as lines and angles. The words complement and supplement are not yet introduced in order that all the emphasis may fall on the axioms and that all results may be written as equations.

IV. *The Discoveries.* The pupil's attitude should now be that of a detective unraveling a mystery for which he has been furnished a few clues. He is seeking new relations between angles and lines; but during the first month we should not expect to see the relative importance of his various discoveries. That the equality of one pair of lines is more important than some other equality is decided by the future uses to which we can put the equality. Hence, whenever triangles are proved equal, the pupil may as well write down all of his new discoveries or equations.

Thus, in that fundamental proposition in which the vertex angle of an isosceles triangle is bisected (if ABC is the triangle, $AB=BC$, and BD is the bisector of the angle B) the pupil makes three discoveries: $AD=DB$, $\angle ADB=\angle CDB$, $\angle A=\angle C$. These he next translates into English: (1) The bisector of the vertex angle of an isosceles triangle bisects the base. (2) The bisector of the vertex angle of an isosceles triangle is perpendicular to the base. (3) The base angles of an isosceles triangle are equal. Here all the discoveries are worth remembering—a thing which will not be true in every exercise. Later, but not now, we can expect the pupil to judge the relative values of the discoveries.

The above theorem is one which can profitably be studied for a long time as it is useful in making the pupil see that the

object of geometry is to draw conclusions from given facts. Write, for example, on the blackboard the six equations between the twelve elements of the two triangles: $AB=BC$, $AD=DB$, $\angle A = \angle C$, and ask: Given any five of these equations, can I always prove the sixth? Given any four, can I prove the remaining two? Given any three, can I prove the other three? Given any two, etc.¹ The pupil is also required to state each exercise in English: thus, if he chooses to use $AD=DB$ and $AB=CB$ from which to prove the remaining four, he says: If the midpoint of the base of an isosceles triangle is joined to the vertex, then etc. This enables us to introduce the word *median* not to give the pupil one more word to memorize, but because the word is useful in simplifying our language. In fact, only after the pupil has stumbled through several circumlocutory phrases do I introduce him to a word like *midpoint*.

V. *The Presentation*. Nothing has been said in the class so far about the manner of presenting a proof. The proofs of the fundamental theorems are written in paragraphs and the directions in any construction problem are written in paragraphs; much as the pupil might write an exposition for his English class on how to erect a tent when camping or how to bake a sponge cake. But now that the pupil has solved the mystery he is no longer a detective; to shift the metaphor and use terms from the commercial world, he is now a salesman trying to sell his idea to a skeptical customer; and just as a salesman is not sent out on the road to sell goods until he has become acquainted with them by experience in the stockroom, so the pupil should not be expected to present a standard proof until the various theorems have become a part of his knowledge. We are now ready to introduce words like hypothesis and congruent and for the first time we write a proof with one column for statements and another for reasons. The pupil may wonder why we did not do the same thing in studying algebra the previous year. I have myself asked the same question of many teachers and the answers are so various that I wonder if we have analyzed the question carefully. My own answer, in part, is that in algebra we deal with a very few fundamental and simple operations; in geometry a theorem is frequently a statement of a result obtained from many complex operations which we do not wish

¹And if the class is a bright one, discuss: How many exercises are involved in such a discussion? And why does the third of the three conclusions in the above theorem omit the words "bisector of the vertex angle?" And can it be proved without drawing BD?

to repeat in detail on every occasion that we use the result, and our column of reasons is a summary or reminder of these details. Algebra is the hammer and saw and screw driver of our tool chest; geometry is a collection of machines which swallow some boards and nails at one end and turn out a finished article of furniture at the other end.

The numbering of all the steps of a proof is discouraged as they are not all of equal importance. Some steps are written with further indentations from the margin than other steps (particularly when lines or angles are added, subtracted or equated to find some more important equality) such as we see in briefs for a debate. Important steps are labeled 1, 2, 3, and the three steps which are always necessary to prove triangles congruent are labeled 1, 2, 3.

The analysis of an exercise is a test of the pupils' logical abilities, and the presentation is or should be a test only of his good judgment in presenting facts; but have we fully realized how a study of the presentation can be used as a means of introducing some of the theorems and words which are so frequently merely thrust upon the pupil? The pupil can be taught to see that, like a salesman, his presentation can be improved by an increase of his vocabulary. The words complement and supplement are introduced at this point for the same reason that words like *median* and *midpoint* were introduced above. Similarly many subordinate equations are eliminated when we introduce the corollary, "if two angles of a triangle were equal, the third angles are equal." Many theorems exist merely to shorten the proof of some future theorem. Nothing illustrates this better than to have some exercise proved without using certain theorems and words and then rewritten, using the theorems. The resulting simplicity of our presentations is the reason for the existence of many theorems. But how often do we call attention to that simple fact? If the beginnings of the subject appear ridiculous to the pupil, isn't it because we have shown him too many details without showing the why and wherefore of the details? The writer seldom assigns a theorem without first studying with the class some exercise for whose solution the theorem is of value in shortening the presentation. The exercises are not put in the text to illustrate the theorems; the theorems are inserted to help solve the exercises.

To summarize, this introduction to geometry is based on the following principle: Instead of making the subject strictly

logical from the beginning, let us first acquaint the pupil with the subject matter, then develop his reasoning abilities without too much emphasis on how he presents his facts paying due attention, of course, to his grammar and English) and last, when he is ready to present his results let us begin to emphasize the details.

A MINERAL ALMANAC.

A preliminary summary of the mineral production of the United States in 1919 has just been issued by the United States Geological Survey, Department of the Interior. This pamphlet, which consists of 128 pages, shows in compact form the output and value of about 70 minerals. Many of the figures are necessarily estimates, but the cooperation of the mineral producers and the long experience of the Geological Survey in this work gives assurance that the estimates represent very nearly the actual production. The tables show the quantity and value of the output of each mineral for five or ten years and the imports and exports of many minerals for several years.

A brief introduction points out the salient features of the mineral industry in 1919. One of the tables compares the values of commodities produced in 1919 with the values that they would have had at the average prices prevailing in 1913. This summary shows the leading mineral products of each state and the world's production of some minerals for a number of years.

A copy of this report may be obtained on application to the Director U. S. Geological Survey, Washington. D. C.

MANGANESE DEPOSITS IN COLORADO AND WYOMING.

During 1917 and 1918, when there was a shortage of manganese ore and a possibility of greater shortage, the deposits throughout the country were actively exploited, and every one reported was examined, in order to determine its probable size and the grade of material it would yield. Geologists of the State and National surveys examined about 1,200 deposits. Some valuable deposits were discovered, but many that were examined can not yield much ore.

The United States Geological Survey has just published two reports that describe manganese deposits in Colorado and Wyoming. The only deposit that was known in Wyoming in 1918 consists of manganese oxides in limestone, and lies near the head of Sheep Creek, Albany County, 38 miles north of Medicine Bow. A small quantity of rather high grade ore was shipped from this deposit.

Nine groups of deposits in Colorado were also examined, and seven of them are described in one of the reports just issued. Reports on the deposits of manganese ore at Leadville and Red Cliff have already been published. Four groups of deposits lie near Salida, Chaffee County; one near Iola, Gunnison County; one near Moffat, Saguache County, and one near Cedar, San Miguel County. Ore has been shipped from some of these deposits, and others could yield small quantities of high-grade ore, but probably none can be worked with profit except when, as in 1917 and 1918, a great shortage exists and prices are much higher than those now offered.